Electromagnetics

<Chap. 9> Transmission Lines

Section 9.6 ~ 9.7

(1st of **week 12**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 9 | Contents for 1st class of week 11

Sec 6. The Smith Chart

- Arbitrary impedance termination
- Introduction, construction and interpretation
- Examples

Chap. 9 | Arbitrary termination of TR-line (1/2)

- **"Resistive" termination ($Z_L = R_L$)**
  - Voltage minima ($R_L < R_0$) or maxima ($R_L > R_0$) at the load end

- **"Arbitrary" termination ($Z_L = R_L + jX_L$)**
  - Voltage minima or maxima shifted by d from the load end
  - If, additional line extended by l_m with resistive termination (R_m)
  - → voltage shape does not change! → *Circuit I* = *Circuit II* (Equivalent)

- **How do we identify Z_L experimentally?**
  - Given condition: we measured S (SWR) and knew R_0
  - **Step 1)** Express Z_L in terms of R_0 and Γ

$$\boxed{Z_L = \left.\frac{V(z')}{I(z')}\right|_{z'=0} = R_0 \left.\frac{1+|\Gamma|e^{j(\theta_\Gamma - 2\beta z')}}{1-|\Gamma|e^{j(\theta_\Gamma - 2\beta z')}}\right|_{z'=0}}$$

We have to obtain

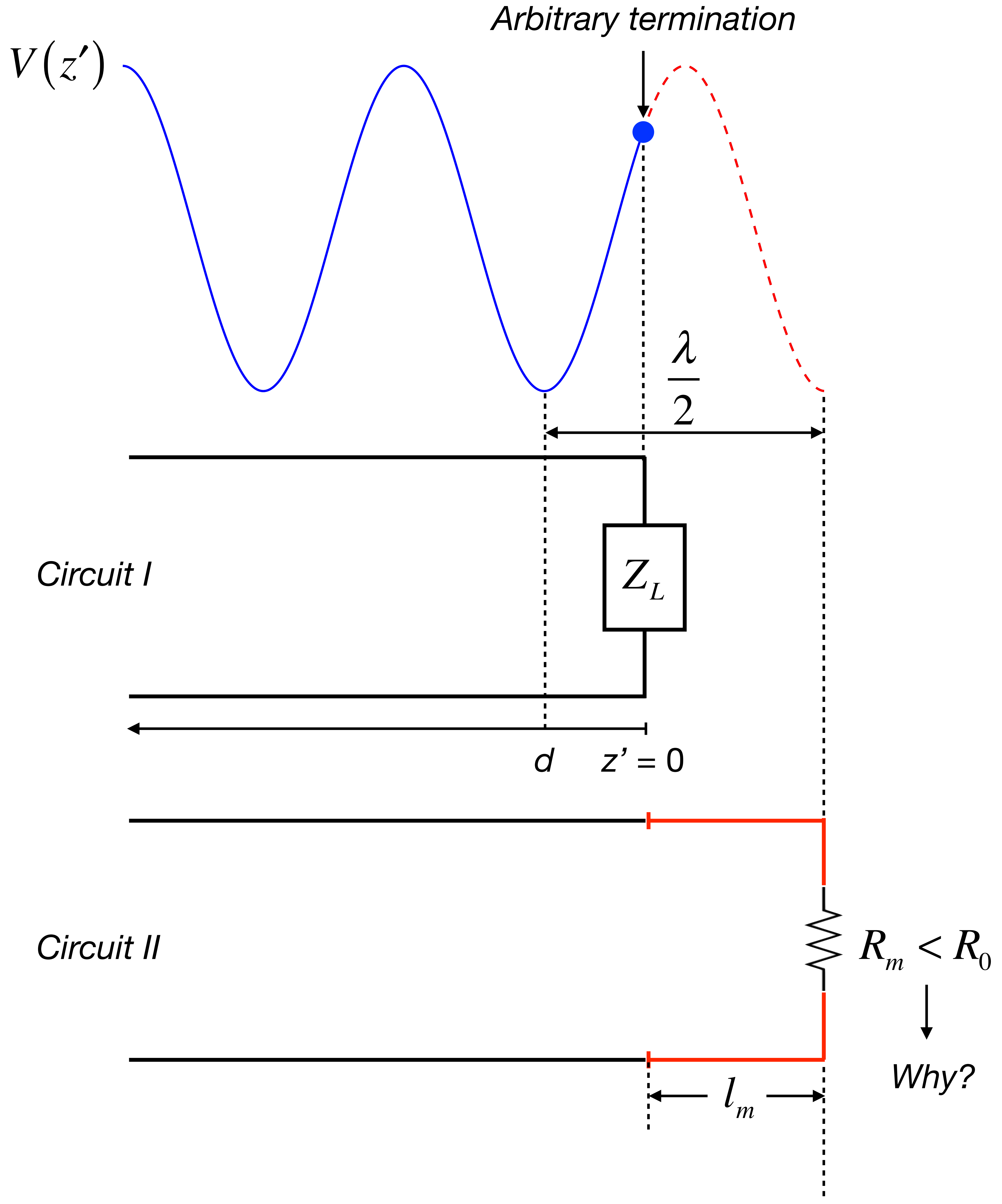
$$\Gamma = |\Gamma| e^{-j\theta_\Gamma}$$

through step 2), 3)

  - **Step 2)** At z' = d, we should have ***first voltage minima*** as

$$\theta_\Gamma - 2\beta d = -(2n+1)\pi\Big|_{n=0} \qquad \boxed{\rightarrow \quad \theta_\Gamma = 2\beta d - \pi}$$

  - **Step 3)** By measuring S, we can get $|\Gamma|$ as $\boxed{|\Gamma| = \frac{S-1}{S+1}}$ $\left(\because S = \frac{1+|\Gamma|}{1-|\Gamma|}\right)$



Chap. 9 | Arbitrary termination of TR-line (2/2)

Engineering example We measured S = 3 for lossless TR-line of R_0 = 50 (Ω). d = 5 (cm) of the first voltage minima for arbitrary terminated TR-line. *Distance between successive voltage minima = 20 (cm)*. What is an arbitrary load impedance Z_L? What is R_m and l_m for equivalent *Circuit II*?

- **Step 1) Step 1)** Express Z_L in terms of R_0 and Γ

$$Z_L = \left.\frac{V(z')}{I(z')}\right|_{z'=0} = R_0 \frac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}}$$

- **Step 2)** At z' = d, we should have ***first voltage minima*** as

$$\theta_\Gamma - 2\beta d = -(2n+1)\pi\Big|_{n=0} \quad \rightarrow \quad \theta_\Gamma = 2\beta d - \pi$$

Here, $\beta = \frac{2\pi}{\lambda}$ where $\frac{\lambda}{2} = 20 \text{ (cm)}$ — *Distance between successive voltage minima*

$$= \frac{2\pi}{0.4} = 5\pi \text{ (rad/m)} \quad \rightarrow \theta_\Gamma = 2\times 5\pi \times 0.05 - \pi = -0.5\pi \text{ (rad)}$$

- **Step 3)** By measuring S, we can get $|\Gamma|$ as

$$|\Gamma| = \frac{S-1}{S+1} = \frac{1}{2}$$

$$\boxed{\therefore Z_L = R_0 \frac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}} = 50\frac{1-j0.5}{1+j0.5} = 30 - j40 \text{ } (\Omega)}$$

- Recall previous slides that if $R_m < R_0$,

$$R_m = \frac{R_0}{S} = \frac{50}{3} = 16.7 \text{ } (\Omega)$$

- From the relation as below (see voltage graph in previous slide)

$$l_m + d = \frac{\lambda}{2} \quad \rightarrow \quad l_m = \frac{\lambda}{2} - d = 0.2 - 0.05 = 0.15 \text{ (m)}$$

