Electromagnetics

<Chap. 9> Transmission Lines

Section 9.4 ~ 9.5

(1st of **week 11**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 9 | Contents for 1st class of week 11

Sec 4. Wave characteristics on Finite Transmission Lines (Cont'd)

- Standing wave ratio
- Resistive termination & arbitrary termination of TR-line
- Wave behavior observed from source

Voltage reflection coefficient

- **Voltage reflection coefficient**

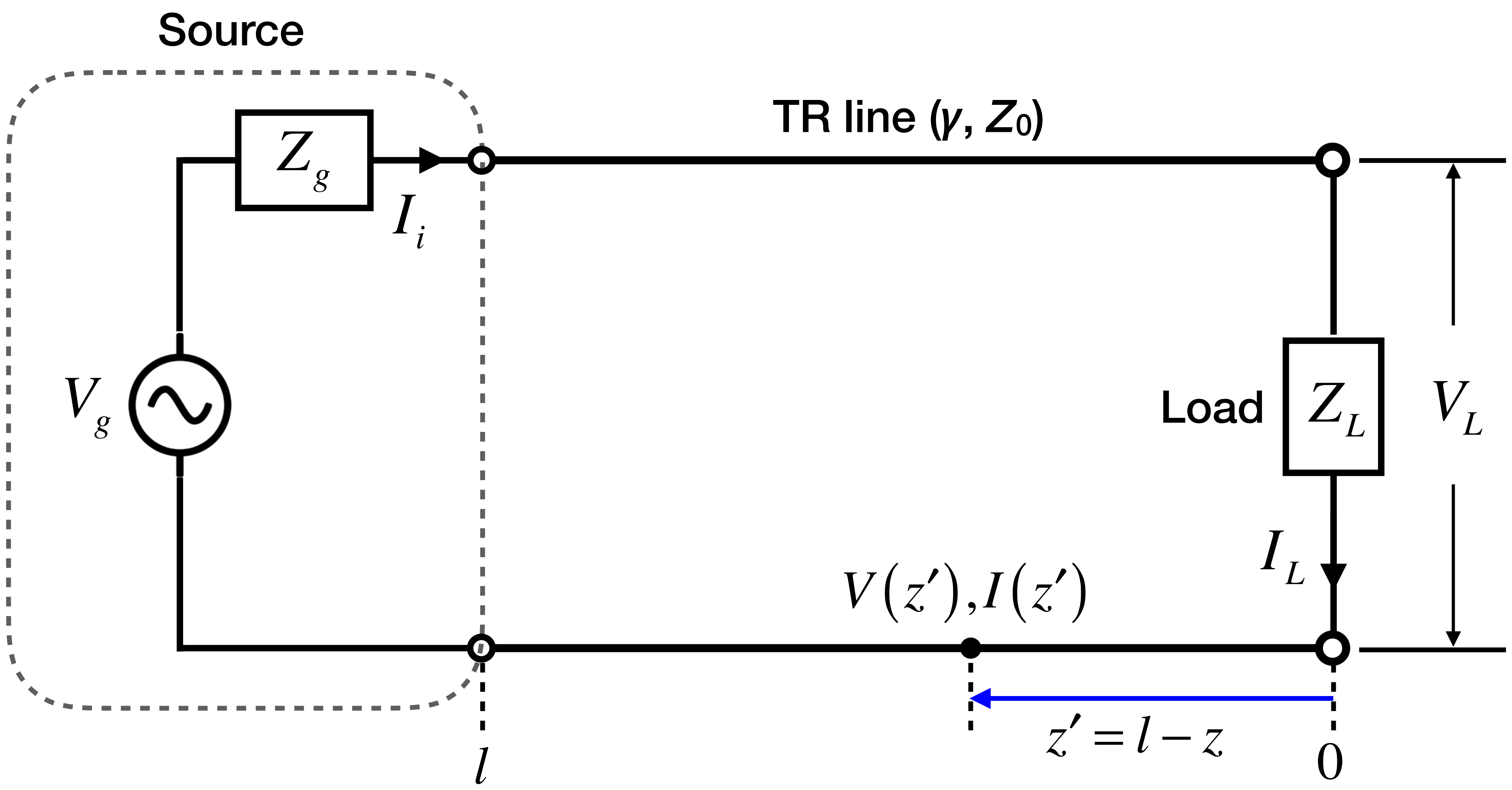
  - Under non-matching condition ($Z_L \neq Z_0$)

$$V(z') = \frac{I_L}{2}\Big[\underbrace{(Z_L + Z_0)e^{\gamma z'}}_{\text{Incident}} + \underbrace{(Z_L - Z_0)e^{-\gamma z'}}_{\text{Reflected}}\Big]$$

$$= \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + \frac{Z_L - Z_0}{Z_L + Z_0}e^{-2\gamma z'}\right]$$

$$\therefore V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + \Gamma e^{-2\gamma z'}\right]$$

Where $\Gamma \triangleq \dfrac{Z_L - Z_0}{Z_L + Z_0}$ *: Voltage reflection coefficient*



Here, $\Gamma \triangleq \dfrac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma}$

- *Ratio* of complex amplitude of ***reflected / incident wave***
- $|\Gamma| \leq 1$ (magnitude)

Standing wave ratio (SWR) [1/2]

- **Standing wave ratio (SWR) for lossless TR-line**

  - Lossless TR-line → $\gamma = j\beta$ and $Z_0 = R_0$

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1+\Gamma e^{-2\gamma z'}\right]$$

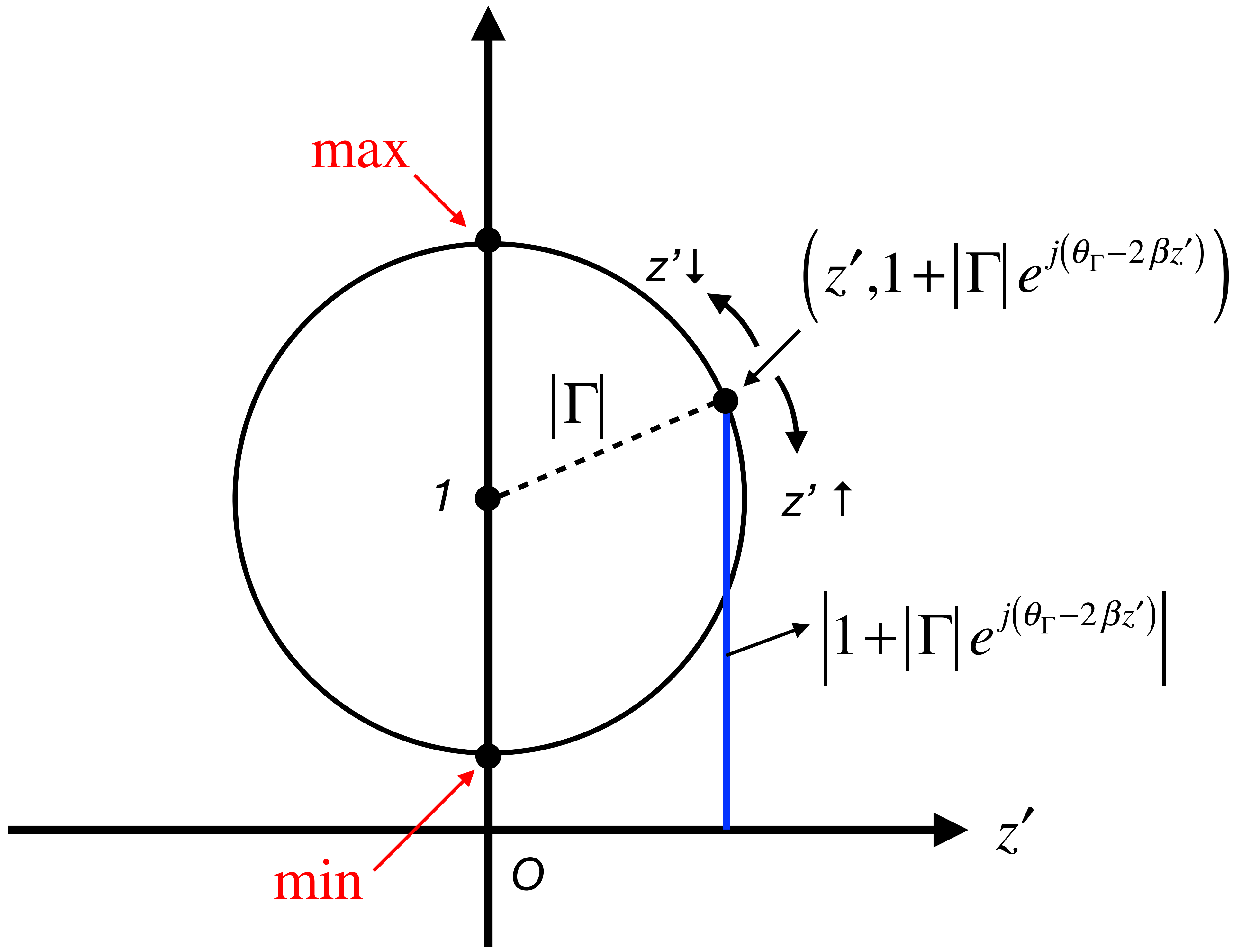
$$\rightarrow \quad V(z') = \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}\left[1+\Gamma e^{-j2\beta z'}\right] \quad (\because \gamma = j\beta \text{ and } Z_0 = R_0)$$

$$= \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}\left[1+|\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right] \quad (\because \Gamma = |\Gamma| e^{-j\theta_\Gamma})$$

  - Magnitude of voltage $V(z')$ oscillating between its maxima and minima

$$|V(z')| = \underbrace{\left|\frac{I_L}{2}\right|}_{\text{Const.}} \cdot \underbrace{|Z_L + R_0|}_{\text{Const.}} \cdot \underbrace{\left|e^{j\beta z'}\right|}_{?} \cdot \underbrace{\left|1+|\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right|}_{\text{Oscillating due to } z'}$$

$$\begin{cases} \max|V(z')| = V_{\max} = \left|\frac{I_L}{2}\right| \cdot |Z_L + R_0| \cdot (1+|\Gamma|) \\ \min|V(z')| = V_{\min} = \left|\frac{I_L}{2}\right| \cdot |Z_L + R_0| \cdot (1-|\Gamma|) \end{cases}$$



$|\Gamma|$: Radius of a circle

$\left|1+|\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right|$: distance from origin to the circle

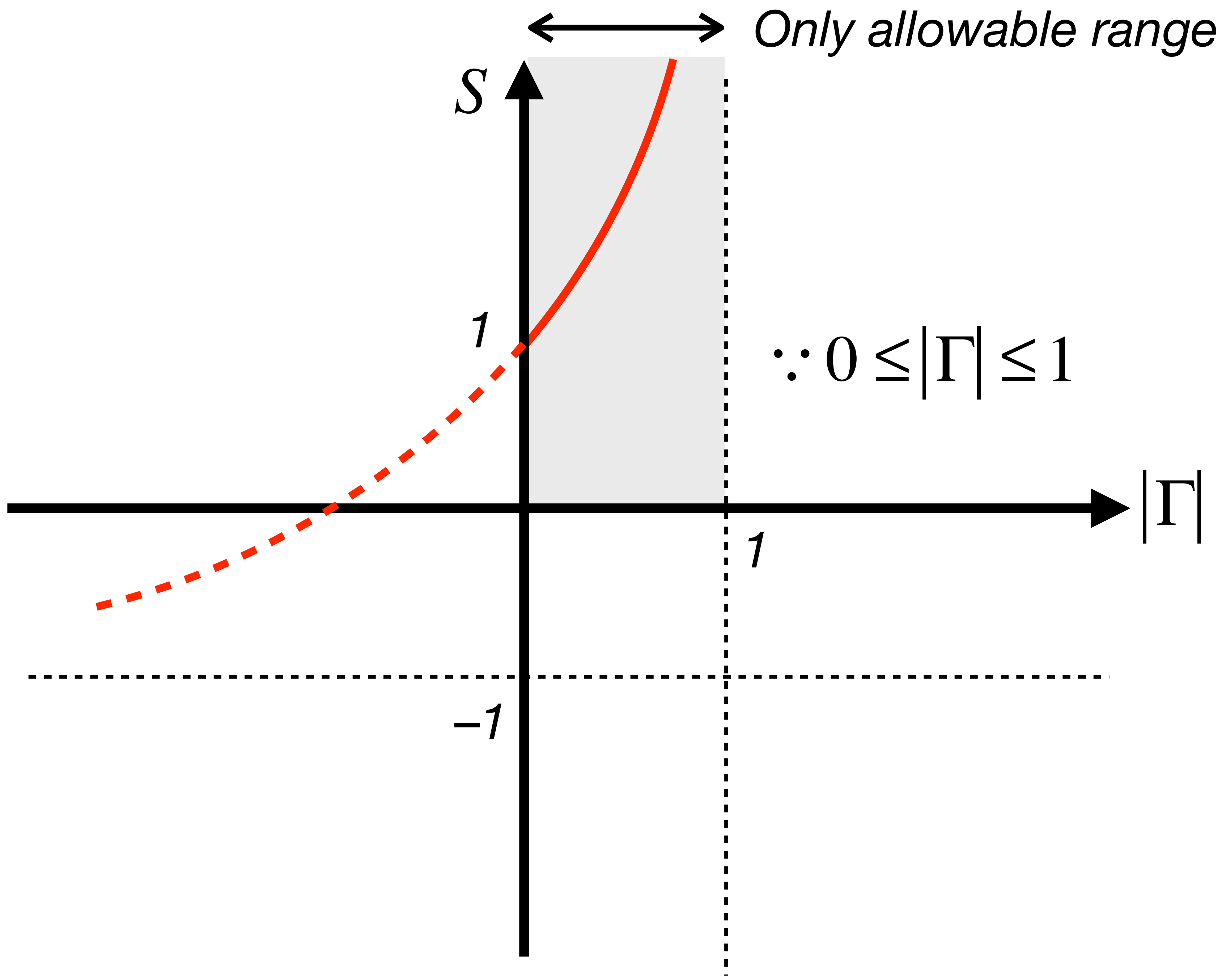
$$\boxed{SWR \text{ or } S \triangleq \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \Longleftrightarrow |\Gamma| = \frac{S-1}{S+1}}$$

Chap. 9 | Standing wave ratio (SWR) [2/2]

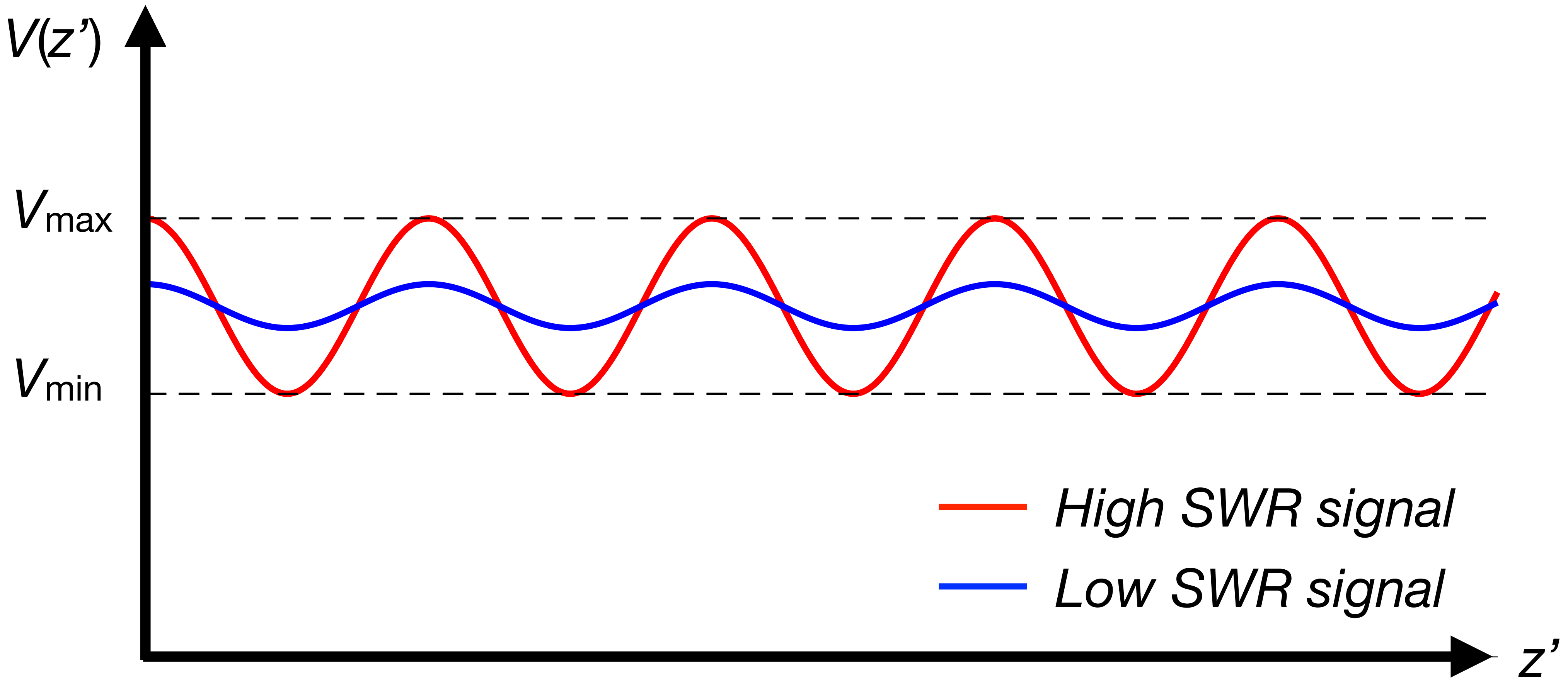
- **Standing wave ratio (*S*) vs. voltage reflection coefficient (*Γ*)**
  - S can be expressed in terms of *Γ*
  - What is the relationship between two like?

$$S = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \quad \rightarrow \quad S = -1 + \frac{2}{1-|\Gamma|}$$

  - *S monotonically increases from "1" to "infinity"* with |*Γ*|!
    - |*Γ*|↑ → *S*↑ (high reflection = high SWR)
    - |*Γ*|↑: ***High reflection*** of the wave
      - ✓ ***Low*** TR-line efficiency
      - ✓ ***High power loss*** → Undesirable!



$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \begin{cases} Z_L = Z_0 : \ \Gamma = 0 \ \rightarrow \ S = 1 & \textit{Minimum!} \\ \text{(matched)} & \\ Z_L = \infty : \ \Gamma = 1 \ \rightarrow \ S = \infty & \textit{Undesirable} \\ \text{(Open Circuit)} & \\ Z_L = 0 : \ \Gamma = -1 \ \rightarrow \ S = \infty & \textit{Undesirable} \\ \text{(Short Circuit)} & \end{cases}$$



∴ S used as a measure how well impedance matched (i.e. $Z_L = Z_0$)

Periodicity of *V* and *I* [1/2]

- **Periodicity of V_{max}, V_{min}, I_{max}, I_{min}**

- *V* and *I* are periodic functions with respect to *z'*

$$\begin{cases} V(z') = \dfrac{I_L}{2}(Z_L + R_0)e^{j\beta z'}\left[1 + |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right] \\ I(z') = \dfrac{I_L}{2R_0}(Z_L + R_0)e^{j\beta z'}\left[1 - |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right] \end{cases}$$

Even multiple

$$\text{(Cond. A)}\ \theta_\Gamma - 2\beta z'_M = -2n\pi, \quad (n = 0, 1, 2, \cdots) \quad \rightarrow \quad V_{max}, I_{min}$$

$$\text{(Cond. B)}\ \theta_\Gamma - 2\beta z'_m = -(2n+1)\pi, \quad (n = 0, 1, 2, \cdots) \quad \rightarrow \quad V_{min}, I_{max}$$

Odd multiple

- Resistive termination ($Z_L = R_L$): On TR-line, where we have max and min?

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \frac{R_L - R_0}{R_L + R_0}$$

Case 1) $R_L > R_0$ → ***Γ* > 0: Positive real** & $\theta_\Gamma = 0$

- At *z'* = 0 (i.e. at the load end)

$$\cancel{\theta_\Gamma} - 2\beta \cancel{z'} = 0$$

→ Satisfying **(Cond. A)** when *n* = 0

→ First V_{max}, I_{min}

- At *z'* ≠ 0

$$\cancel{\theta_\Gamma} - 2\beta z' = -2n\pi, \quad (n = 1, 2, \cdots)$$

$$\rightarrow z'_M = \frac{n\pi}{\beta} = \frac{\lambda\pi}{2\pi}n = n\frac{\lambda}{2}$$

→ Higher order V_{max}, I_{min}

Case 2) $R_L < R_0$ → ***Γ* < 0: Negative real** & $\theta_\Gamma = -\pi$

- At *z'* = 0 (i.e. at the load end)

$$\theta_\Gamma - 2\beta \cancel{z'} = -\pi$$

→ Satisfying **(Cond. B)** when *n* = 0

→ First V_{min}, I_{max}

- At *z'* ≠ 0

$$\theta_\Gamma - 2\beta z' = -(2n+1)\pi, \quad (n = 1, 2, \cdots)$$

$$\rightarrow z' = \frac{n\pi}{\beta} = \frac{\lambda\pi}{2\pi}n = n\frac{\lambda}{2}$$

→ Higher order V_{min}, I_{max}

Chap. 9 | Periodicity of *V* and *I* [2/2]

Case 1) $R_L > R_0$ → **$\Gamma > 0$: Positive real** & $\theta_\Gamma = 0$

- At $z' = 0$ (i.e. at the load end)

$\cancel{\theta_\Gamma} - 2\beta \cancel{z'} = 0$ → Satisfying **(Cond. A)** when $n = 0$

→ First $\boldsymbol{V_{max}}$, $\boldsymbol{I_{min}}$

- At $z' \neq 0$

$\cancel{\theta_\Gamma} - 2\beta z' = -2n\pi, \quad (n = 1, 2, \cdots)$

$\rightarrow z'_M = \frac{n\pi}{\beta} = \frac{\lambda\pi}{2\pi} n = n\frac{\lambda}{2}$

→ Higher order $\boldsymbol{V_{max}}$, $\boldsymbol{I_{min}}$

Case 2) $R_L < R_0$ → **$\Gamma < 0$: Negative real** & $\theta_\Gamma = -\pi$

- At $z' = 0$ (i.e. at the load end)

$\theta_\Gamma - 2\beta \cancel{z'} = -\pi$ → Satisfying **(Cond. B)** when $n = 0$

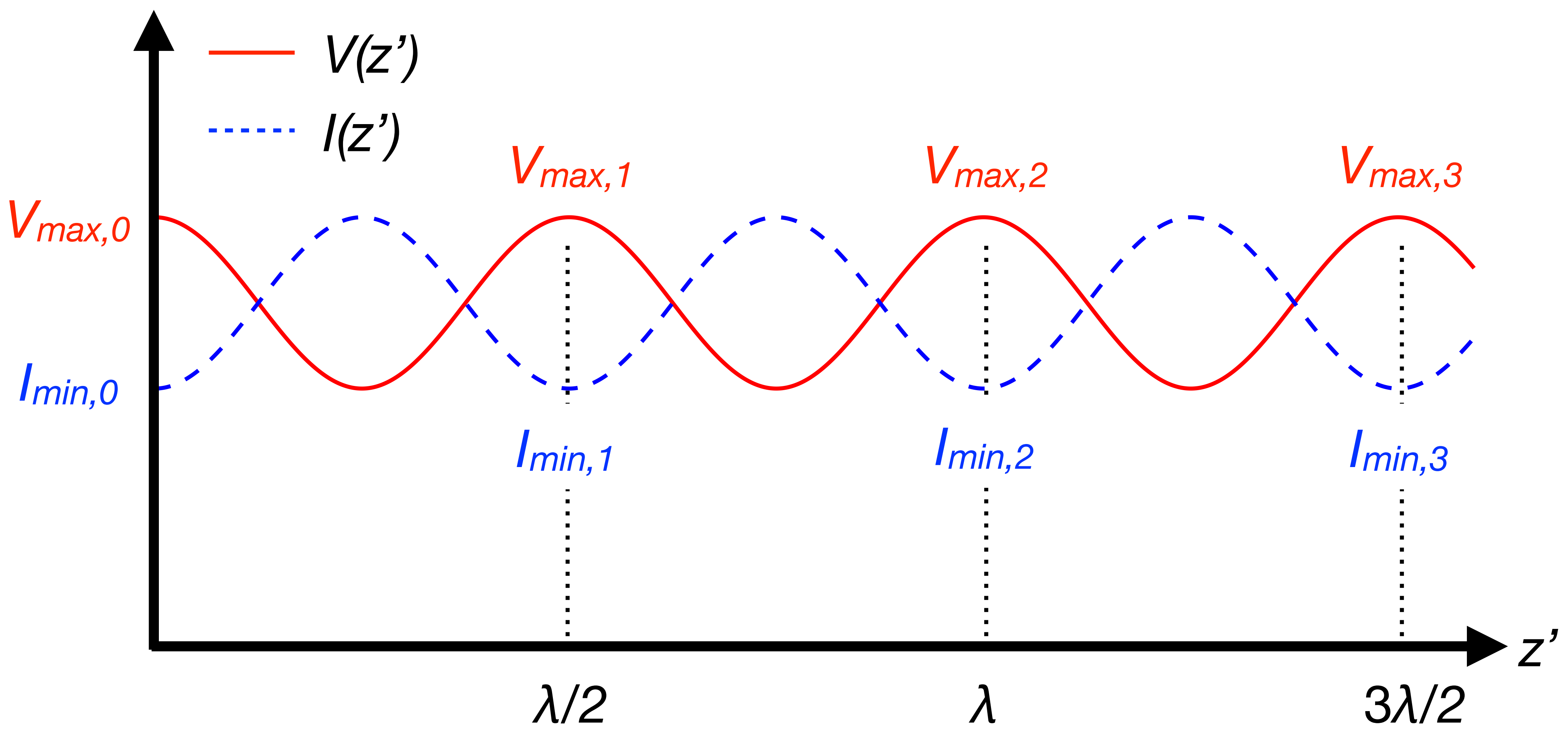
→ First $\boldsymbol{V_{min}}$, $\boldsymbol{I_{max}}$

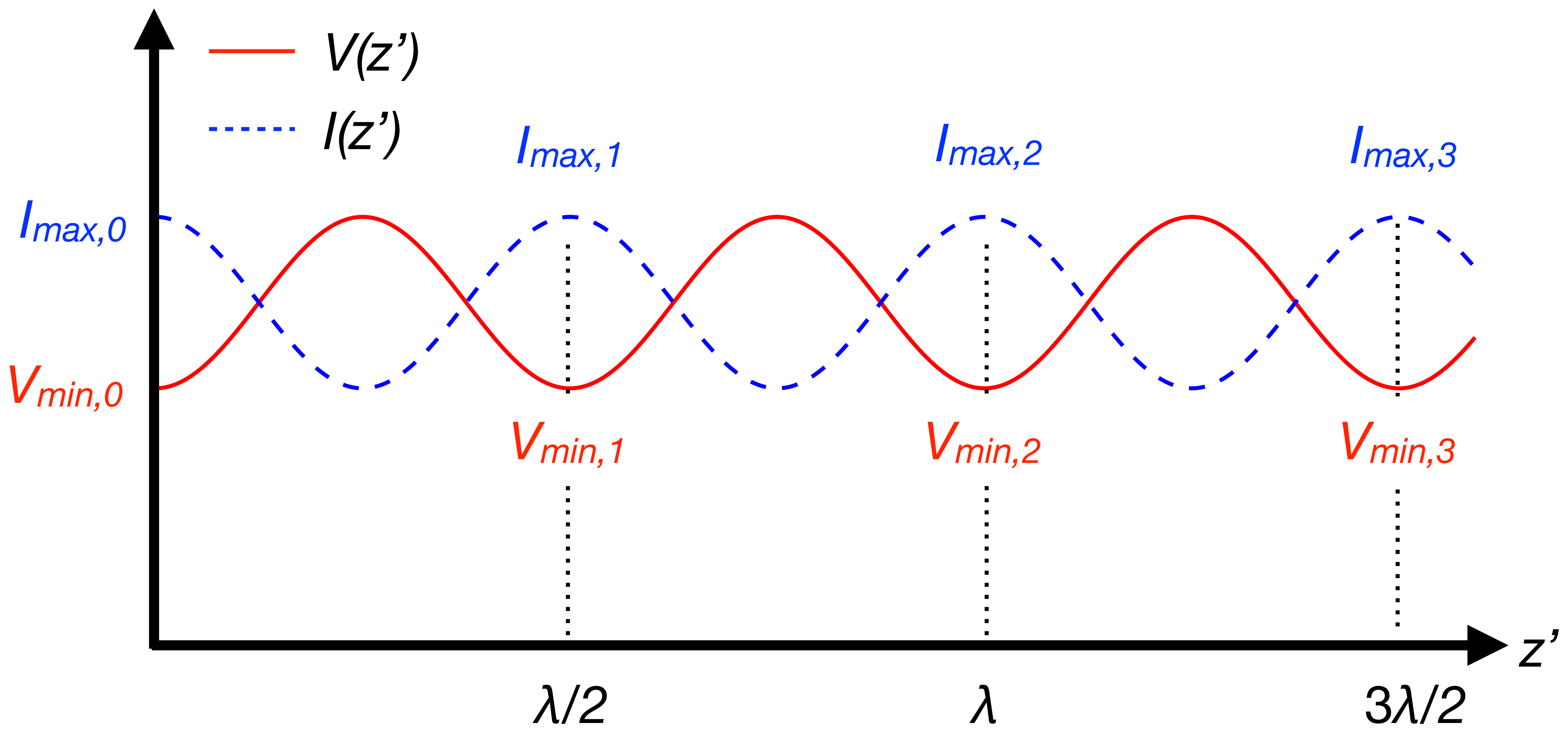
- At $z' \neq 0$

$\theta_\Gamma - 2\beta z' = -(2n+1)\pi, \quad (n = 1, 2, \cdots)$

$\rightarrow z' = \frac{n\pi}{\beta} = \frac{\lambda\pi}{2\pi} n = n\frac{\lambda}{2}$

→ Higher order $\boldsymbol{V_{min}}$, $\boldsymbol{I_{max}}$





Chap. 9 | Standing wave ratio (SWR) Example

Engineering example — We can easily ***identify an arbitrary load ($Z_L = R_L$)*** at the end of loss TR-line (with characteristic impedance of R_0) ***by measuring S***. How to express R_L in terms of S and R_0?

$$V(z') = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}\right] \rightarrow V(z') = \frac{I_L}{2}\left[(R_L + R_0)e^{j\beta z'} + (R_L - R_0)e^{-j\beta z'}\right] = I_L\left(R_L \cos\beta z' + jR_0 \sin\beta z'\right)$$

Case 1) If $R_L > R_0$ → **Γ > 0: Positive real** & $\theta_\Gamma = 0$

- A *first voltage* maxima: z' = 0

$$\cancel{\theta_\Gamma} - 2\beta \cancel{z'} = 0 = -2n\pi\Big|_{n=0}$$

$$\beta z' = 0 \quad \rightarrow \quad \sin\beta z' = 0 \quad \rightarrow \quad V(0) = V_{\max} = I_L R_L$$