Chap. 9 | The Smith Chart: Introduction

- **Discussion so far**

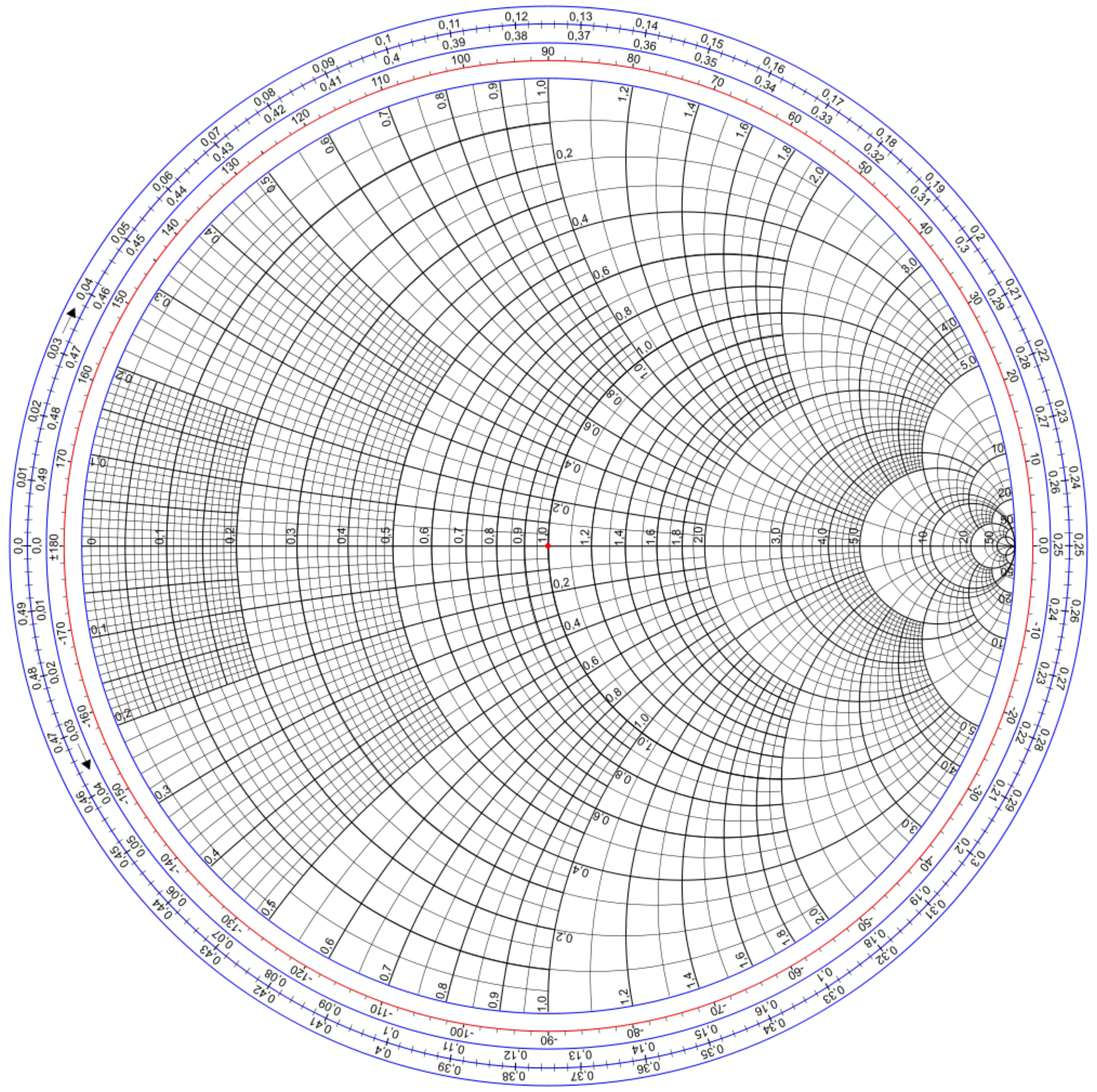
  - Tedious TR-line calculations involving Z_i (input impedance), Γ (reflection coefficient), Z_L (load impedance)

$$Z_i = R_0 \frac{Z_L + jR_0 \tan \beta z'}{R_0 + jZ_L \tan \beta z'} \qquad Z_L = R_0 \frac{1+\Gamma}{1-\Gamma} \quad \text{where} \quad \Gamma = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\Theta_\Gamma}$$

- **The Smith Chart**

  - A graphical representation of Z_i, Z_L and Γ
  - ***"Easy"*** to visualize complex-valued quantities and obtain them

  - Commonly used to identify ***load characteristics***
    - Check how capacitive or inductive a load is
    - Check How well impedance-matched a load is
    - and many more in RF engineering



<The Smith Chart>

Philip Hagar Smith
(1905-1987)
At Bell lab

Chap. 9 | Construction of Smith Chart (1/3)

- **How Smith Chart constructed for lossless TR-line?**

- Starting with reflection coefficient as

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \frac{Z_L/R_0 - 1}{Z_L/R_0 + 1} = \frac{z_L - 1}{z_L + 1} \quad \text{where} \quad z_L \triangleq \frac{Z_L}{R_0} = \frac{R_L + jX_L}{R_0} = r + jx$$

: Normalized load impedance w.r.t. R₀

- Conversely, z_L expressed in terms of Γ as $(\because \Gamma = \Gamma_r + j\Gamma_i)$

$$z_L = \frac{1+\Gamma}{1-\Gamma} \quad \rightarrow \quad \text{(lhs) } z_L = r + jx, \quad \text{(rhs) } \frac{1+\Gamma}{1-\Gamma} = \frac{1+(\Gamma_r + j\Gamma_i)}{1-(\Gamma_r + j\Gamma_i)} = \frac{1-\Gamma_r^2 - \Gamma_i^2 + j2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$\begin{cases} r = \dfrac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} & \cdots(1) \\[2ex] x = \dfrac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} & \cdots(2) \end{cases}$$

⇨ *Load impedance (r, x)* vs. *Reflection coefficients (Γ_r, Γ_i)* ⇨ ∴*The Smith Chart*

Determining load impedance (r, x) in Reflection coefficient plane (Γ_r, Γ_i)

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \quad \cdots(1)'$$

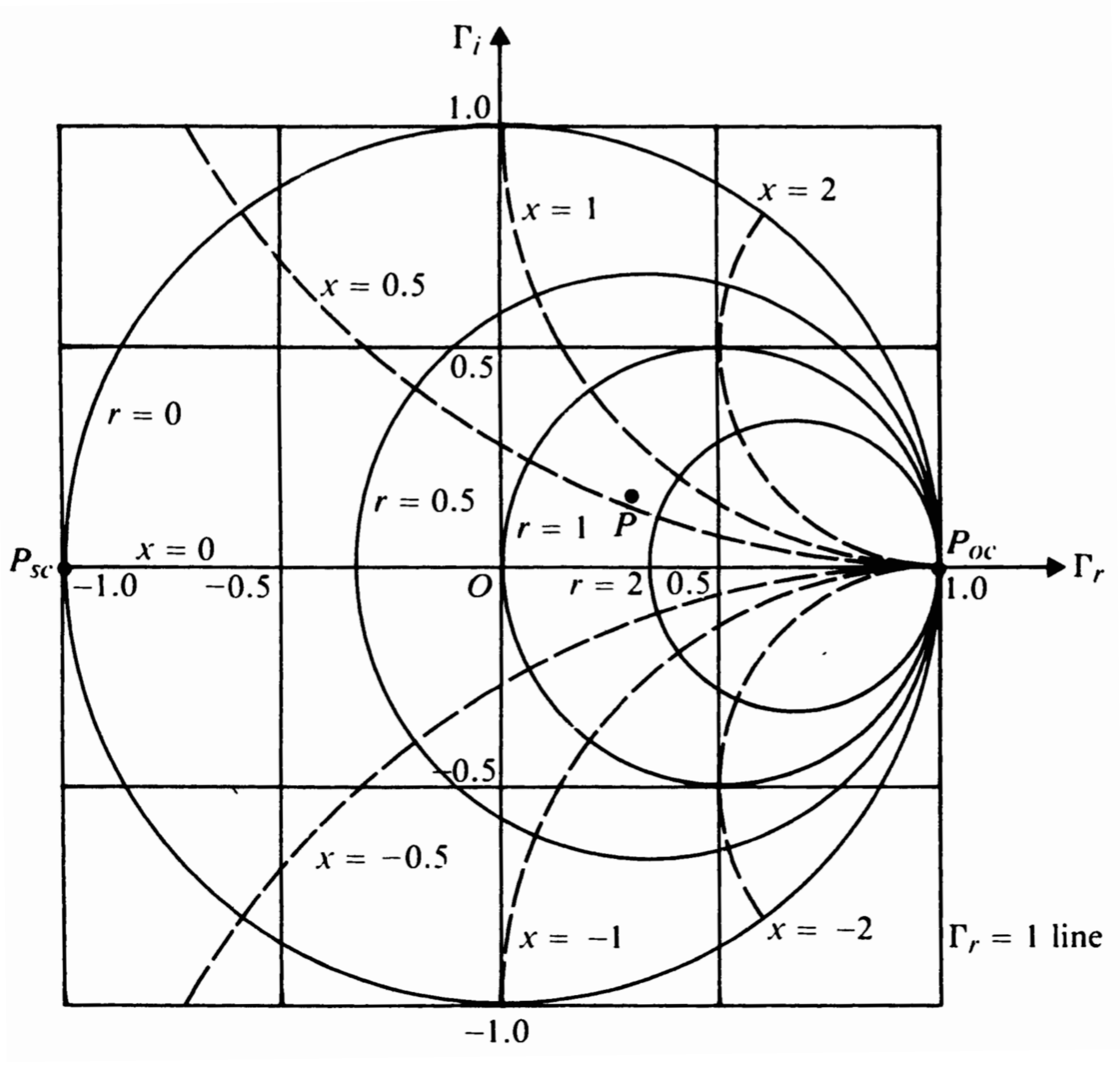
Circle of radius 1/(1+r) and centered at (r/(1+r), 0)