- A *first voltage* minima: z' = λ/4

$$\beta z' = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2} \quad \rightarrow \quad \cos\beta z' = 0 \quad \rightarrow \quad V\left(\frac{\lambda}{4}\right) = V_{\min} = I_L R_0$$

$$\boxed{S = \frac{V_{\max}}{V_{\min}} = \frac{I_L R_L}{I_L R_0} = \frac{R_L}{R_0} \quad \rightarrow \quad \therefore R_L = SR_0}$$

Case 2) If $R_L < R_0$ → **Γ < 0: Negative real** & $\theta_\Gamma = -\pi$

- A *first voltage* minima: z' = 0

$$\theta_\Gamma - 2\beta \cancel{z'} = -\pi = -(2n+1)\pi\Big|_{n=0}$$

$$\beta z' = 0 \quad \rightarrow \quad \sin\beta z' = 0 \quad \rightarrow \quad V(0) = V_{\min} = I_L R_L$$

- A *first voltage* maxima: z' = λ/4

$$\beta z' = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2} \quad \rightarrow \quad \cos\beta z' = 0 \quad \rightarrow \quad V\left(\frac{\lambda}{4}\right) = V_{\max} = I_L R_0$$

$$\boxed{S = \frac{V_{\max}}{V_{\min}} = \frac{I_L R_0}{I_L R_L} = \frac{R_0}{R_L} \quad \rightarrow \quad \therefore R_L = \frac{R_0}{S}}$$

Chap. 9 | Arbitrary termination of TR-line

- **"Resistive" termination ($Z_L = R_L$, Previous slides)**

  - Voltage minima or maxima at the load end

- **"Arbitrary" termination ($Z_L = R_L + jX_L$)**

  - Voltage minima or maxima shifted by *d* from the load end

  - If, additional line extended by l_m with resistive termination (R_m)

  → voltage shape does not change! → *Circuit I* = *Circuit II* (Equivalent)

- **How do we identify Z_L experimentally?**

  - Given condition: we measured *S* (SWR) and already know R_0

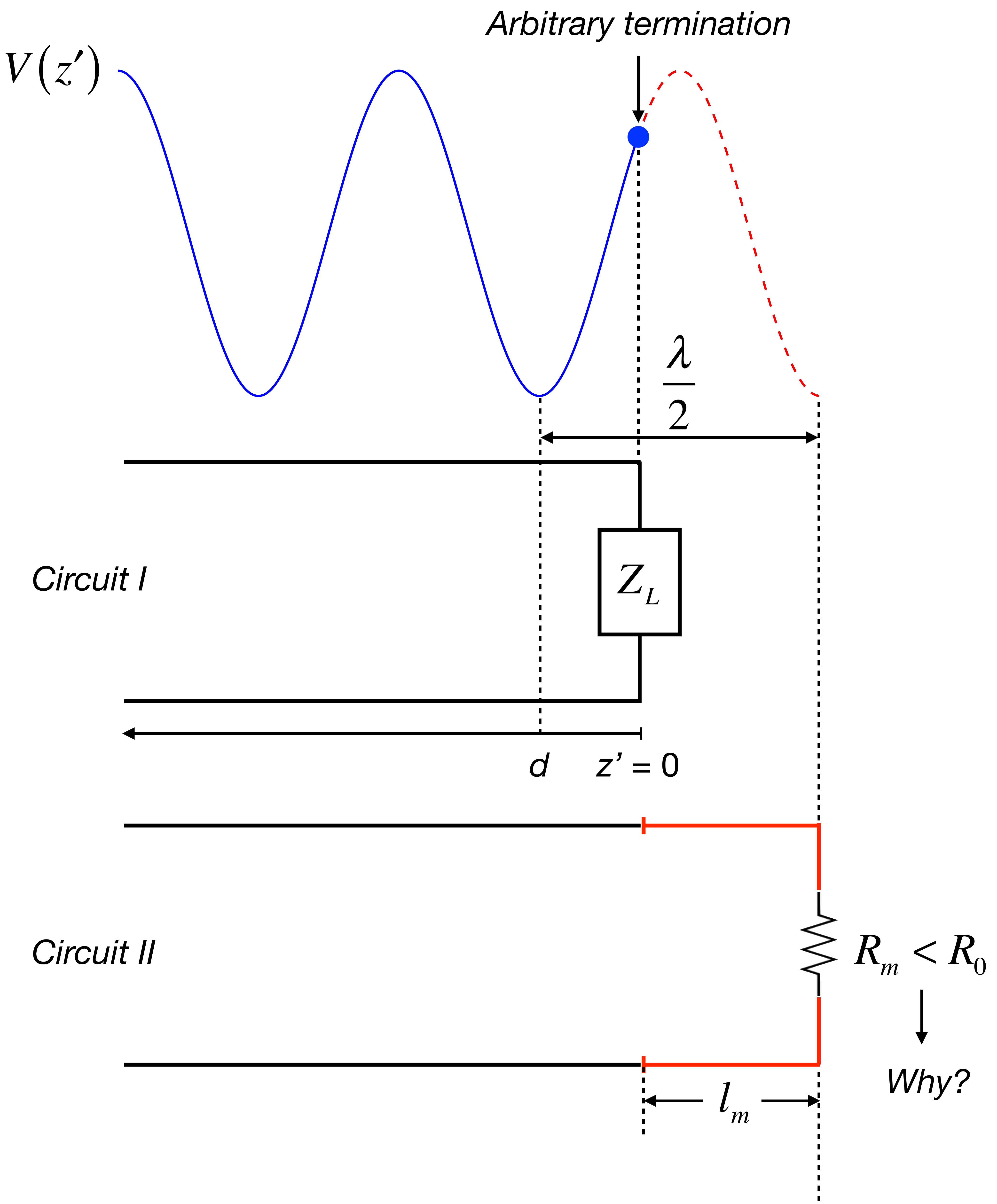
  - **Step 1)** Express Z_L in terms of R_0 and voltage reflection coefficient Γ

$$\boxed{Z_L = \left.\frac{V(z')}{I(z')}\right|_{z'=0} = R_0 \frac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}}}$$

  - **Step 2)** By applying **(Cond. B)**, we can find θ_Γ for first voltage minima

$$\theta_\Gamma - 2\beta d = -(2n+1)\pi\Big|_{n=0} \qquad \boxed{\rightarrow \quad \theta_\Gamma = 2\beta d - \pi}$$

  - **Step 3)** By measuring *S*, we can get $|\Gamma|$ as $\boxed{|\Gamma| = \frac{S-1}{S+1}}$



Chap. 9 | Arbitrary termination of TR-line

Engineering example We measured S = 3 for lossless TR-line of R_0 = 50 (Ω). d = 5 (cm) of the first voltage minima for arbitrary terminated TR-line. *Distance between successive voltage minima = 20 (cm)*. What is an arbitrary load impedance Z_L? What is R_m and l_m for equivalent *Circuit II*?

- **Step 1)** Express Z_L in terms of R_0 and voltage reflection coefficient Γ

$$Z_L = \left.\frac{V(z')}{I(z')}\right|_{z'=0} = R_0 \frac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}}$$

- **Step 2)** By applying **(Cond. B)**, we can find θ_Γ for the first voltage minima

Distance between successive voltage minima

$$\theta_\Gamma - 2\beta d = -(2n+1)\pi\big|_{n=0} \quad \rightarrow \quad \theta_\Gamma = 2\beta d - \pi \qquad \text{Here, } \beta = \frac{2\pi}{\lambda} \text{ where } \frac{\lambda}{2} = 20 \text{ (cm)}$$

$$= \frac{2\pi}{0.4} = 5\pi \text{ (rad/m)} \qquad \rightarrow \theta_\Gamma = 2\times 5\pi \times 0.05 - \pi = -0.5\pi \text{ (rad)}$$

- **Step 3)** By measuring S, we can get $|\Gamma|$ as

$$|\Gamma| = \frac{S-1}{S+1} = \frac{1}{2}$$

$$\boxed{\therefore Z_L = R_0 \frac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}} = 50\frac{1-j0.5}{1+j0.5} = 30 - j40 \ (\Omega)}$$

- Recall Case 2) in slide p.8,

$$R_m = \frac{R_0}{S} = \frac{50}{3} = 16.7 \ (\Omega)$$

- From the relation as below (see voltage graph in previous slide)

$$l_m + d = \frac{\lambda}{2} \quad \rightarrow \quad l_m = \frac{\lambda}{2} - d = 0.2 - 0.05 = 0.15 \text{ (m)}$$

Chap. 9 | Wave behavior observed from source

- **Discussion so far**

  - Effect of load (Z_L) on (V, I) characteristics

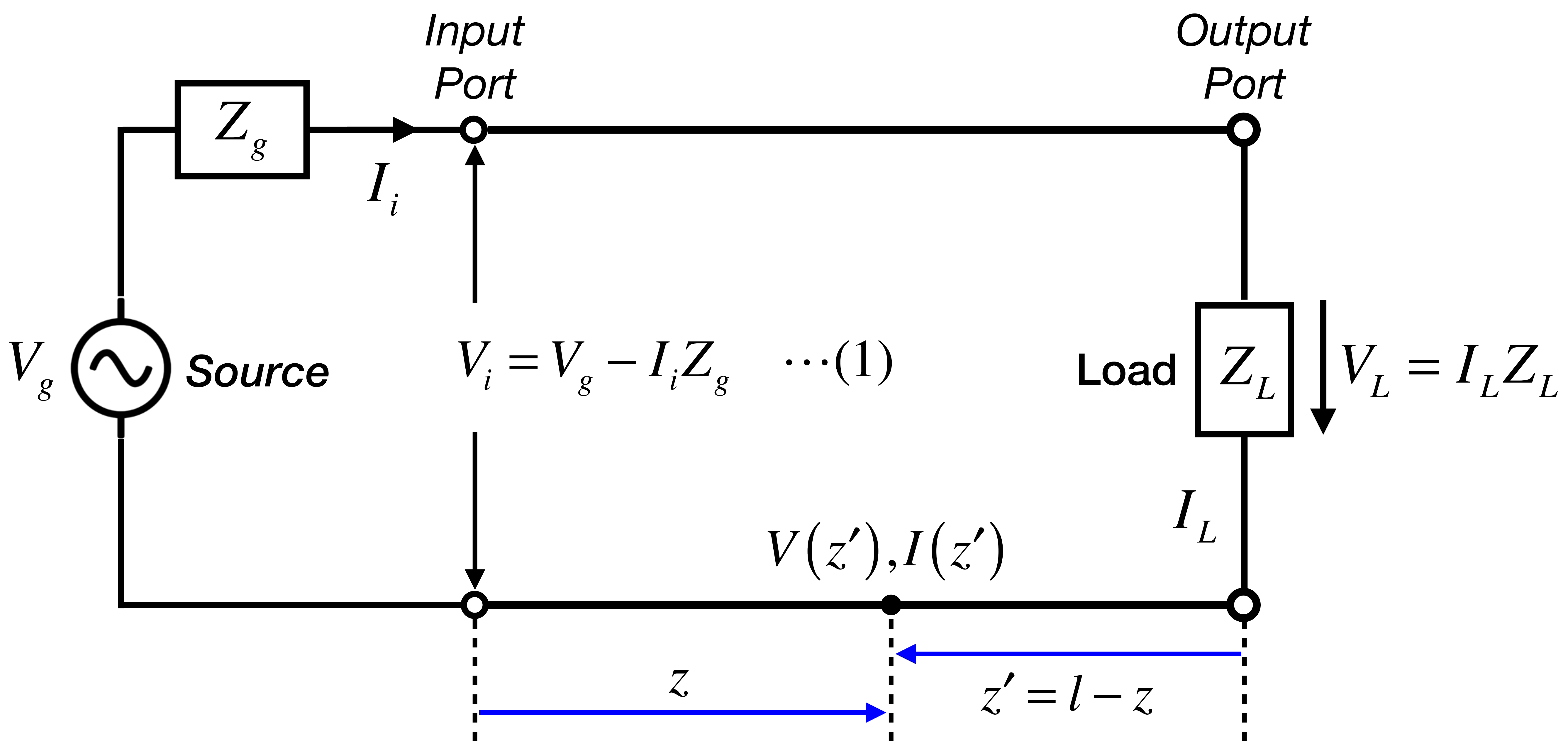
- **Effect of source (V_g and Z_g) on (V, I) characteristics**

> ***Purpose***
>
> **By using source characteristics (V_g, Z_g) &**
>
> **line characteristics (γ, Z_0, l) &**
>
> **load impedance Z_L,**
>
> **We want to determine V, I at any z of the line**

$$\left[\begin{array}{l} V(z') = \dfrac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left(1 + \Gamma e^{-2\gamma z'}\right) \\ I(z') = \dfrac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}\left(1 - \Gamma e^{-2\gamma z'}\right) \end{array}\right. \quad \cdots(2)$$

- At $z' = l$ (source end)

$$\left[\begin{array}{l} V(l) \triangleq V_i = \dfrac{I_L}{2}(Z_L + Z_0)e^{\gamma l}\left(1 + \Gamma e^{-2\gamma l}\right) \\ I(l) \triangleq I_i = \dfrac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma l}\left(1 - \Gamma e^{-2\gamma l}\right) \end{array}\right. \quad \cdots(3)$$



- If we plug ***eqn. (3)*** into ***eqn. (1)***, we get

$$\frac{I_L}{2}(Z_L + Z_0)e^{\gamma l} = \frac{Z_0 V_g}{Z_0 + Z_g}\frac{1}{\left[1 - \Gamma_g \Gamma e^{-2\gamma l}\right]} \quad \cdots(4)$$

where $\boxed{\Gamma_g = \dfrac{Z_g - Z_0}{Z_g + Z_0}}$ *Reflection coefficient at the source end*

c.f.) $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ *Reflection coefficient at the load end*

- If we plug ***eqn. (4)*** into $V(z')$ of ***eqn. (2)***, we get

$$\boxed{V(z') = \frac{Z_0 V_g}{Z_0 + Z_g}e^{-\gamma z}\left(\frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma l}}\right)}$$

Chap. 9 | Wave behavior observed from source

$$V(z') = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}\left(1+\Gamma e^{-2\gamma z'}\right)\left(1-\Gamma_g \Gamma e^{-2\gamma l}\right)^{-1}$$

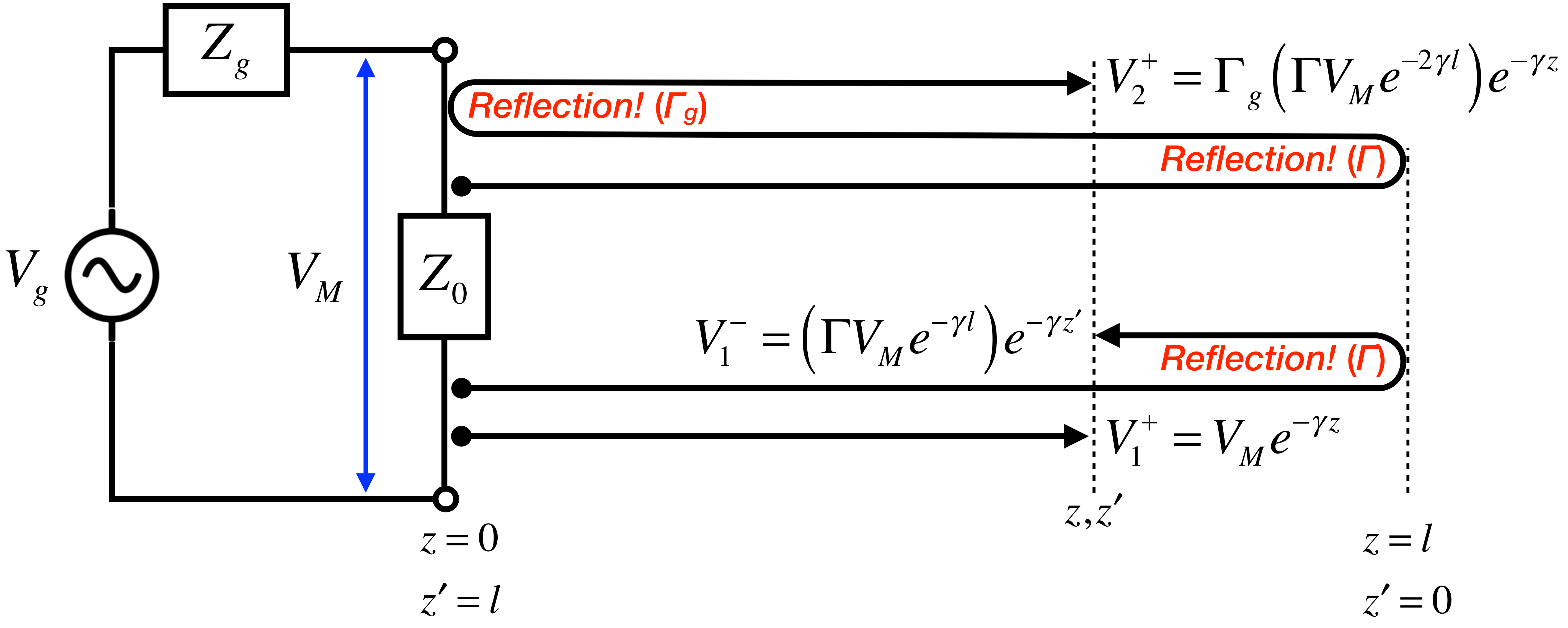
$$= \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}\left(1+\Gamma e^{-2\gamma z'}\right)\left[1+\Gamma_g \Gamma e^{-2\gamma l} - \left(\Gamma_g \Gamma e^{-2\gamma l}\right)^2 + \cdots\right]$$ ***Taylor expansion***

$$= \frac{Z_0 V_g}{Z_0 + Z_g}\left[e^{-\gamma z} + \left(\Gamma e^{-\gamma l}\right)e^{-\gamma z'} + \Gamma_g\left(\Gamma e^{-2\gamma l}\right)e^{-\gamma z} + \cdots\right]$$

$$V(z') = V_1^+ + V_1^- + V_2^+ + V_2^- + \cdots = \begin{cases} V_1^+ = \dfrac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z} = V_M e^{-\gamma z}, \\ V_1^- = \left(\Gamma V_M e^{-\gamma l}\right) e^{-\gamma z'}, \\ V_2^+ = \Gamma_g\left(\Gamma V_M e^{-2\gamma l}\right) e^{-\gamma z}, \\ \vdots \end{cases}$$

where $\boxed{V_M = \dfrac{Z_0 V_g}{Z_0 + Z_g}}$

Voltage initially sent down to TR-line at the input port



Trajectory of each voltage wave

- V_1^+ : Initial wave traveling by *z* in +z direction
- V_1^- : V_1^+ reached at $z = l$ (or $z' = 0$), reflected (Γ), and then traveling by *z'* in –z direction
- V_2^+ : V_1^- reached at $z' = l$ (or $z = 0$), reflected (Γ_g), and then traveling by *z* in +z direction
- …

∴ ***Resulting standing wave V(z') →***
= Sum of all waves traveling in both directions!

* In the real case ($\gamma = \alpha + j\beta$)

- Amplitude of reflected waves diminishes each time it transverses the line

Some special cases

* Matched condition ($Z_L = Z_0$)

- $\Gamma = 0$ → Only V_1^+ exists, *no reflected wave*

* $Z_L \neq Z_0$, but $Z_g = Z_0$

- $\Gamma_g = 0$ → V_1^+ and V_1^- exists, *no higher-order reflected waves*

Question

Distortionless line에서 리액턴스가 0이라는 말이 i(z)와 v(z)의 phasor차이가 없이 동등하게 진행됨은 이해를 했습니다.
그런데 I(z)와 v(z)의 phasor차이가 없다는 사실이 어떤 물리적인 의미를 갖고 전송선에서 이것이 어떤 profit을 가지는지 궁금합니다.

Lossy TR-line

- Propagation constant (γ)

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{(RG-\omega^2 LC)+j\omega(LG+RC)}$$

$$= \left[\kappa(\omega)e^{j\Theta(\omega)}\right]^{\frac{1}{2}} = \sqrt{\kappa(\omega)}e^{j\frac{\Theta(\omega)}{2}}$$

$$= \sqrt{\kappa(\omega)}\left(\cos\frac{\Theta(\omega)}{2}+j\sin\frac{\Theta(\omega)}{2}\right) = \alpha(\omega)+j\beta(\omega)$$

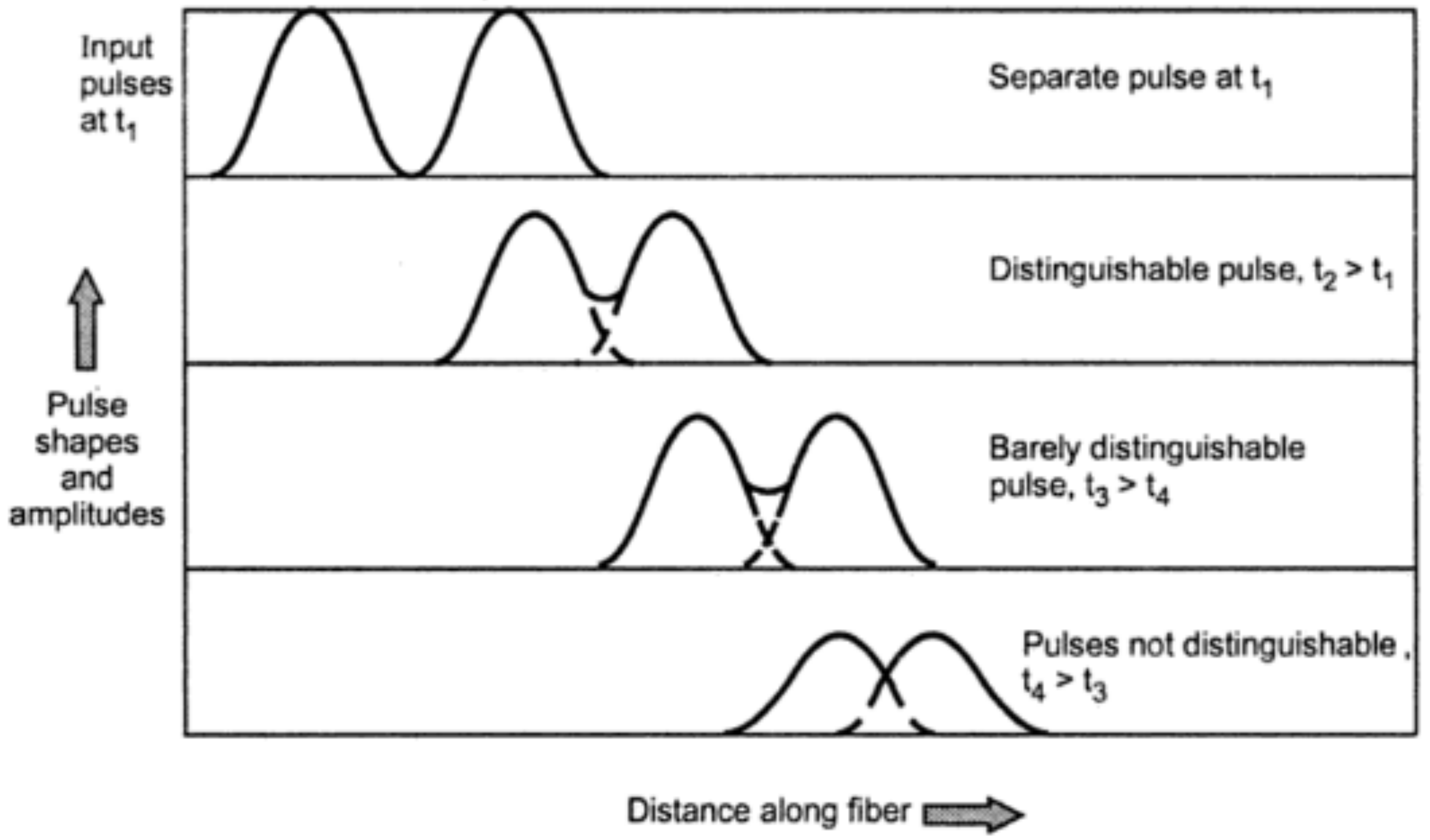


Fig. 1 Dispersion and attenuation in fiber

<Signal distortion>

- Phase velocity

$$u_p = \frac{\omega}{\beta(\omega)} = \frac{\omega}{\sqrt{\kappa(\omega)}\sin\frac{\Theta(\omega)}{2}}$$

→ *"Dispersive system"*

signal at different ω travel at different u_p

→ *Signal distortion*

- Characteristic impedance (Z_0)

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \Re(\omega)+j\Lambda(\omega)$$

→ *Phase shift between V and I*

Img src: Your Electrical Home

Question

Distortionless line에서 리액턴스가 0이라는 말이 i(z)와 v(z)의 phasor차이가 없이 동등하게 진행됨은 이해를 했습니다.
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Distortionless TR-line (R/L = G/C)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{\frac{C}{L}}(R + j\omega L) \longrightarrow u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Non-dispersive

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}}$$

No phase shift

From characteristic impedance (Z_0)

To have reactance to be zero,

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{(R + j\omega L)(G - j\omega C)}{(G + j\omega C)(G - j\omega C)}} = \sqrt{\frac{RG + \omega^2 LC + j\omega(LG - RC)}{G^2 + \omega^2 C^2}} \longrightarrow LG - RC = 0 \rightarrow \boxed{\therefore \frac{R}{L} = \frac{G}{C}}$$

∴ ***Zero reactance condition*** *(no phase shift between V and I)*	⇔	***Distortionless condition*** *(u_p = constant)*

Electromagnetics

<Chap. 9> Transmission Lines

Section 9.4 ~ 9.5

(2nd of **week 11**)

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Chap. 9 | Contents for 2nd class of week 11

Sec 5. Transients on Transmission Lines

- Signal reflection at source & load ends
- Non-oscillating signals (step-function, pulse)
- Transient response for TR-line with resistive vs. reactive termination

Chap. 9 | Wave behavior observed from source (1/2)

- **Discussion so far**

  - Effect of load (Z_L) on (V, I) characteristics

- **Effect of source (V_g and Z_g) on (V, I) characteristics**

Purpose

By using source characteristics (V_g, Z_g) &

line characteristics (γ, Z_0, l) &

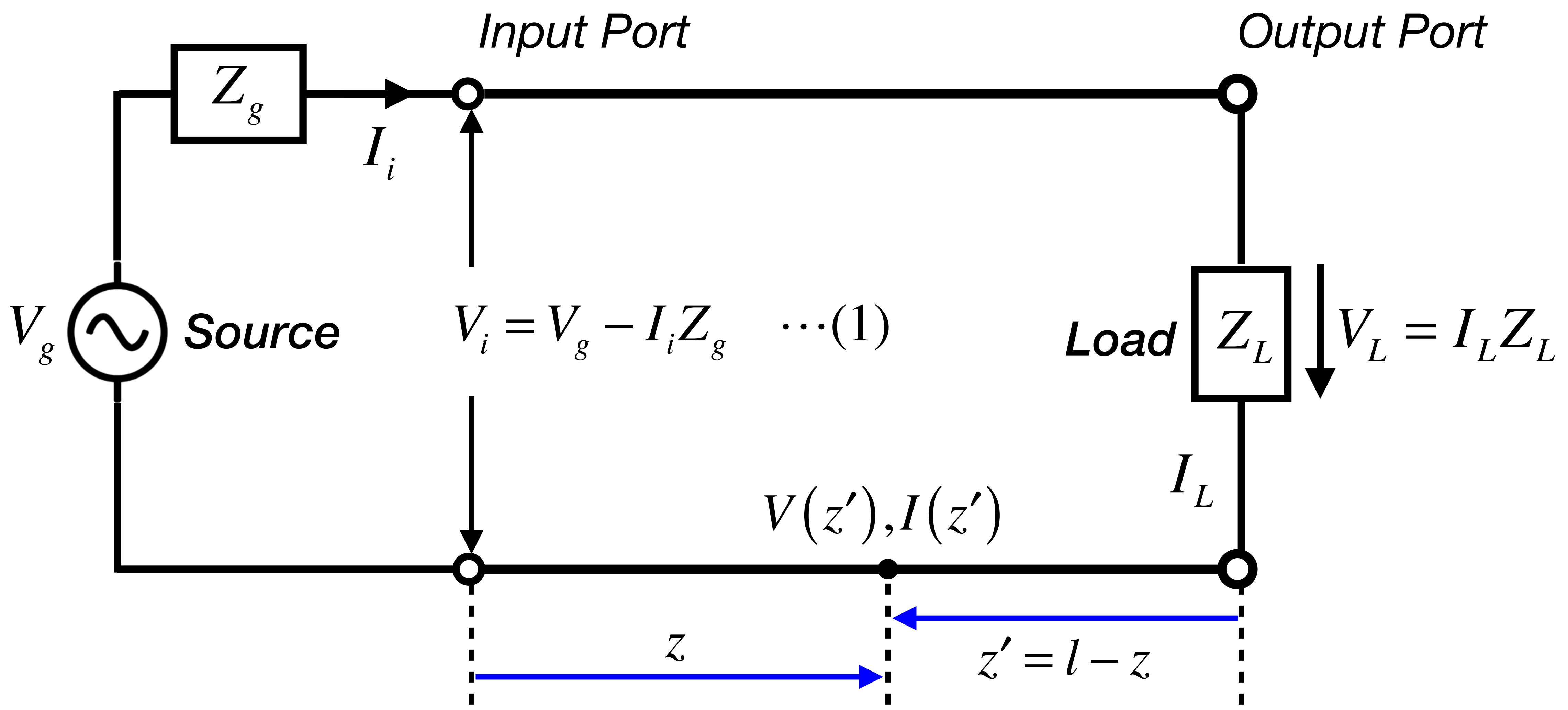
load impedance Z_L,

We want to determine (V, I) at any z of the line

$$\begin{cases} V(z') = \dfrac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left(1 + \Gamma e^{-2\gamma z'}\right) \\ I(z') = \dfrac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}\left(1 - \Gamma e^{-2\gamma z'}\right) \end{cases} \quad \cdots(2)$$

- At $z' = l$ (source end)

$$\begin{cases} V(l) \triangleq V_i = \dfrac{I_L}{2}(Z_L + Z_0)e^{\gamma l}\left(1 + \Gamma e^{-2\gamma l}\right) \\ I(l) \triangleq I_i = \dfrac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma l}\left(1 - \Gamma e^{-2\gamma l}\right) \end{cases} \quad \cdots(3)$$



- If we plug ***eqn. (3)*** into ***eqn. (1)***, we get

$$\frac{I_L}{2}(Z_L + Z_0)e^{\gamma l} = \frac{Z_0 V_g}{Z_0 + Z_g} \frac{1}{\left[1 - \Gamma_g \Gamma e^{-2\gamma l}\right]} \quad \cdots(4)$$

where $\boxed{\Gamma_g = \dfrac{Z_g - Z_0}{Z_g + Z_0}}$ ***Reflection coefficient at the source end***

c.f.) $\boxed{\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}}$

Reflection coefficient at the load end

- If we plug ***eqn. (4)*** into $V(z')$ of ***eqn. (2)***, we get

$$\boxed{V(z') = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}\left(\frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma l}}\right)}$$