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \quad \cdots(2)'$$

Circle of radius 1/|x| and centered at (1,1/x)

Chap. 9 | Construction of Smith Chart (2/3)

$$z_L \triangleq \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \quad \leftrightarrow \quad \Gamma = \Gamma_r + j\Gamma_i$$



<Smith Chart in reflection coefficient plane>

- **Circles with solid-lines**

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$$

center: $(\Gamma_r, \Gamma_i) = \left(\frac{r}{1+r}, 0\right)$

radius: $\frac{1}{1+r}$

- Different r values → circles of ***different radii centered at different positions*** ($r/(1+r)$, 0) ***on*** Γ_r ***axis***

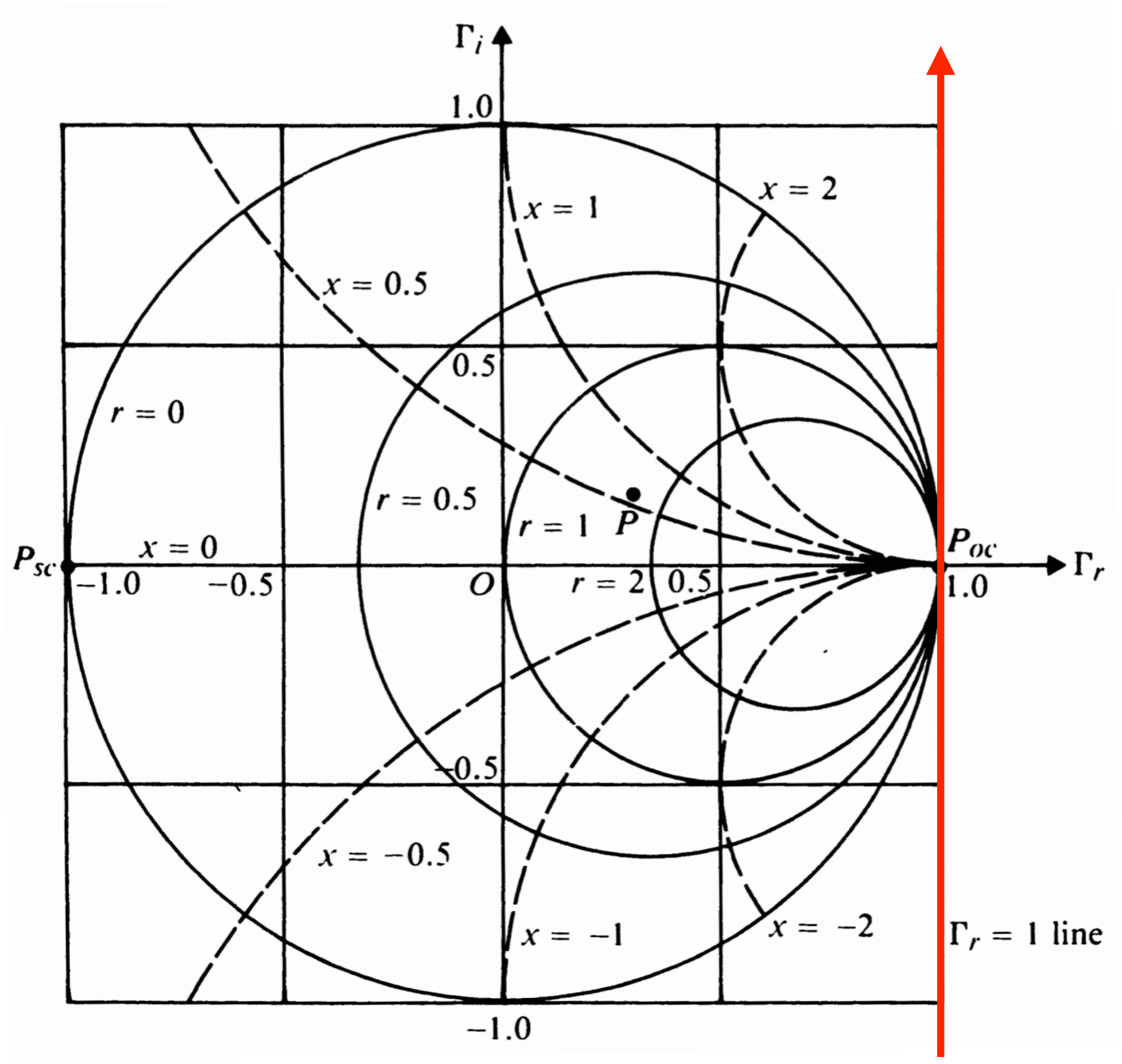
- Since $|\Gamma| \leq 1$, only those within a unit box meaningful
  - All circles passing through $(\Gamma_r, \Gamma_i) = (1, 0)$ → ($\therefore \Gamma = 1$) *What condition?*

$\left(\text{Hint: } \Gamma = \frac{Z_L - R_0}{Z_L + R_0}\right)$

- Circles vs. r value
  - At $r = 0$: a circle, centered at origin, is largest
  - As r increases, circle gets smaller
  - As r → ∞, circle ends at $(\Gamma_r, \Gamma_i) = (1, 0)$

Construction of Smith Chart (3/3)

$$z_L \triangleq \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \quad \leftrightarrow \quad \Gamma = \Gamma_r + j\Gamma_i$$



<Smith Chart in reflection coefficient plane>

- **Circles with dashed-lines**

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

center: $(\Gamma_r, \Gamma_i) = \left(1, \frac{1}{x}\right)$

radius: $\frac{1}{|x|}$

- Different *x* values → circles of ***different radii 1/|x| centered at different positions* (1, 1/*x*) *on* Γr = 1 *line* (red line)**
  - Centers of all the circles lie on $\Gamma_r = 1$ line