Chap. 9 | Wave behavior observed from source (2/2)

$$V(z') = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}\left(1 + \Gamma e^{-2\gamma z'}\right)\left(1 - \Gamma_g \Gamma e^{-2\gamma l}\right)^{-1}$$

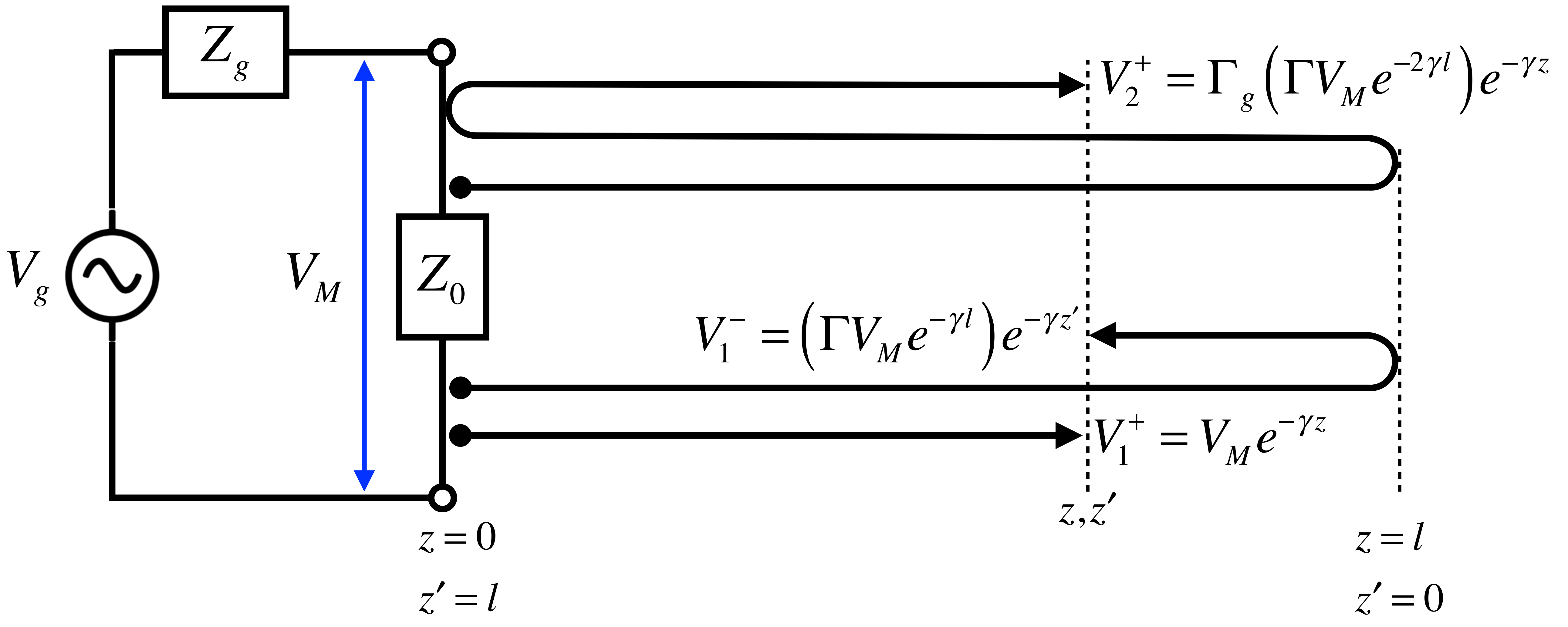
$$= \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}\left(1 + \Gamma e^{-2\gamma z'}\right)\left[1 + \Gamma_g \Gamma e^{-2\gamma l} - \left(\Gamma_g \Gamma e^{-2\gamma l}\right)^2 + \cdots\right] \quad \textit{Taylor expansion}$$

$$= \frac{Z_0 V_g}{Z_0 + Z_g}\left[e^{-\gamma z} + \left(\Gamma e^{-\gamma l}\right)e^{-\gamma z'} + \Gamma_g\left(\Gamma e^{-2\gamma l}\right)e^{-\gamma z} + \cdots\right]$$

$$V(z') = V_1^+ + V_1^- + V_2^+ + V_2^- + \cdots = \begin{cases} V_1^+ = \dfrac{Z_0}{Z_0 + Z_g} V_g e^{-\gamma z} = V_M e^{-\gamma z}, \\ V_1^- = \left(\Gamma V_M e^{-\gamma l}\right)e^{-\gamma z'}, \\ V_2^+ = \Gamma_g\left(\Gamma V_M e^{-2\gamma l}\right)e^{-\gamma z}, \\ \vdots \end{cases}$$

where $\boxed{V_M = \dfrac{Z_0}{Z_0 + Z_g} V_g}$

Voltage initially sent down to TR-line at the input port



Trajectory of each voltage wave

- V_1^+ : Initial wave traveling by z in +z direction
- V_1^- : V_1^+ reached at $z = l$ (or $z' = 0$), reflected (Γ), and then traveling by z' in –z direction
- V_2^+ : V_1^- reached at $z' = l$ (or $z = 0$), reflected (Γ_g), and then traveling by z in +z direction
- …

∴ ***Resulting standing wave V(z') →***
= Sum of all waves traveling in both directions!

* Amplitude of reflected waves decreases each time it transverses the line
∵ $|\Gamma| < 1$, $|\Gamma_g| < 1$, and $\gamma = \boldsymbol{\alpha} + j\beta$

Special cases

* $Z_L = Z_0$ (Matched)
- $\Gamma = 0$ → Only V_1^+ exists, *no reflected wave*

* $Z_L \neq Z_0$, but $Z_g = Z_0$
- $\Gamma_g = 0$ → V_1^+ and V_1^- exists, *no higher-order reflected waves*

Chap. 9 | Transient response: step-function (1/3)

- **Discussion so far**
  - Steady-state, single-frequency ***time-harmonic (i.e. oscillating)*** input & output

- **Transient response for *non-harmonic* signals**
  - Example: Pulse, step-function, ramp, and so on
  - Reactance (X), λ, k, β (due to oscillation) lose their meanings

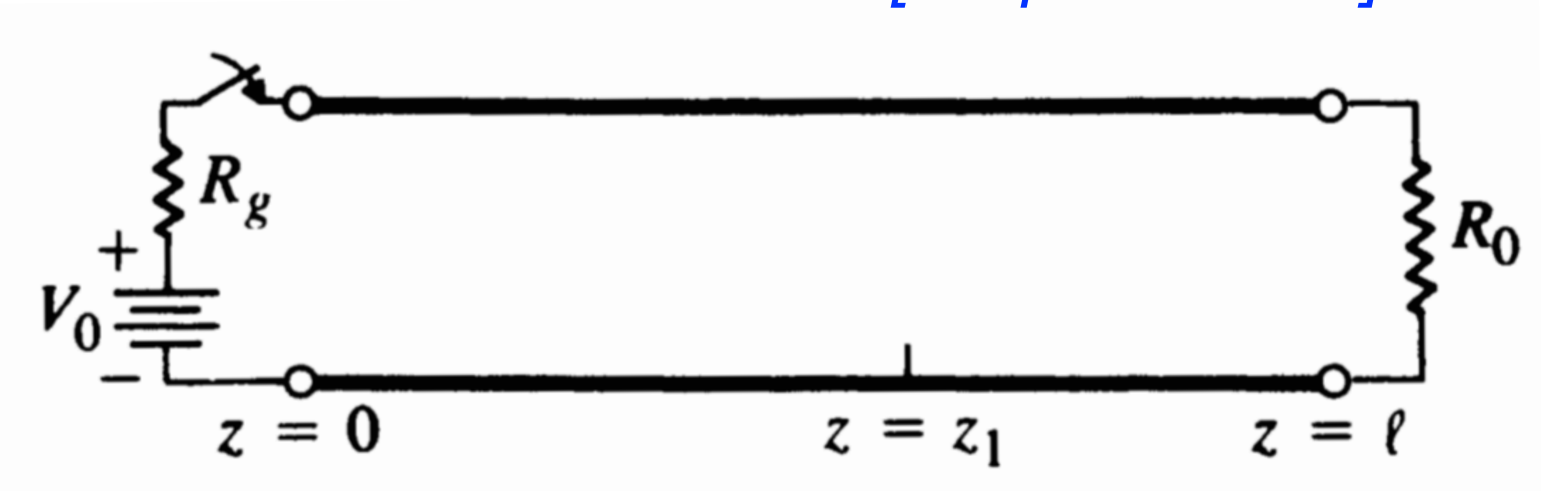
- **Simplest example: step-function signal**
  - DC voltage V_0 applied at $t = 0$
  - R_g: internal (series) resistance

  - <Case 1> $R_L = R_0$
    - ***Matched*** condition → ***No reflection***
    - Impedance looking into TR-line: R_0 (independent of z)

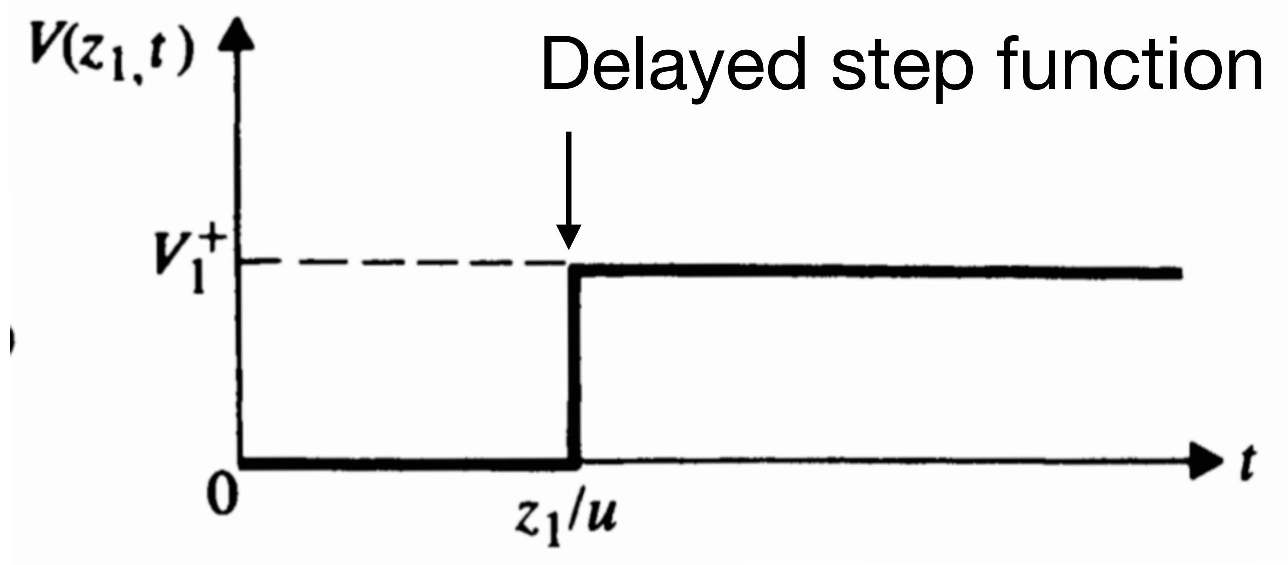
    ∴ Voltage signal traveling in +z-direction (V_1^+) with velocity of u

$$V_1^+ = \frac{R_0}{R_0 + R_g} V_0 \quad \text{and} \quad u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Switch closed at t = 0 [Step-function]



Voltage vs. time at z = z_1



- Takes $t = \dfrac{z_1}{u}$ for V_1^+ traveling from $z = 0$ to z_1
- When V_1^+ reaches $z = l$ (load end)
  - ***No reflected wave***
  - Entire line charged at V_1^+

    (i.e. *steady-state* established)

$$\left(\because \Gamma_L = \frac{R_L - R_0}{R_L + R_0} \right)$$

Chap. 9 | Transient response: step-function (2/3)

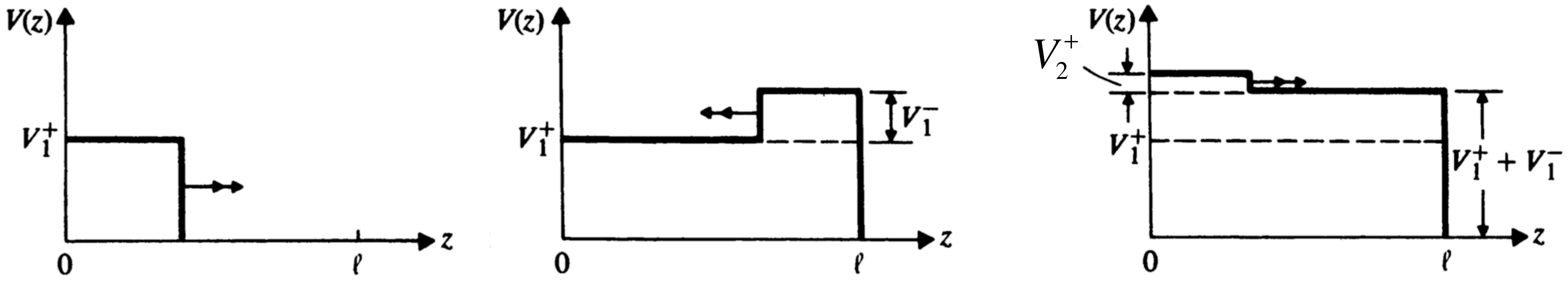
- Simplest example: step-function signal

  - <Case 2> $R_g \neq R_0$ and $R_L \neq R_0$

At $t = 0$: $V_1^+ = \dfrac{R_0}{R_0 + R_g} V_0$ travels in +z direction with $u = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{\mu\varepsilon}}$

At $t = T$: $\left(T = \dfrac{l}{u}\right)$ V_1^+ reaches at $z = l$ and reflected $\left(\Gamma_L = \dfrac{R_L - R_0}{R_L + R_0}\right)$. Then, $V_1^- = \Gamma_L V_1^+$ travels in -z direction with u

At $t = 2T$: V_1^- reaches at $z = 0$ and reflected $\left(\Gamma_g = \dfrac{R_g - R_0}{R_g + R_0}\right)$. Then, $V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$ travels in +z direction with u



$0 \leq t \leq T$ $\quad$ $T \leq t \leq 2T$ $\quad$ $2T \leq t \leq 3T$

$|\Gamma_L| < 1$ and $|\Gamma_g| < 1$

Chap. 9 | Transient response: step-function (3/3)

- Simplest example: step-function signal

  - <Case 2> $R_g \neq R_0$ and $R_L \neq R_0$

At $t = \infty$: At a load end ($z = l$), we have the ***steady-state voltage*** as

$$V_L = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \cdots$$

$$= V_1^+ \left(1 + \Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L^2 + \Gamma_g^2\Gamma_L^2 + \Gamma_g^2\Gamma_L^3 + \cdots\right)$$

$$= V_1^+ \left(1 + \Gamma_L\right)\left(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \cdots\right)$$

$$\boxed{= V_1^+ \left(\frac{1 + \Gamma_L}{1 - \Gamma_g\Gamma_L}\right)} \quad \left(\because \left|\Gamma_L\Gamma_g\right| < 1\right)$$