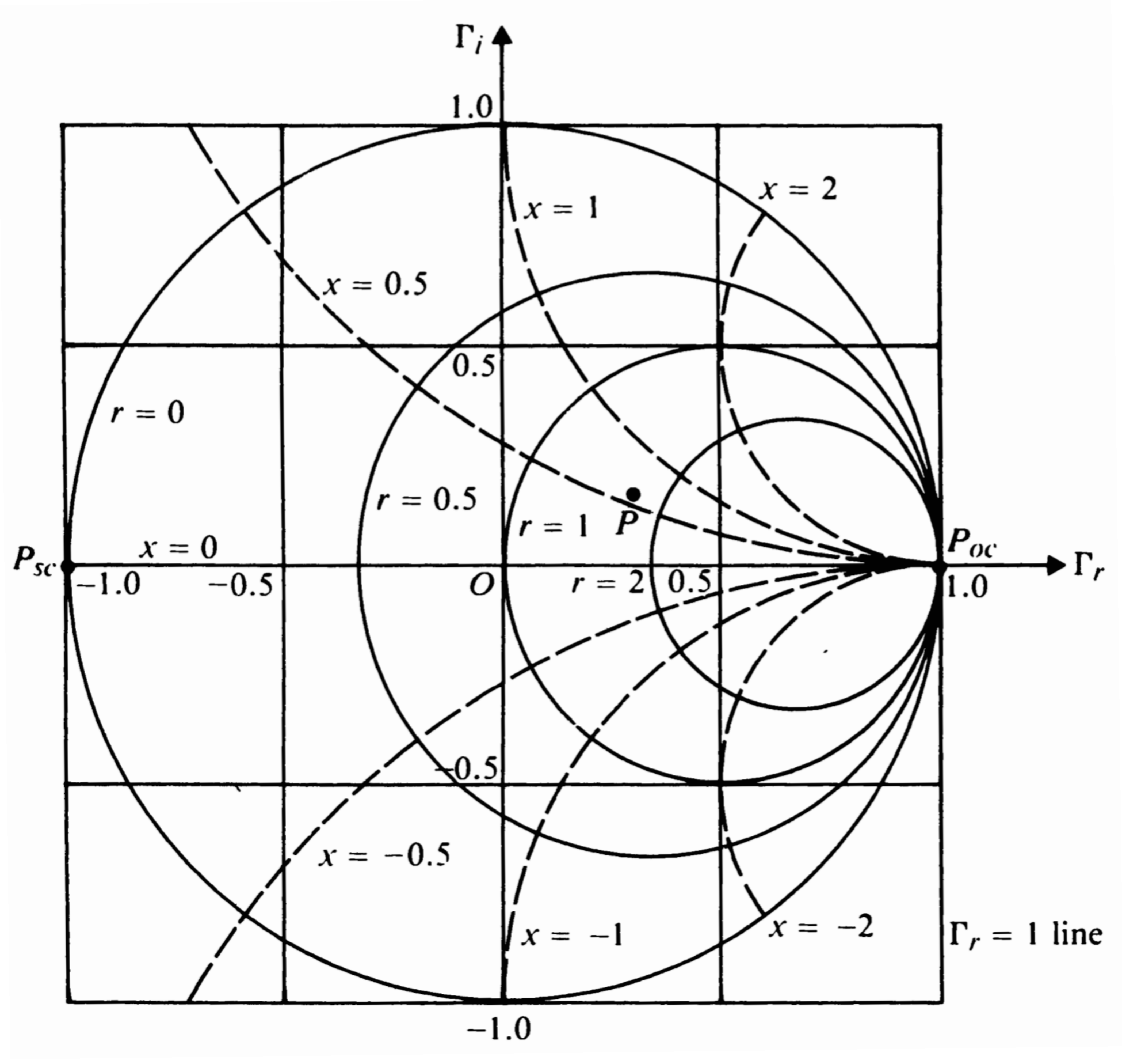
- Since |Γ| ≤ 1, only those lying within a unit box meaningful

- Circles vs. *x* value
  - If x > 0 (inductive), circles lie above Γ_r axis
  - If x < 0 (capacitive), circles lie below Γ_r axis

  - At *x* = 0, circles become Γ_r axis itself
  - As |x| increases, circles progressively become smaller
  - As |x| → ∞, circles end at $(\Gamma_r, \Gamma_i) = (1, 0)$ *What condition?*

Interpretation of Smith Chart

$$z_L \triangleq \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \quad \leftrightarrow \quad \Gamma = \Gamma_r + j\Gamma_i$$



<*Smith Chart in reflection coefficient plane*>

- **How to read it then?**
  - Intersection of *r*- and *x*-circles = Normalized load impedance, $z_L = r + jx$
    - ∴ Actual impedance $Z_L = R_0 \cdot (r + jx)$
  - Point P
    - Intersections of [$r = 1.7$] circle and [$x = 0.6$] circle
    - $z_L = 1.7 + j0.6$
  - Point P_{sc}: $(\Gamma_r, \Gamma_i) = (-1, 0)$
    - Intersections of [$r = 0$] circle and [$x = 0$] circle
    - $z_L = 0$ (→ short-circuit)
  - Point P_{oc}: $(\Gamma_r, \Gamma_i) = (1, 0)$
    - Represents infinite impedance ***(why?)*** $\left(\text{Hint: } \Gamma = \frac{Z_L - R_0}{Z_L + R_0}\right)$
    - $z_L = \infty$ (→ Open-circuit)

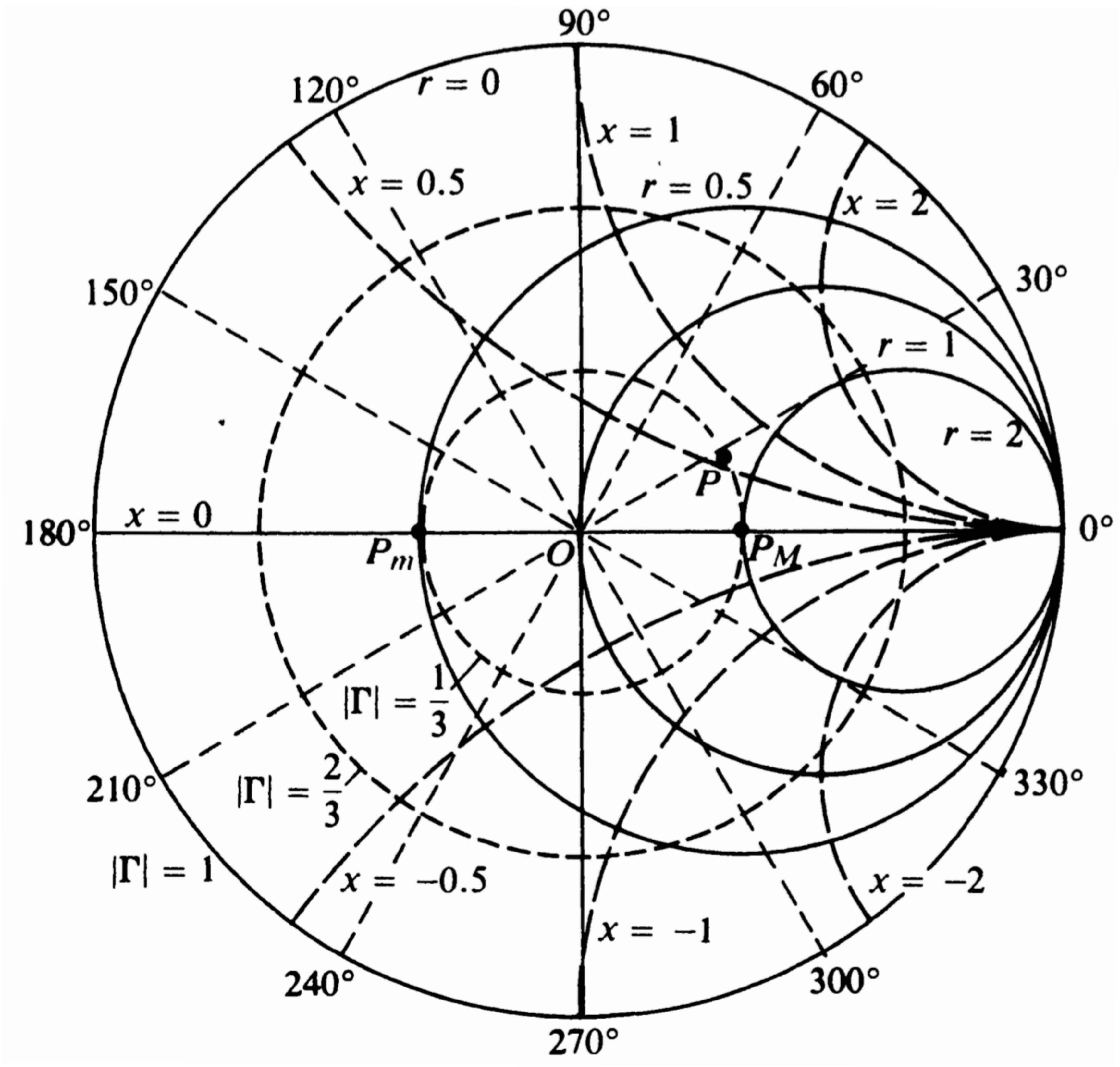
Chap. 9 | Smith Chart in Polar Coordinate

$$z_L \triangleq \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \quad \leftrightarrow \quad \Gamma = \Gamma_r + j\Gamma_i$$

- **Smith Chart in Polar coordinate**
  - All the points in Γ plane can be represented as

$$\Gamma = \Gamma_r + j\Gamma_i \triangleq |\Gamma| e^{j\Theta_\Gamma}$$

  - Centered at origin (O) with radius of $0 \le |\Gamma| \le 1$ & phase angle θ_Γ
  - e.g.) At P with a load $z_L = r + jx$, we can obtain Γ for that load



- Two intersections with Γ_r axis (**P_M** and **P_m**)
  - **P_M**: ***Positive*** real $\Gamma > 0$