We have the ***steady-state current*** as

$$I_L = I_1^+ + I_1^- + I_2^+ + I_2^- + I_3^+ + I_3^- + \cdots$$

$$= \frac{V_1^+}{R_0}\left(1 - \Gamma_L + \Gamma_g\Gamma_L - \Gamma_g\Gamma_L^2 + \Gamma_g^2\Gamma_L^2 - \Gamma_g^2\Gamma_L^3 + \cdots\right) \longleftarrow$$

$$= \frac{V_1^+}{R_0}\left(1 - \Gamma_L\right)\left(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \cdots\right)$$

$$\boxed{= \frac{V_1^+}{R_0}\left(\frac{1 - \Gamma_L}{1 - \Gamma_g\Gamma_L}\right)}$$

Recall the relation

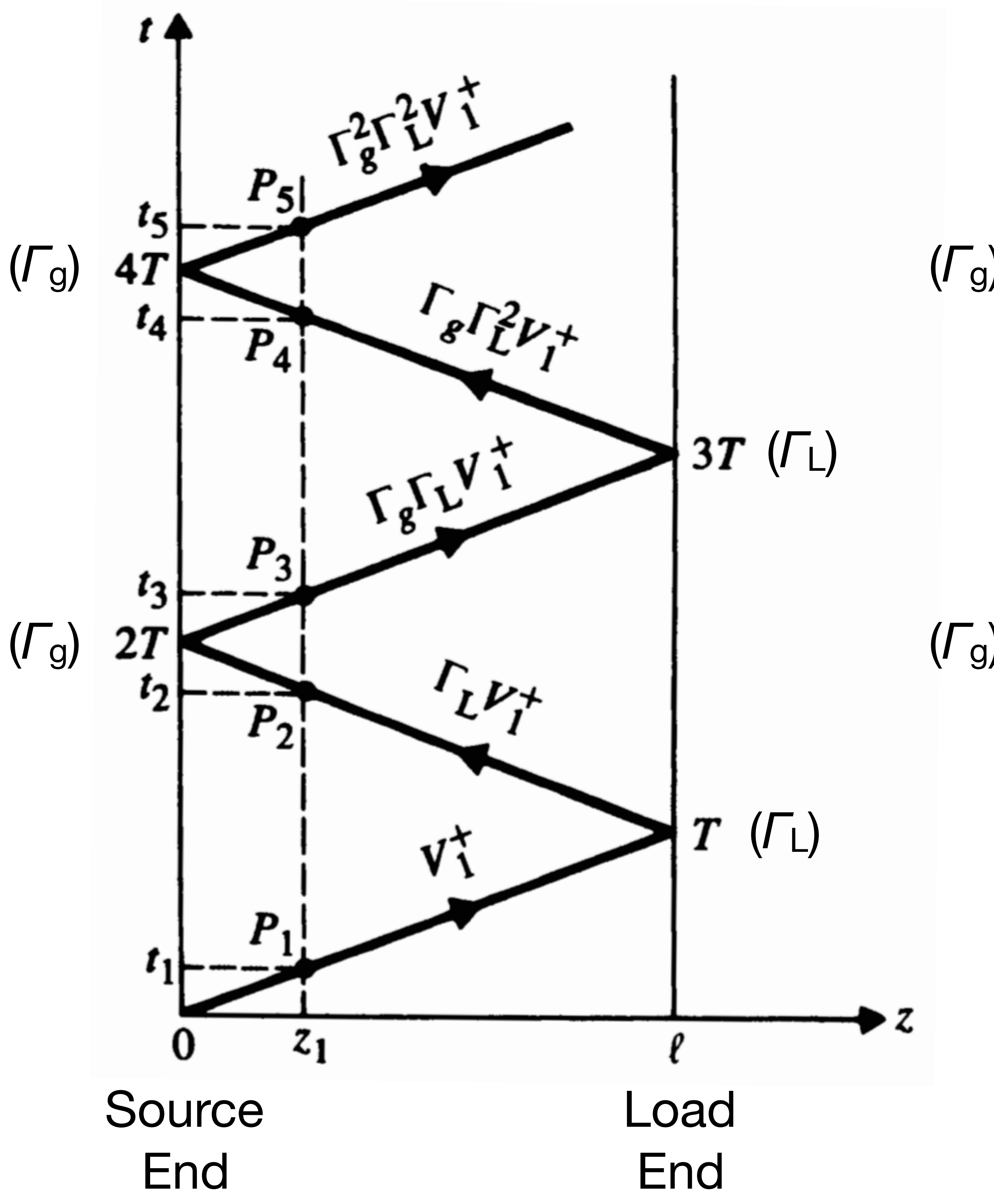
$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0$$

Phase of current changed by π upon reflection

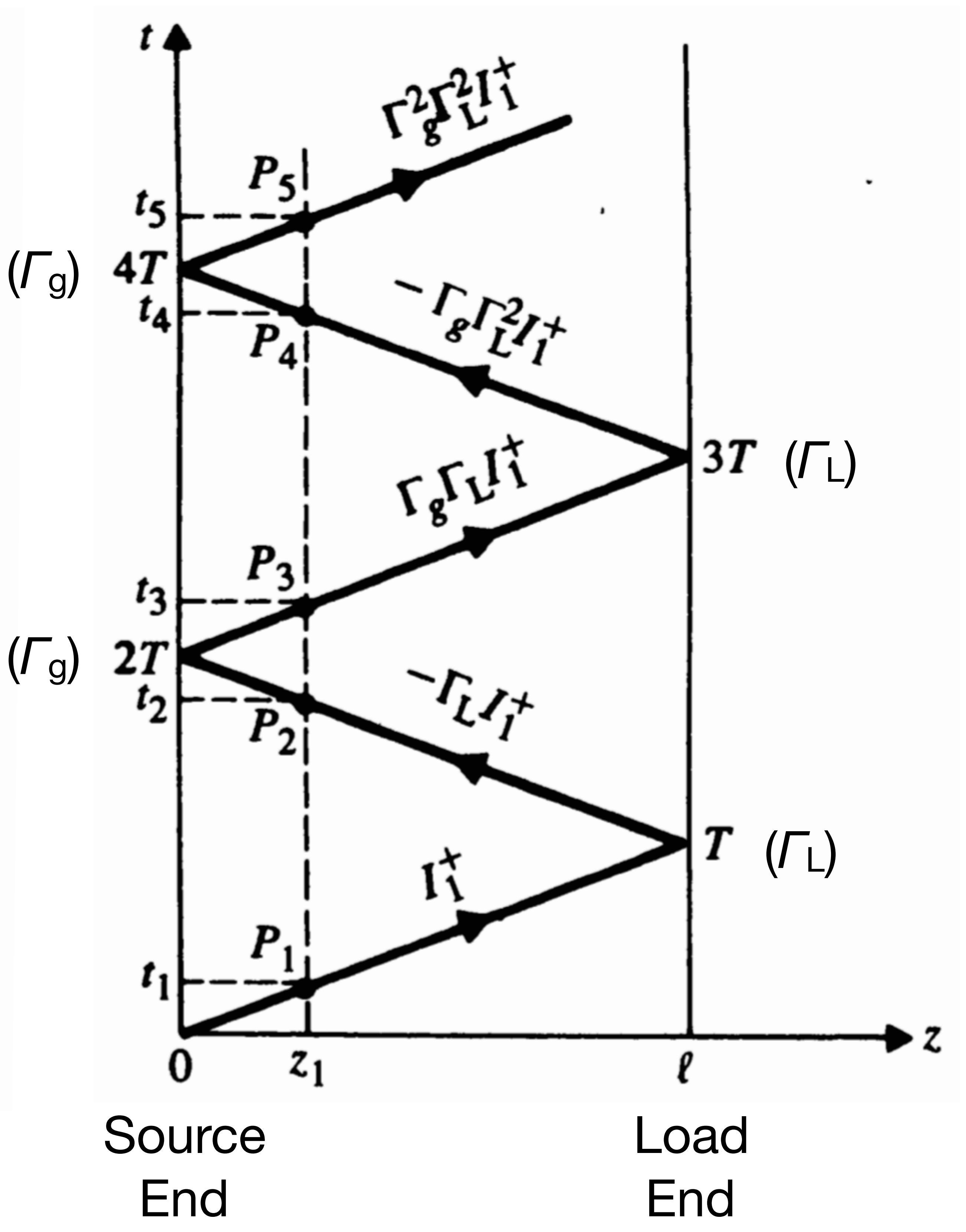
$$\begin{cases} V(z') = \dfrac{I_L}{2}\left(Z_L + R_0\right)e^{j\beta z'}\left[1 + |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right] \\ I(z') = \dfrac{I_L}{2R_0}\left(Z_L + R_0\right)e^{j\beta z'}\left[1 - |\Gamma| e^{j(\theta_\Gamma - 2\beta z')}\right] \end{cases}$$

Chap. 9 | Reflection diagram

- Reflection diagram



<Voltage reflection diagram>



<Current reflection diagram>

- Graphical representation of (V, I) propagation *vs. t or z*
- At $0 \le t \le T$
  - $\boldsymbol{V_1^+}$ travels in **+*z*** direction from $z = 0$ to l
  - $\boldsymbol{I_1^+}$ travels in **+*z*** direction from $z = 0$ to l
- At $T \le t \le 2T$
  - $\boldsymbol{V_1^- = \Gamma_L V_1^+}$ travels in **-*z*** direction from $z = l$ to 0
  - $\boldsymbol{I_1^- = -\Gamma_L I_1^+}$ travels in **-*z*** direction from $z = l$ to 0
- At $2T \le t \le 3T$
  - $\boldsymbol{V_2^+ = \Gamma_g\Gamma_L V_1^+}$ travels in **+*z*** direction from $z = 0$ to l
  - $\boldsymbol{I_1^- = \Gamma_g\Gamma_L I_1^+}$ travels in **+*z*** direction from $z = 0$ to l
- ***(V, I) vs. time (t)*** at any location: algebraic sum ***along vertical line***!
- ***(V, I) vs. location (z)*** at any given time: algebraic sum ***below horizontal line***!

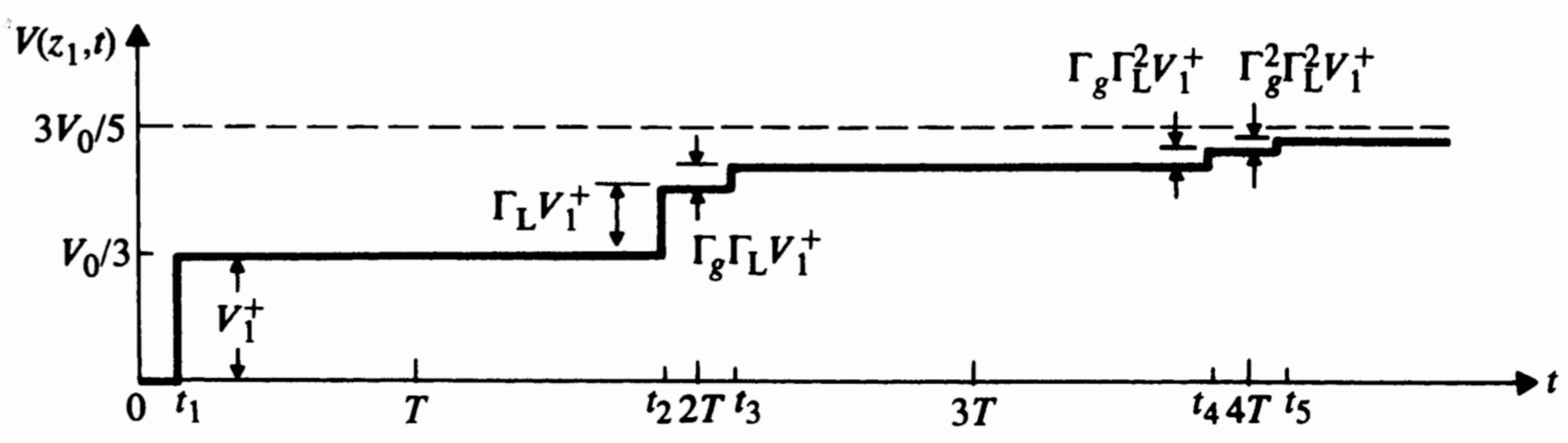
Chap. 9 | Transient response: Example

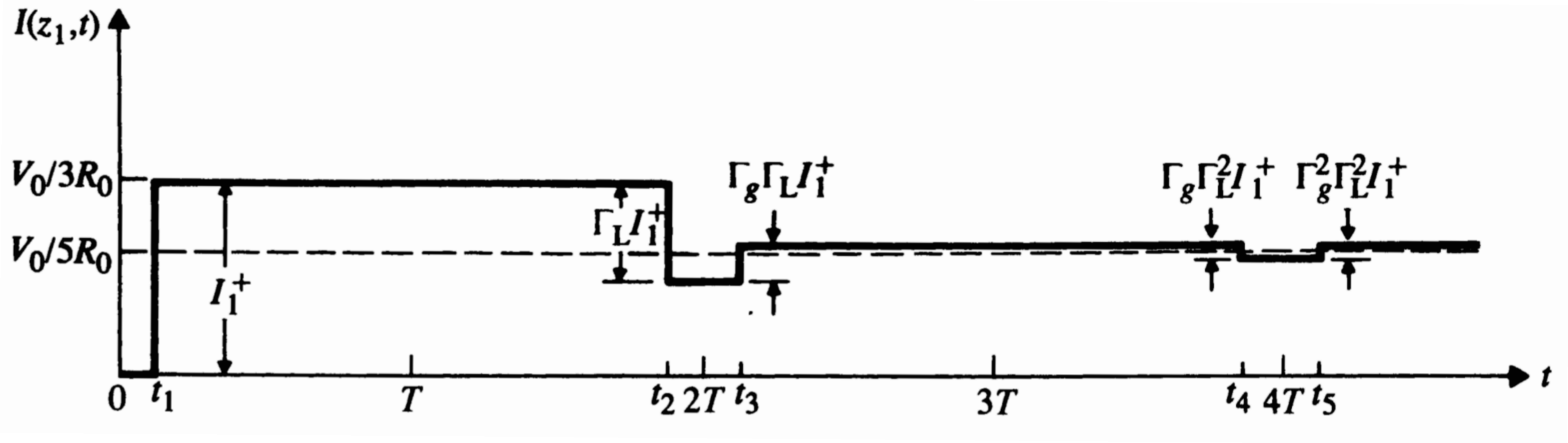
- voltage, current variation vs. time (at $z = z_1$)