    → Purely resistive load $Z_L = R_L$ $\left(\because \Gamma = \dfrac{R_L - R_0}{R_L + R_0}\right)$

    → $R_L > R_0$ or $r = R_L / R_0 > 1$

    Previously, $R_L / R_0 = S$ if $R_L > R_0$ (see slide 13-2)

    ∴ ***r* = *S***

    The value of r-circle passing through P_M
    = Standing-wave ratio, S

  - **P_m**: ***Negative*** real $\Gamma < 0$

    → Purely resistive load $Z_L = R_L$

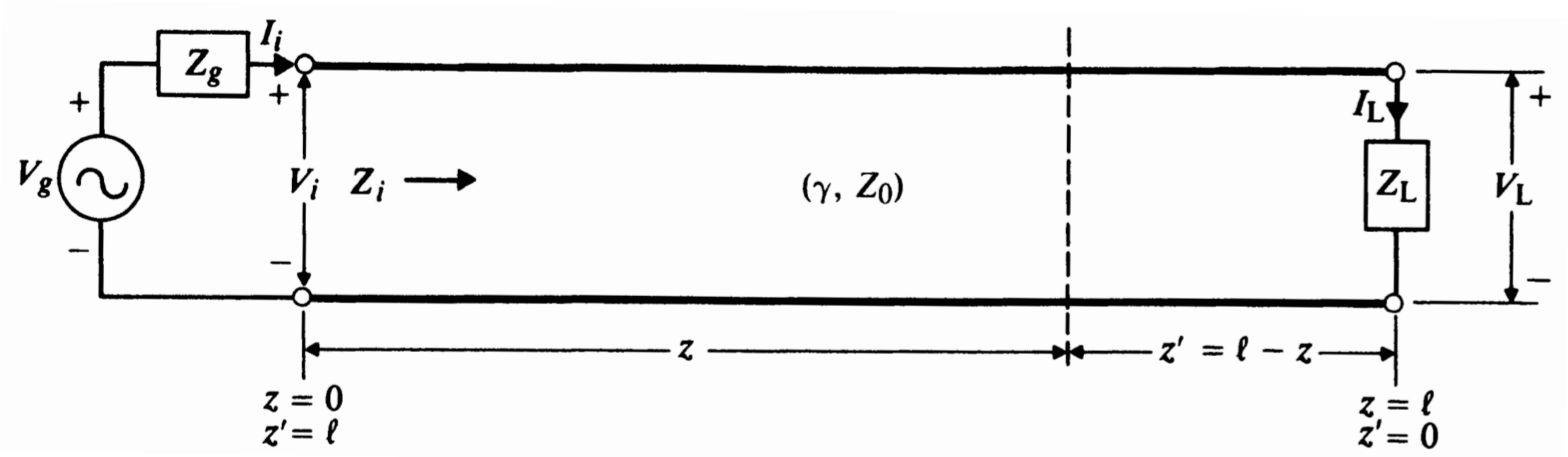
    → $R_L < R_0$ or $r = R_L / R_0 < 1$

    Previously, $R_0 / R_L = S$ if $R_L < R_0$ (see slide 13-2)

    ∴ ***r* = 1/*S***

Chap. 9 | Input impedance in Smith Chart (1/2)

- Smith Chart for input impedance Z_i



- Input impedance Z_i looking toward the load at z'

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0\left[\frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}}\right]$$

- Normalized input impedance z_i given as

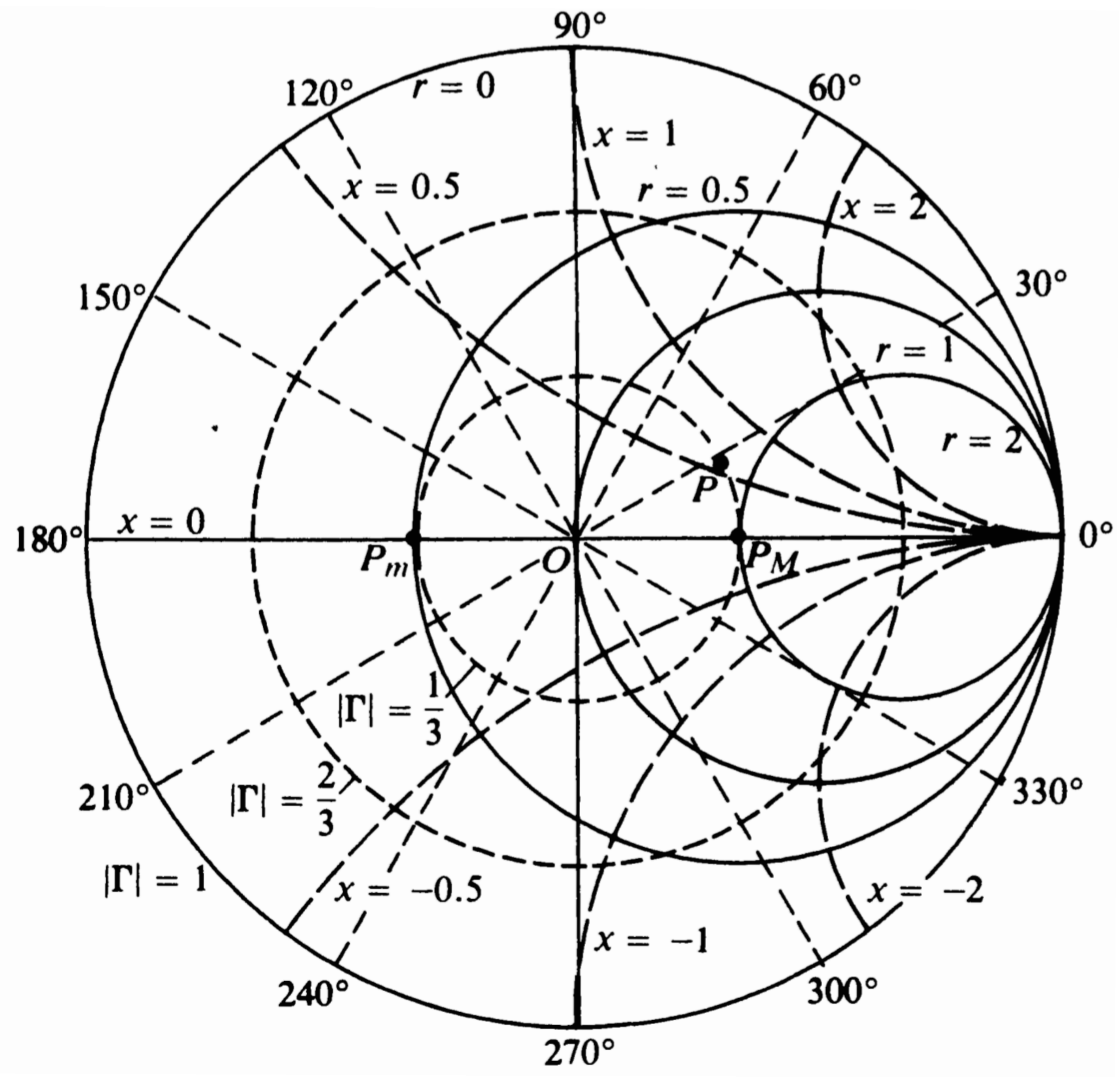
$$z_i(z') = \frac{Z_i(z')}{Z_0} = \frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}} = \frac{1+|\Gamma|e^{j(\Theta_\Gamma - 2\beta z')}}{1-|\Gamma|e^{j(\Theta_\Gamma - 2\beta z')}}$$

∴ Our previous discussion is a special case of $z_i(z')$ where $z' = 0$

$$z_i(0) = z_L = \frac{1+|\Gamma|e^{j\Theta_\Gamma}}{1-|\Gamma|e^{j\Theta_\Gamma}}$$

- Magnitude $|\Gamma|$ and S independent of z', but only *phase angle* $(\theta_\Gamma - 2\beta z')$ varies!

- When calculating $z_i(z')$
  - Find the point ***A*** with $|\Gamma|$ and θ_Γ for a given z_L [$= z_i(0)$]
  - Rotate ***OA*** by an angle $-2\beta z'$ (i.e. clockwise direction)
  - New point ***B*** represents $z_i(z')$



- What is $2\beta z'$?

$$2\beta z' = 2\frac{2\pi}{\lambda}z' = 4\pi\frac{z'}{\lambda}$$

z' → Half-wave length

$$z' = \frac{\lambda}{2}n \rightarrow 2\pi n$$

Full-turns

z' → Quarter-wave length

$$z' = \frac{\lambda}{4}(2n-1) \rightarrow \pi(2n-1)$$

Half-turns

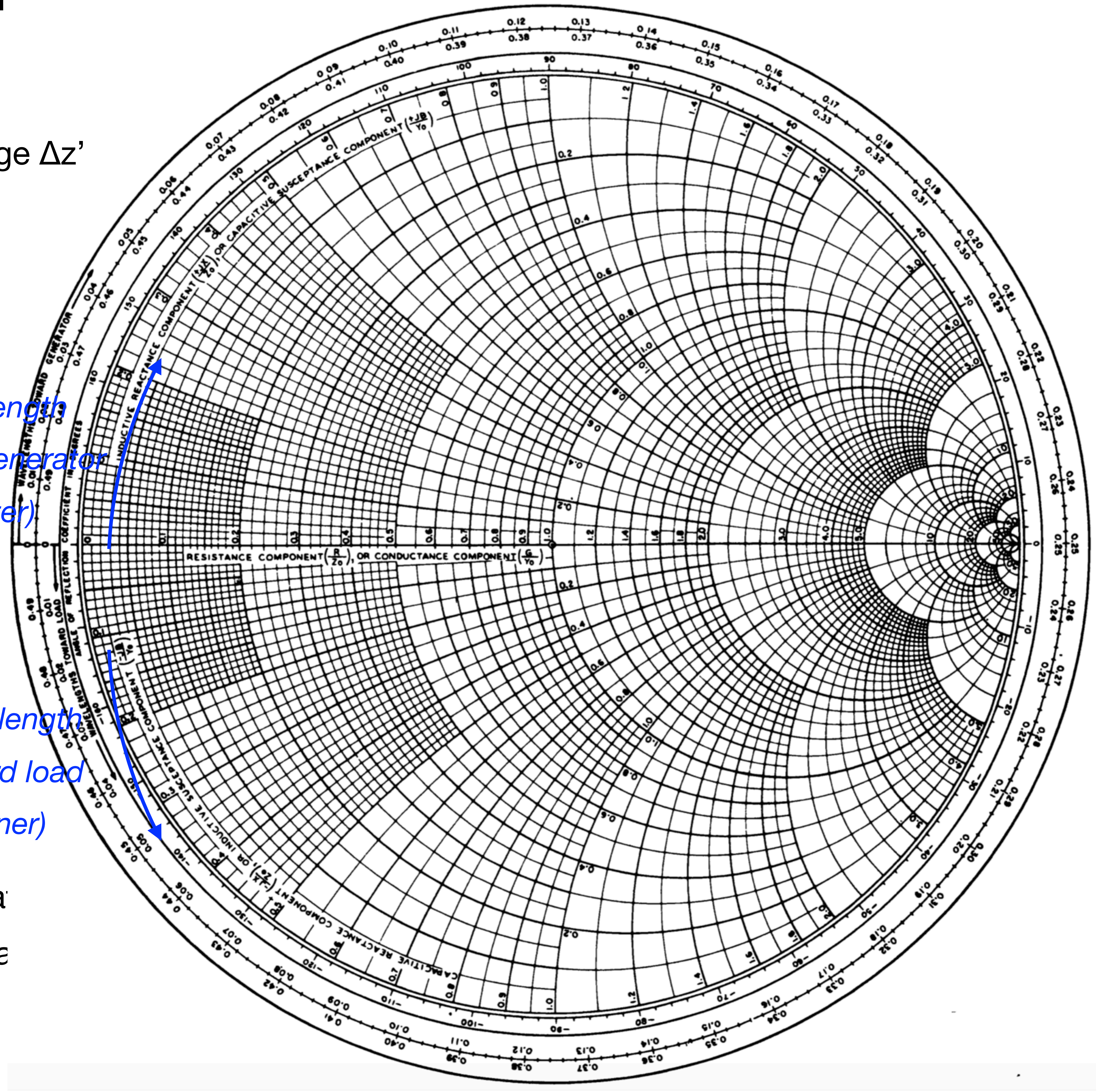
Chap. 9 | Input impedance in Smith Chart (2/2)

- **Scales along the perimeter**
  - Representing *phase angle shift* caused by length change Δz’

$$-2\beta\Delta z' = -4\pi\frac{\Delta z'}{\lambda}$$

  - Values on scale represent **Δz’/λ**
  - Outer scale
    - “Wavelengths toward generator”
    - Increasing z’ → Δz’ > 0 → –2βΔz’ < 0
    - Clockwise rotation
  - Inner scale
    - “Wavelengths toward load”
    - Decreasing z’ → Δz’ < 0 → –2βΔz’ > 0
    - Counter-clockwise rotation

  - If Δz’ = λ/2 → Δz’/λ = 0.5 → –4π(Δz’/λ) = -2π [Full rota
  - If Δz’ = λ/4 → Δz’/λ = 0.25 → –4π(Δz’/λ) = -π [Half rota



Chap. 9 | Examples

- **Example 1**

  - Find the input impedance with given condition as

  $$z_L = 0,\ \ R_0 = 50\ (\Omega),\ \ z' = 0.1\lambda$$

- **Procedure**

  (1) Find intersection of r = 0 and x = 0 circles → $\boldsymbol{P_{sc}}$

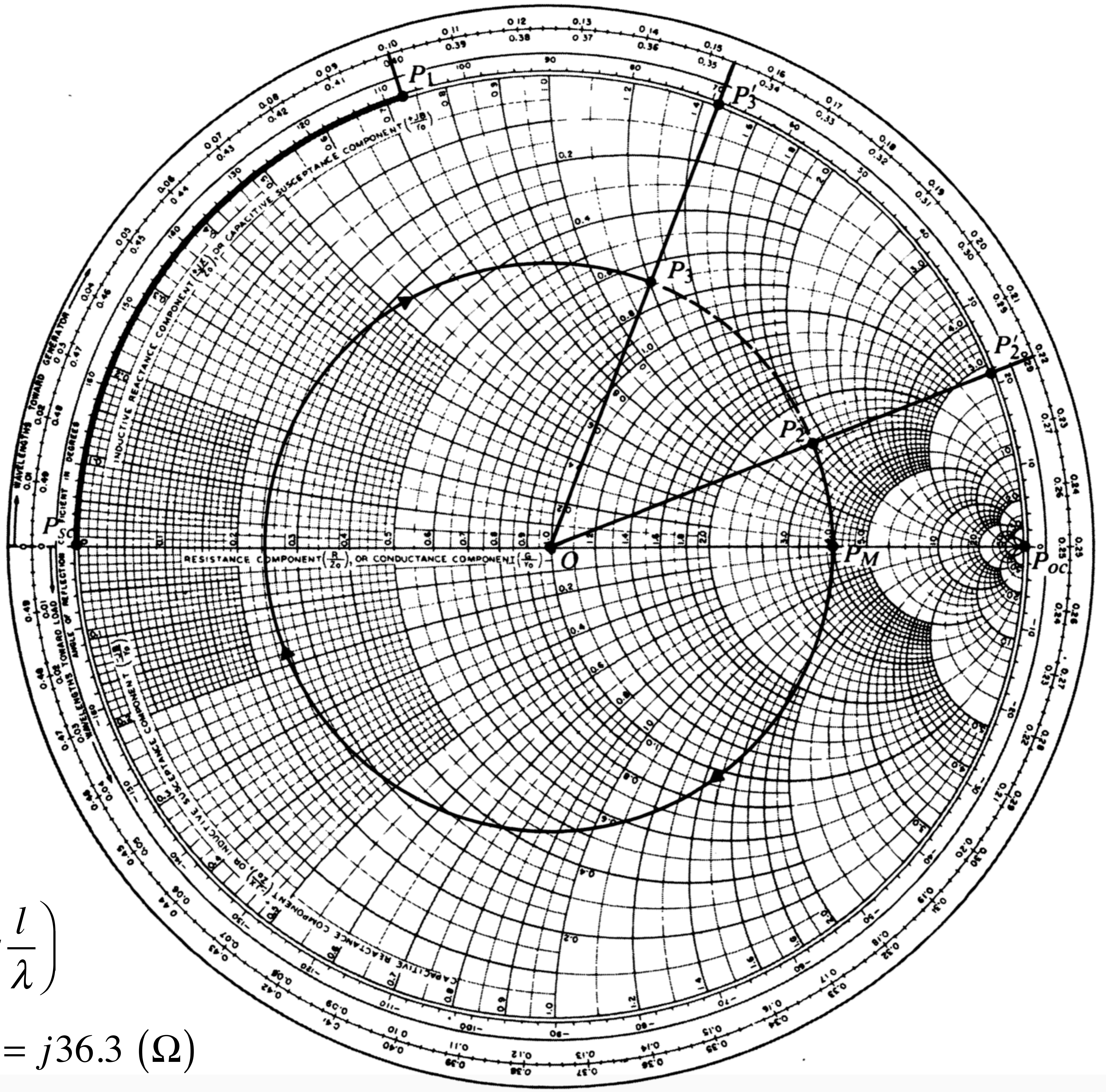
  $$\left(\because z_L = r + jx = 0\right)$$