  - If $R_L = 3R_0$ ($\Gamma_L = 1/2$) and $R_g = 2R_0$ ($\Gamma_g = 1/3$),

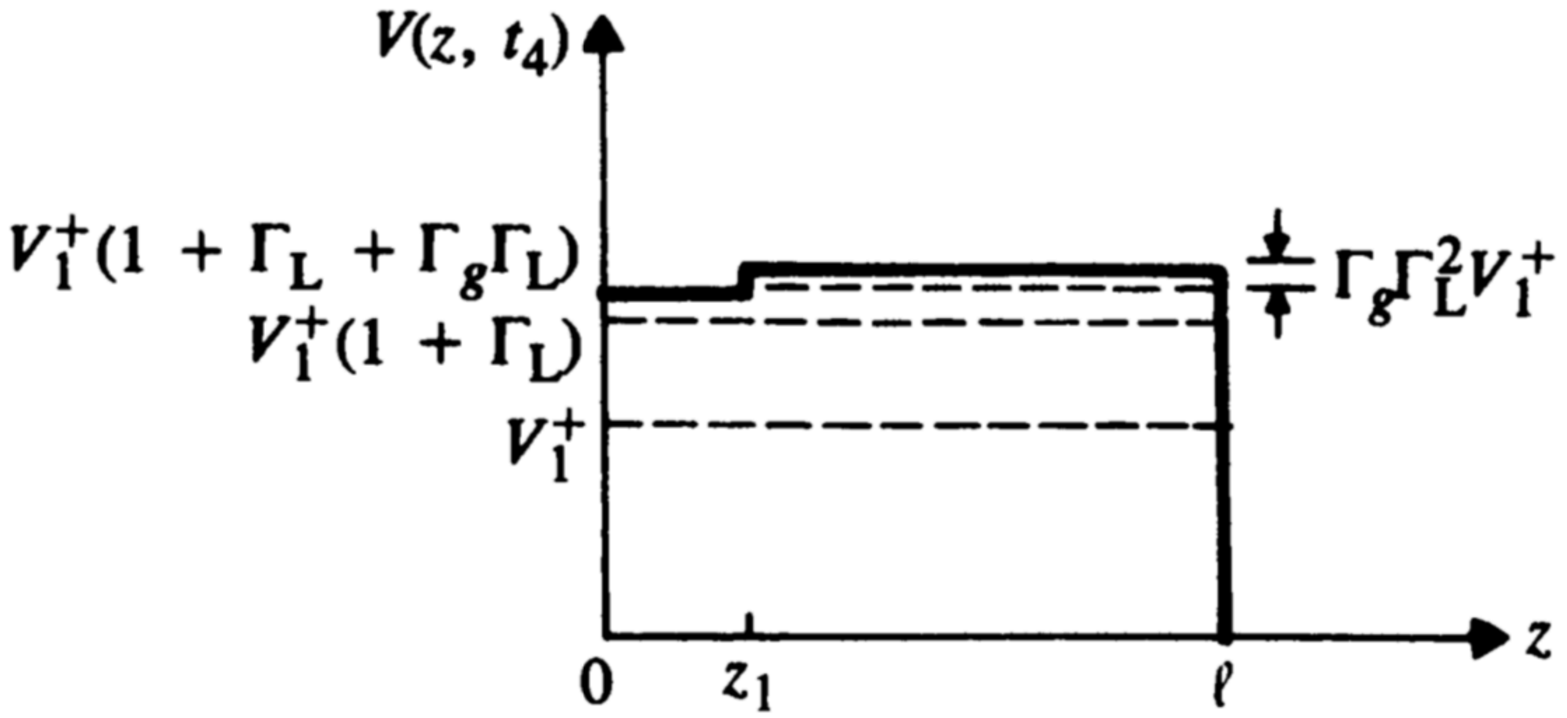
$$V_L = V_1^+\left(\frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L}\right) = \frac{R_0}{R_0+R_g}V_0\left(\frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L}\right) = \frac{3}{5}V_0$$

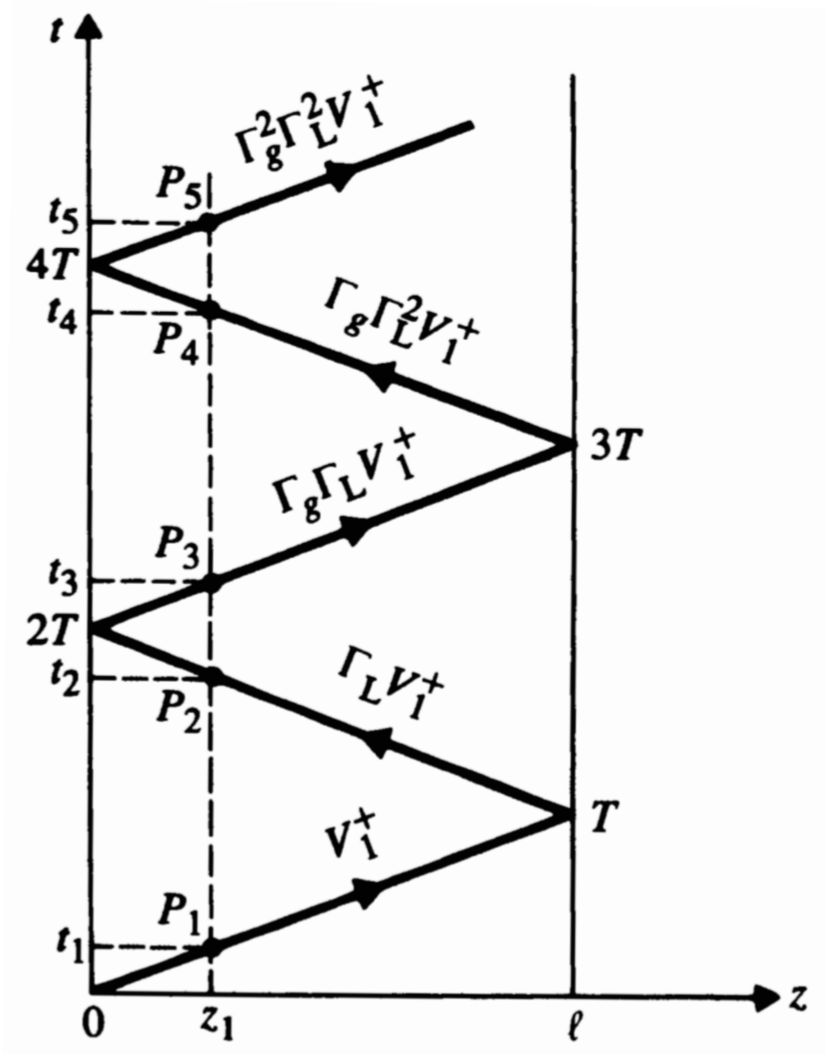
$$I_L = \frac{V_1^+}{R_0}\left(\frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L}\right) = \frac{V_0}{R_0+R_g}\left(\frac{1-\Gamma_L}{1-\Gamma_g\Gamma_L}\right) = \frac{V_0}{5R_0}$$





- voltage distribution vs. location (at $t = t_4$)



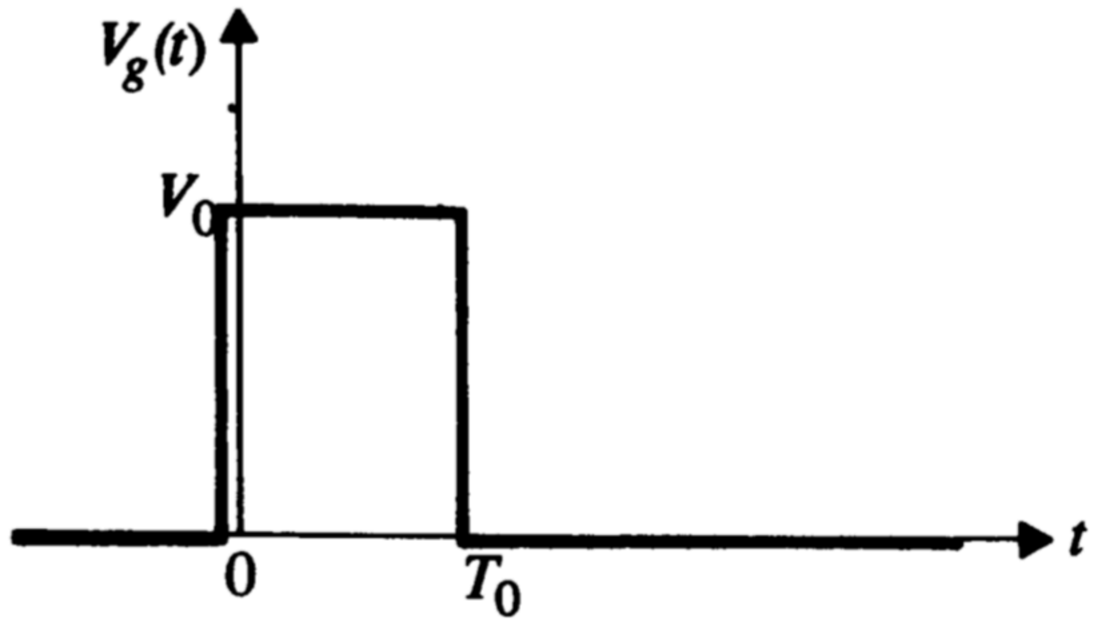


- Some special cases
  - I. $R_L = R_0$
    - $\Gamma_L = 0$ → No reflection at load end
    - After $t = T = l/u$, only $\boldsymbol{V_1^+}$ and $\boldsymbol{I_1^+}$ exist
  - II. $R_L \neq R_0$, $R_g = R_0$
    - $\Gamma_g = 0$ → No reflection at source end
    - After $t = 2T$, only ($\boldsymbol{V_1^+}$, $\boldsymbol{V_1^-}$) and ($\boldsymbol{I_1^+}$, $\boldsymbol{I_1^-}$) exist

Transient response: pulse signal (1/2)

- **Discussion so far**

  - Transient response for ***step-function signal*** represented by $v_g(t) = V_0 U(t) = \begin{cases} 0, & t<0 \\ V_0, & t>0 \end{cases}$



- **Pulse signal**

  - *superposition of two step-functions*: $v_g(t) = V_0\left[U(t) - U(t-T_0)\right]$

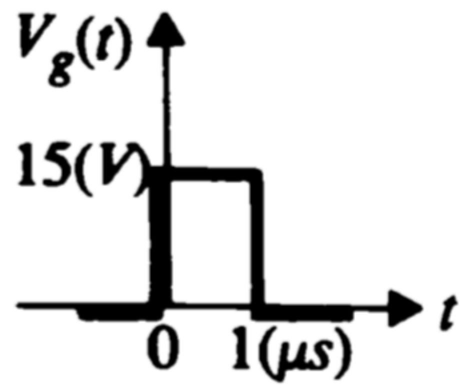
- **Example**

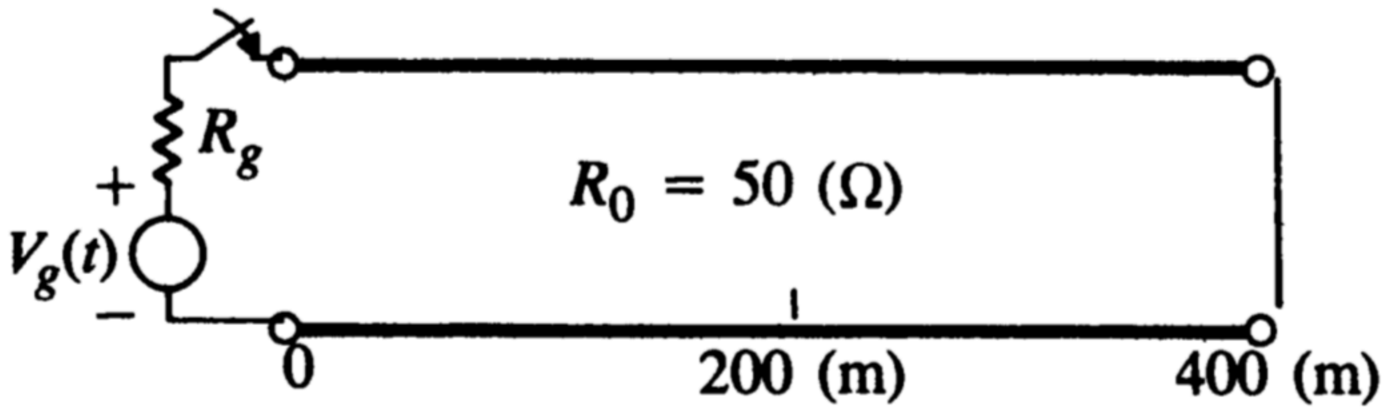
  - Given condition

    - Magnitude: V_0 = 15 (V), duration: T_0 = 1 (μs)

      $$v_g(t) = 15\left[U(t) - U\left(t - 10^{-6}\right)\right] \text{ (V)}$$

    - Series resistance: R_g = 25 (Ω)
    - Characteristic impedance of TR-line: R_0 = 50 (Ω)
    - l = 400 (m), material within TR-line ε = 2.25
    - Load impedance: $Z_L = R_L$ = 0 (Ω) (→Short-circuited!)





  - **Q:** Voltage change at z = l/2 = 200 (m) between 0 ≤ t ≤ 8 (μs)?

    - Reflection coefficient

      $$\Gamma_L = \frac{R_L - R_0}{R_L + R_0} = -1, \quad \Gamma_g = \frac{R_g - R_0}{R_g + R_0} = \frac{25-50}{75} = -\frac{1}{3}$$

    - Propagation speed & transverse time

      $$u = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8 \text{ (m/s)}, \quad T = \frac{u}{l} = 2 \ (\mu\text{s})$$

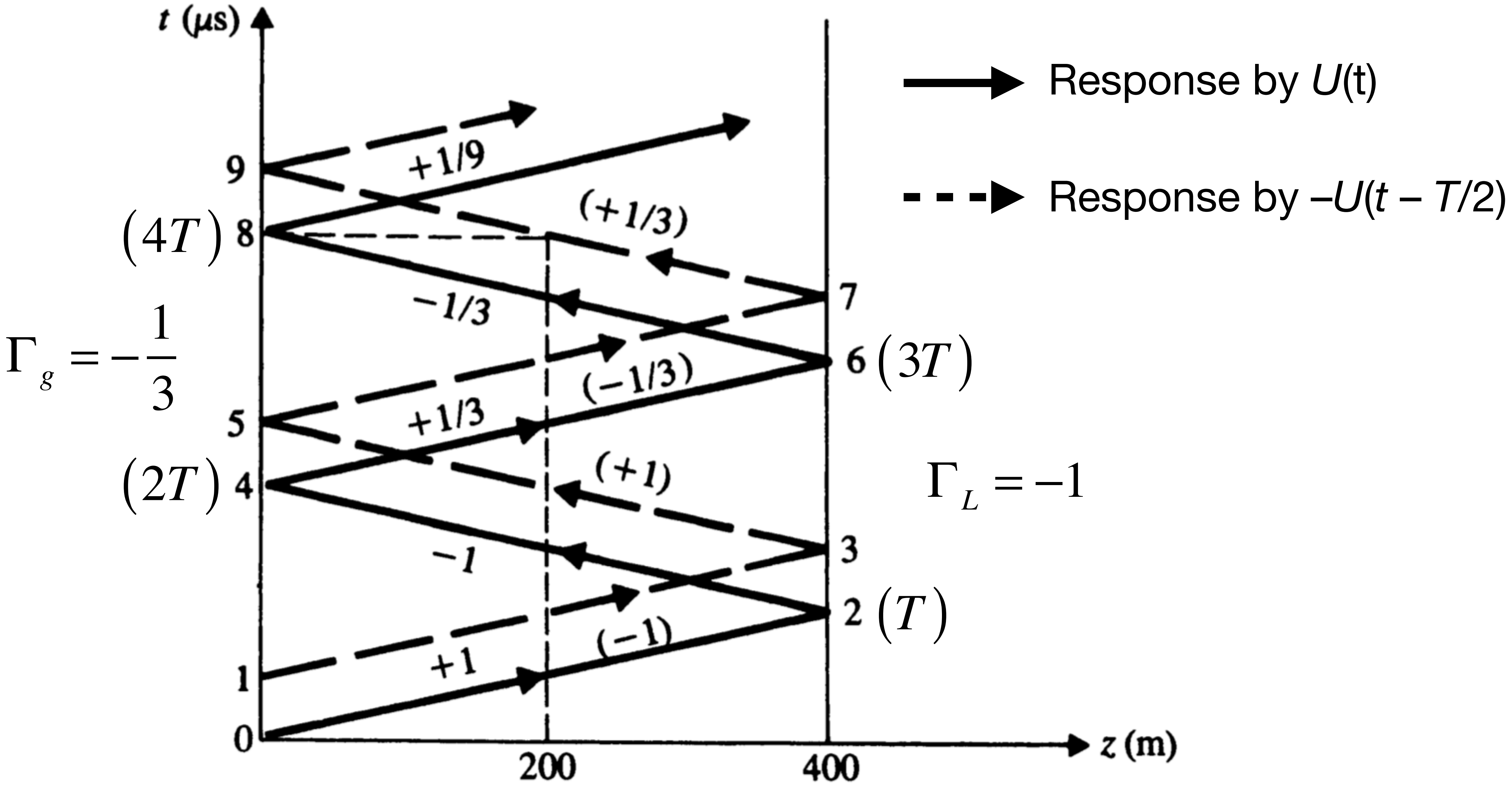
    - Voltage sent down to TR-line

      $$V_1^+ = \frac{R_0}{R_g + R_0} V_0 = \frac{25}{25+50}\times 15 = 10 \ (V)$$

Transient response: pulse signal (2/2)