  (2) Move along perimeter by 0.1 (=z'/λ) *[Clockwise]* → $\mathbf{P_1}$

  $$\left(\because -4\pi\frac{z'}{\lambda} < 0\right)$$

  (3) At $\mathbf{P_1}$, we read r = 0 and x ~ 0.725. Thus, $z_i = j0.725$

  (4) Finally, $Z_i = R_0 \cdot z_i = 50 \cdot j0.725 = j36.3\ (\Omega)$



** Result consistent as previously,

$$Z_i = R_0\frac{Z_L + jR_0\tan(\beta l)}{R_0 + jZ_L\tan(\beta l)} = jR_0\tan(\beta l) = jR_0\tan\left(2\pi\frac{l}{\lambda}\right)$$

$$= j50\tan 36^\circ = j36.3\ (\Omega)$$

Chap. 9 | Examples

- **Example 2**

  - Find Γ, S, Z_i at $z' = l$ and location of a voltage maximum with given condition

$$Z_L = 260 + j180 \ (\Omega), \ R_0 = 100 \ (\Omega), \ l = 0.434\lambda$$

- **Procedure**

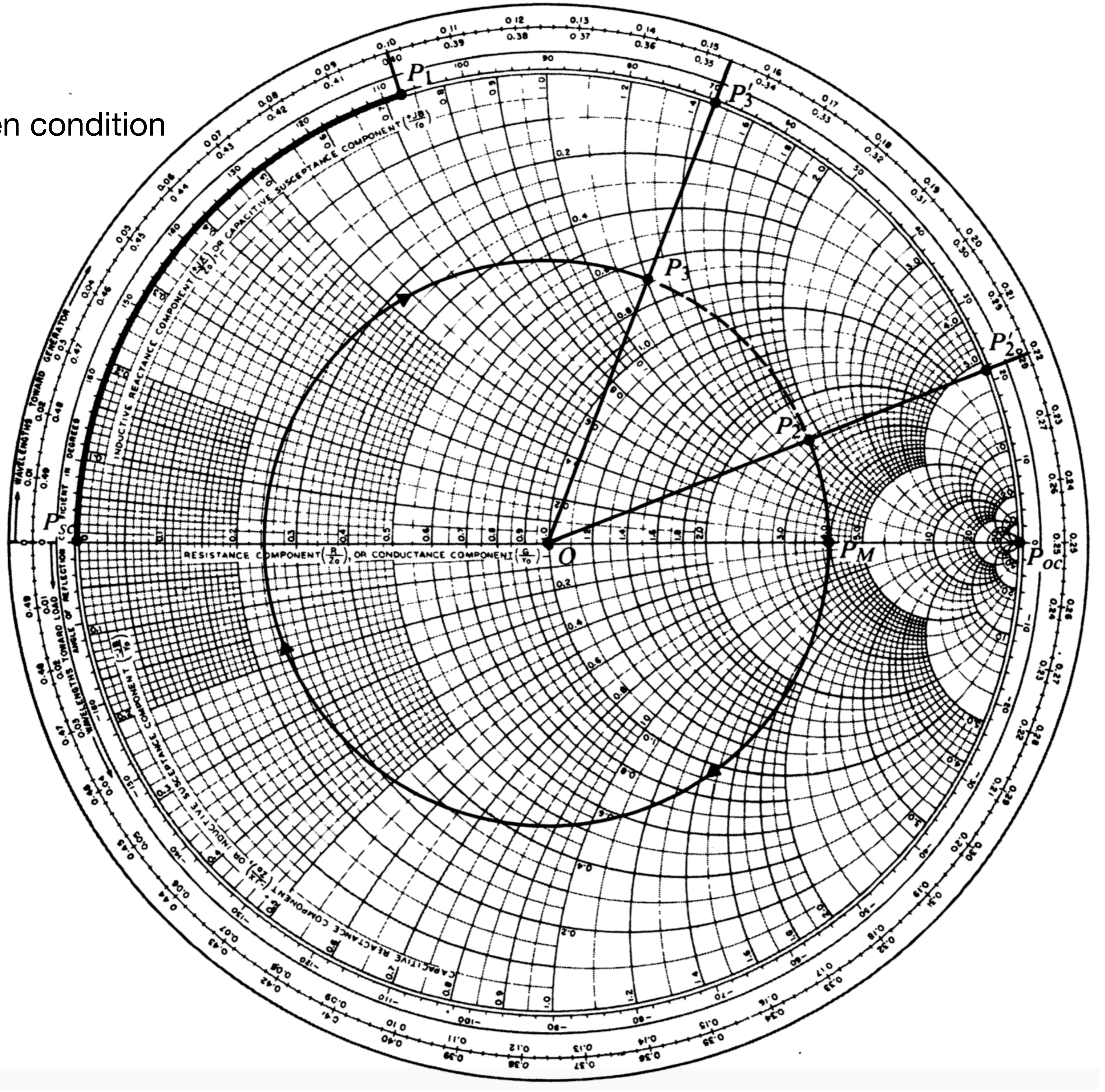
  (1) Find the $z_L = Z_L/R_0 = 2.6 + j1.8$ on the Smith Chart: **P₂**

  (2) Obtain $|\Gamma|$ of a circle centered at the origin and passing **P_2** by simply plugging Z_L and R_0 into $|\Gamma| = \left|\frac{Z_L - R_0}{Z_L + R_0}\right| = 0.6$

  (3) Now, to obtain θ_Γ, draw an extension line of **OP_2** to reach at **P_2'**. Read the value 0.22 (=z'/λ, *w.r.t.* P_{sc}) on the outer scale. Thus, $\theta_\Gamma = \pi - 4\pi\frac{z'}{\lambda} = 0.12\pi \ (\text{rad}) = 21°$

  (4) Now, intercept between the circle and positive-real axis gives r = ***S* = 4.**

  (5) To find input impedance at $z' = l$, extend the line **OP_2** to reach at ***P_2'*** and read 0.220 on outer scale. From there, rotate in the *CW* direction by 0.434 (= l/λ), reaching at 0.154 at **P_3'**.



Chap. 9 | Examples

- **Example 2**

  - Find Γ, S, Z_i at $z' = l$ and location of a voltage maximum with given condition