- Example: pulse signal

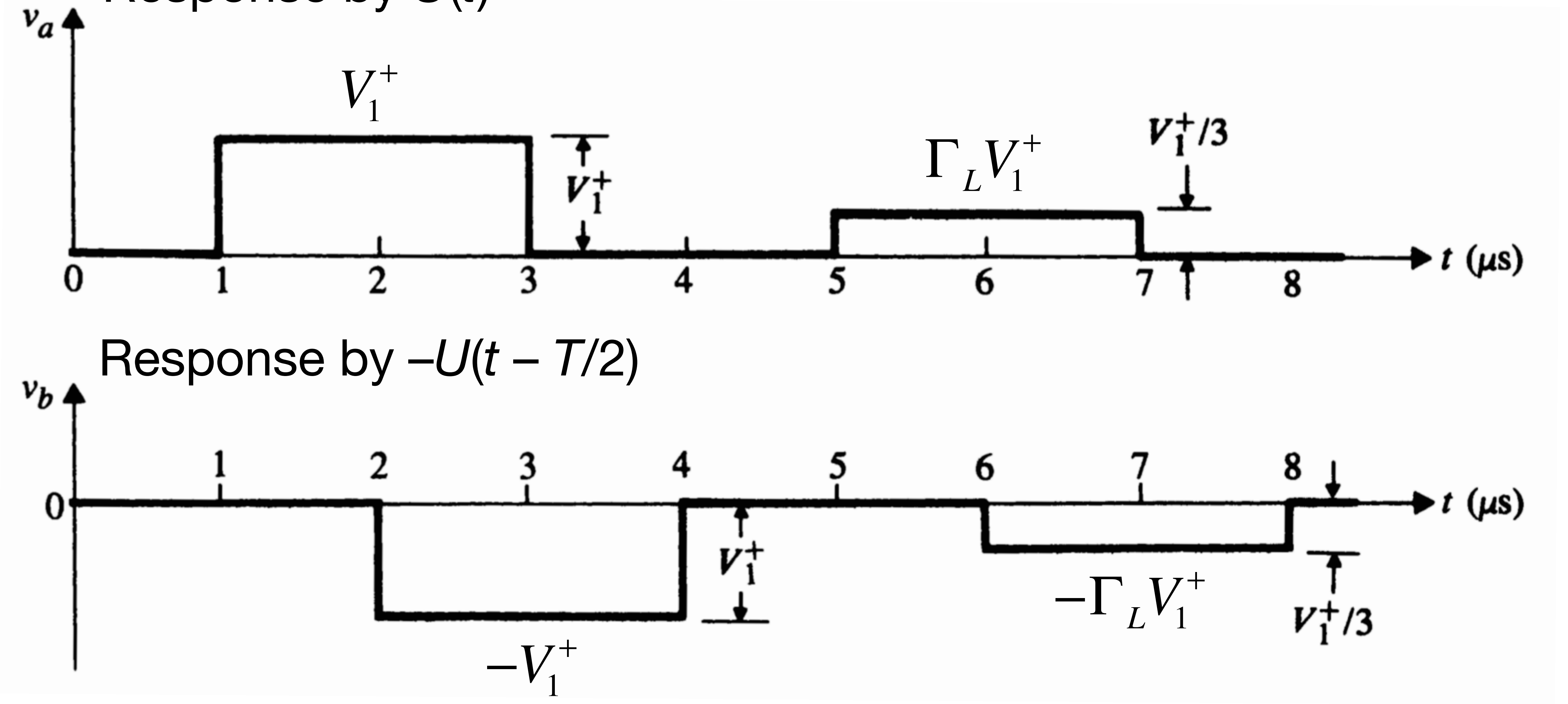
<Reflection diagram for U(t) and –U(t – T/2)>



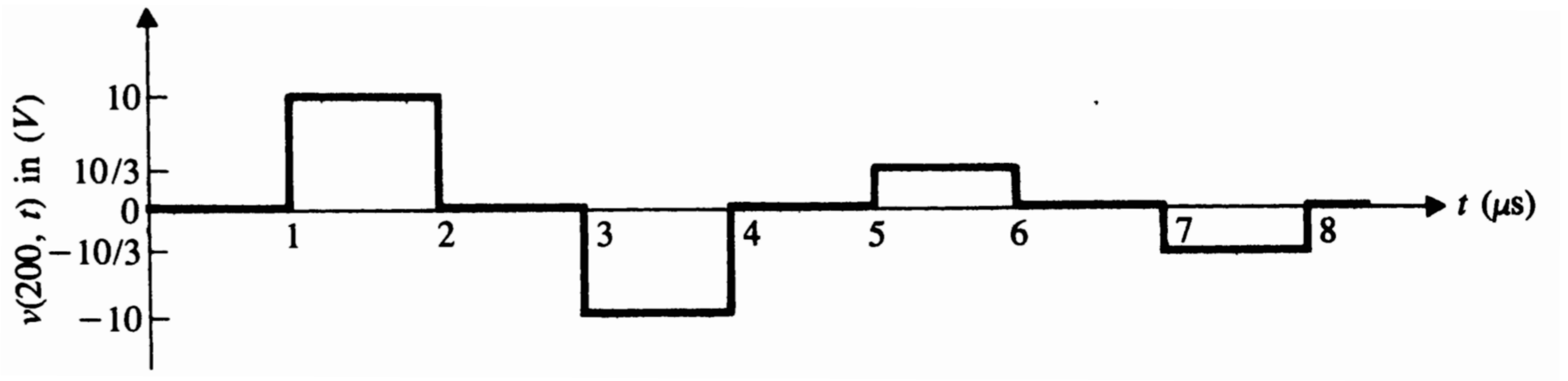
$$v_g(t) = 15\left[U(t) - U\left(t - \frac{T}{2}\right)\right] \text{ (V)}$$

$$\begin{cases} T = 2\ (\mu s) \\ V_1^+ = 10\ (V) \end{cases}$$

Response by *U*(*t*)

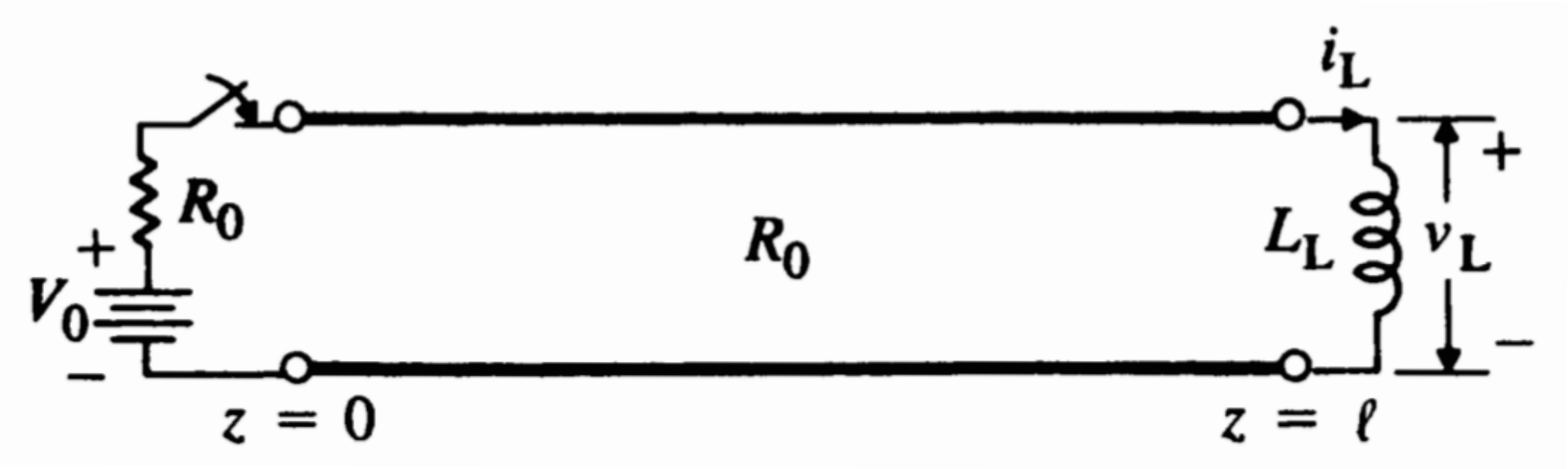


Response by *U*(*t*) – *U*(*t* – *T*/2)



Chap. 9 | TR-line with reactive termination (1/2)

- **TR-line terminated with reactive load**
  - $Z_L = R_L + jX_L$ (Due to X_L, phase shift introduced upon reflection) $\left(\because \Gamma_L = \dfrac{Z_L - R_0}{Z_L + R_0}\right)$
  - Time-dependence of incident wave ≠ reflected wave
  - Not simple as the resistive termination



- **Inductive termination ($X_L > 0$)**
  - Condition
    - TR-line terminated with an inductor load (L_L)
    - Internal (or series) impedance $R_g = R_0$ (What does it mean?)
    - Voltage initially sent down to TR-line:

$$V_1^+ = \frac{R_0}{R_g + R_0} V_0$$

  - At t = T (= u/l)
    - V_1^+ reached at $z = l$, reflected by inductor (Γ_L)
    - $V_1^- = \Gamma_L V_1^+$ generated and travel in -z direction (→ Because Γ_L complex, V_1^- no longer constant, but *time-dependent!*)

- At $z = l$ after reflection (i.e. $t \geq T$)

$$v_L(t) = V_1^+ + V_1^-(t) \quad \cdots (1)$$

- Equivalently, $v_L(t) = L_L \dfrac{di_L(t)}{dt} \quad \cdots (2)$

where $i_L(t) = \dfrac{V_1^+}{R_0} - \dfrac{V_1^-(t)}{R_0} \rightarrow R_0 i_L(t) = V_1^+ - V_1^-(t) \quad \cdots (3)$

- By eliminating V_1^-(t) by combining **eqns. (1) and (3)**, we have

$$v_L(t) = 2V_1^+ - R_0 i_L(t) \quad \cdots (4)$$

- By substituting **eqn. (2)** into **eqn. (4)**, we have

$$\boxed{L_L \frac{di_L(t)}{dt} + R_0 i_L(t) = 2V_1^+, \quad (t \geq T)}$$

Chap. 9 | TR-line with reactive termination (2/2)

- Inductive termination ($X_L > 0$)

  - IVP for first-order differential equation

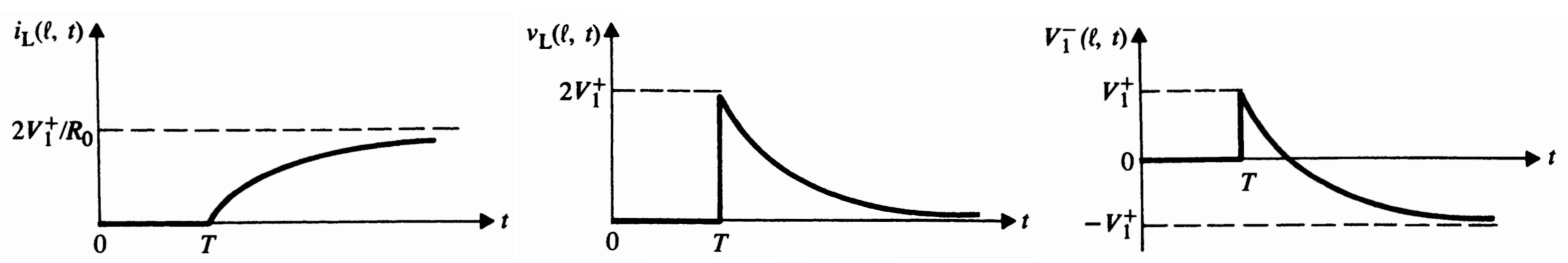
$$\frac{di_L(t)}{dt}+\frac{R_0}{L_L}i_L(t)=\frac{2V_1^+}{L_L},\quad (t\ge T)\quad \text{and}\quad i_L(T)=0$$

  - By applying Laplacian operator,

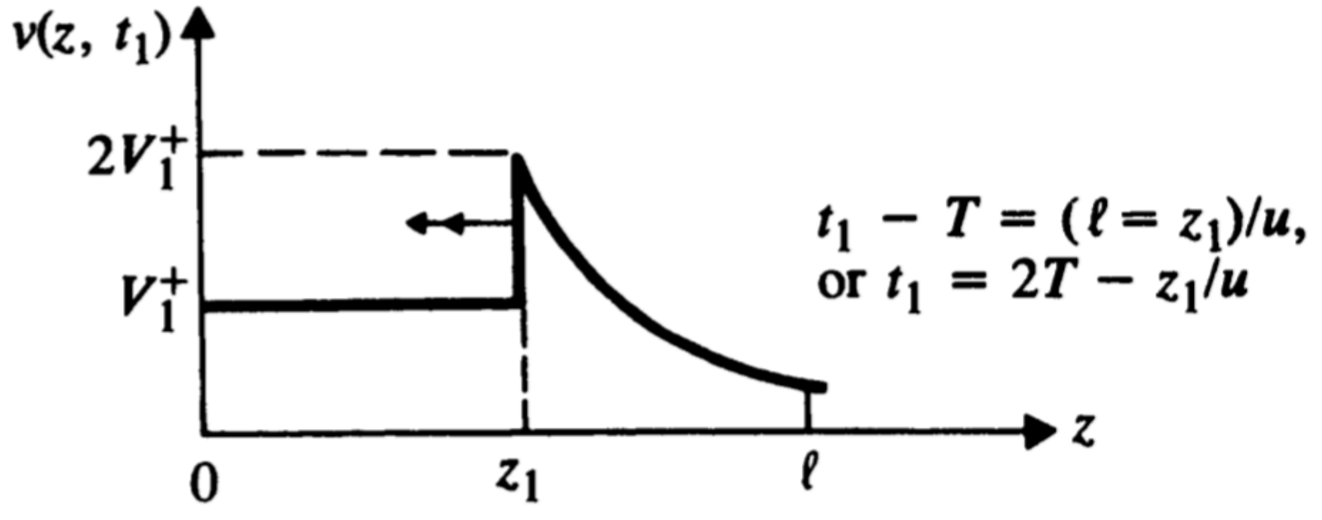
$$sI(s)-\cancel{i_L(T)}+\frac{R_0}{L_L}I(s)=\frac{2V_1^+}{L_L s}\quad\rightarrow\quad I(s)=\frac{2V_1^+}{L_L}\frac{1}{s(s+R_0/L_L)}=\frac{2V_1^+}{R_0}\left[\frac{1}{s}-\frac{1}{s+R_0/L_L}\right]$$

  - By applying inverse Laplacian operator, we get $i_L(t)$ (Current variation at load end)

$$\boxed{i_L(t)=\frac{2V_1^+}{R_0}\left[1-e^{-\frac{R_0}{L_L}(t-T)}\right]}\ \Rightarrow\ \boxed{v_L(t)=L_L\frac{di_L(t)}{dt}=2V_1^+e^{-\frac{R_0}{L_L}(t-T)}}\ \Rightarrow\ \boxed{V_1^-(t)=V_1^+-v_L(t)}$$



Current increases after V_1^+ arrived at $z = l$



<Voltage distribution vs. z at $t = t_1$>
($T < t_1 < 2T$)