$$Z_L = 260 + j180\ (\Omega),\ \ R_0 = 100\ (\Omega),\ \ l = 0.434\lambda$$

- **Procedure (Cont'd)**

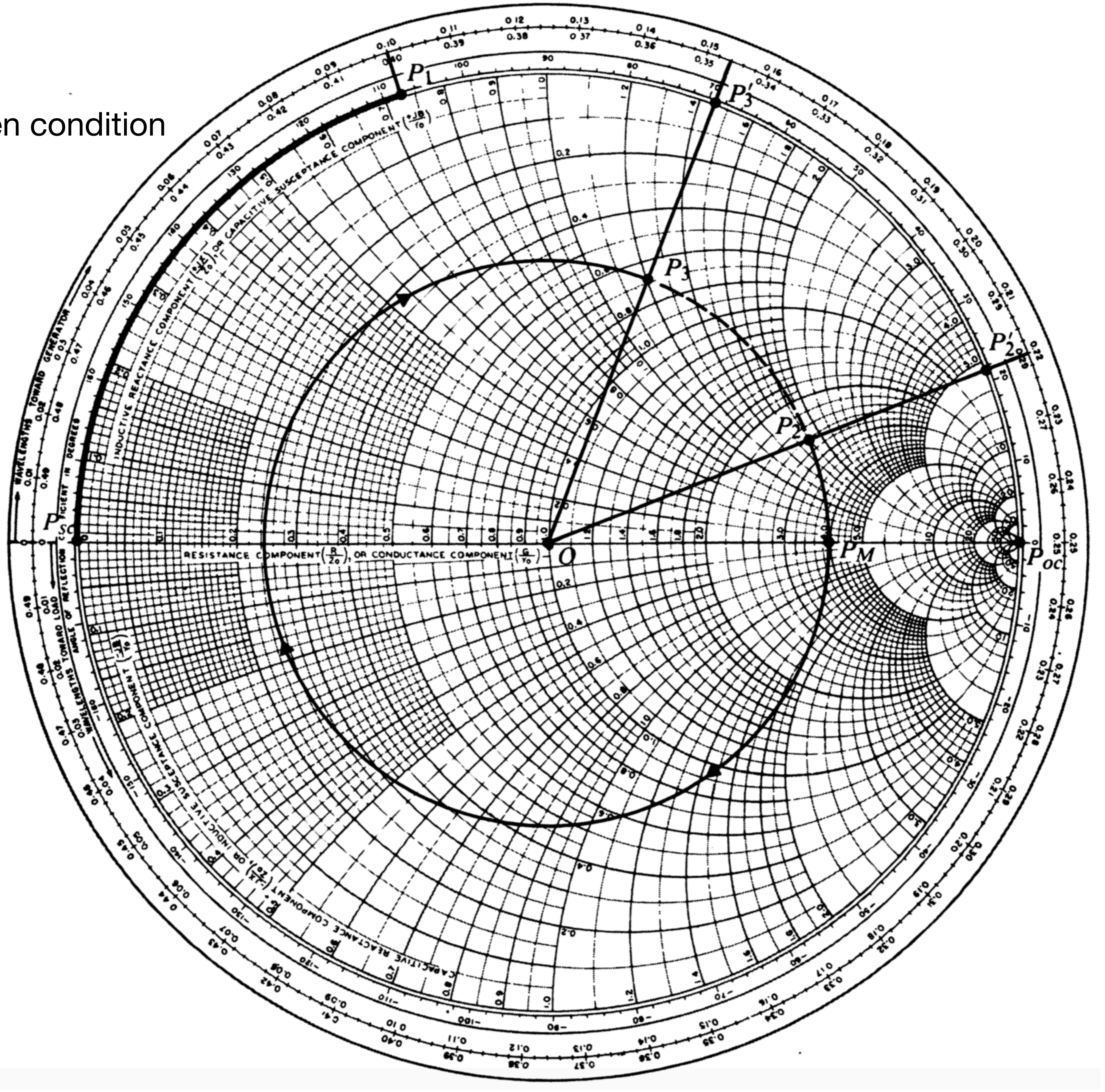
(4) To find input impedance at $z' = l$, extend the line **OP_2** to reach at ***P_2'*** and read 0.220 on outer scale. From there, rotate in the CW direction by 0.434 (= l/λ), reaching at <u>0.154</u> at **P_3'**.
($\because$0.22 + [0.5 – 0.066]) → 0.22 - 0.066) (0.5 is a full-turn)

(5) Find the intercept between the circle and the line **OP_3'** which gives **P_3**.

(6) At **P_3**, read r = 0.69 and x = 1.2.

(7) Thus, **$Z_i = R_0 \cdot z_i = 100 \cdot (0.69 + j1.2) = 69 + j120\ (\Omega)$**

(8) In going from P2 to P3, the circle intersects the positive real axis at PM with voltage maxima. Thus, voltage maxima appears at (0.25-0.22)/λ away from the load.



- **Example 3**

  - Find Γ, Z_L, l_m and R_m using the Smith Chart with given condition as

  $$R_0 = 50\ (\Omega),\quad S = 3.0,\quad \lambda = 0.4\ (m)$$

  First voltage minima at $z'_m = 0.05\ (m)$ *(Meaning?)*

- **Procedure**

  (1) On positive real-axis, **P_M** represents $r = S = 3.0$ ($= R_L/R_0$)

  (2) Then, we have circle of radius $|\Gamma| = 0.5$ (θ_Γ yet unknown)

  $$\left(\because |\Gamma| = \left| \frac{R_L - R_0}{R_L + R_0} \right| = 0.5 \right)$$

  (3) Intersection between negative real-axis and the circle

  (4) : **P_m** [$\Gamma < 0 \rightarrow R_L < R_0$] → ***Voltage minima*** at $z' = 0$

  (see slide 13-2)

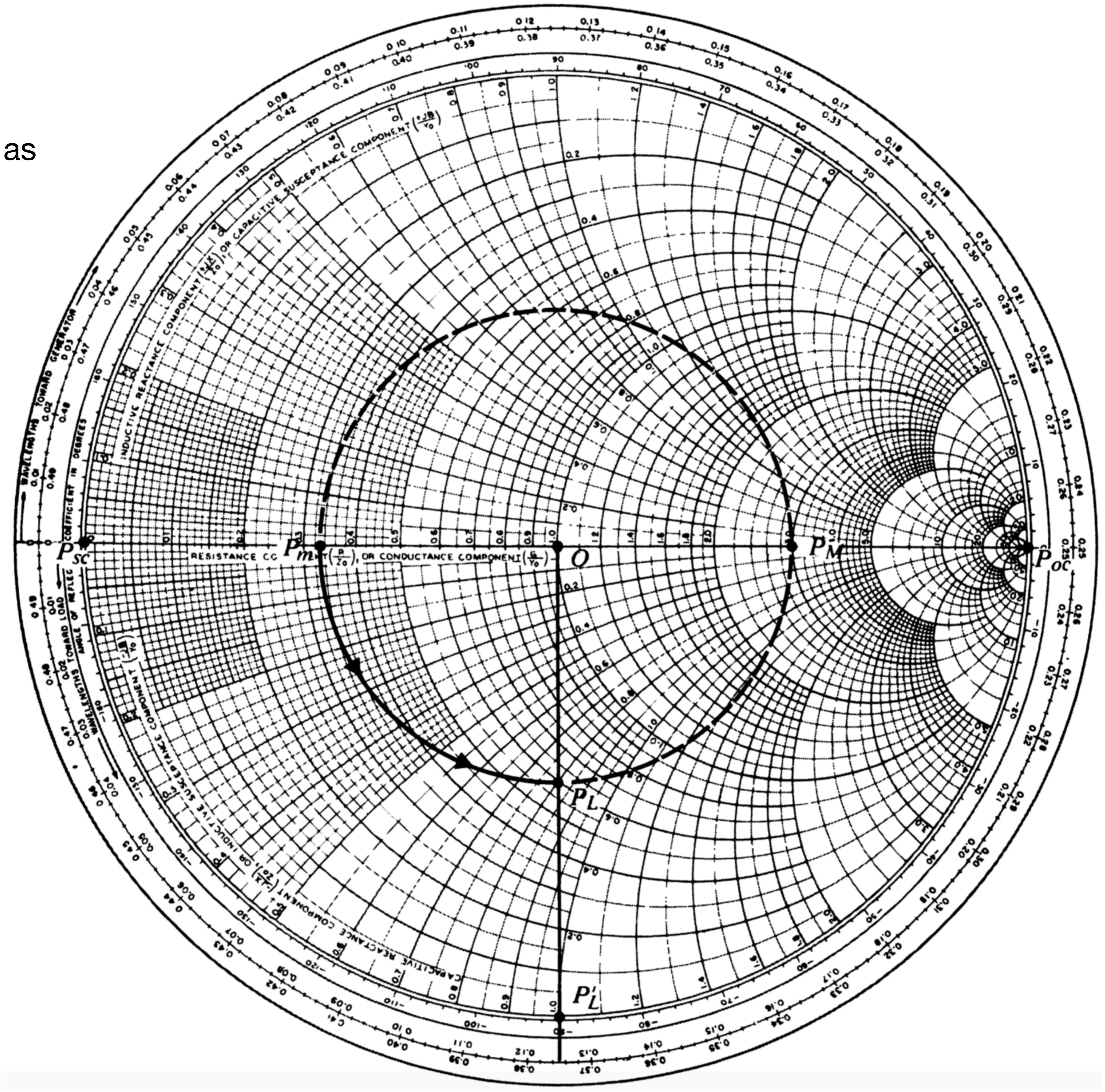
  (5) To find load impedance, move from **P_m** along perimeter

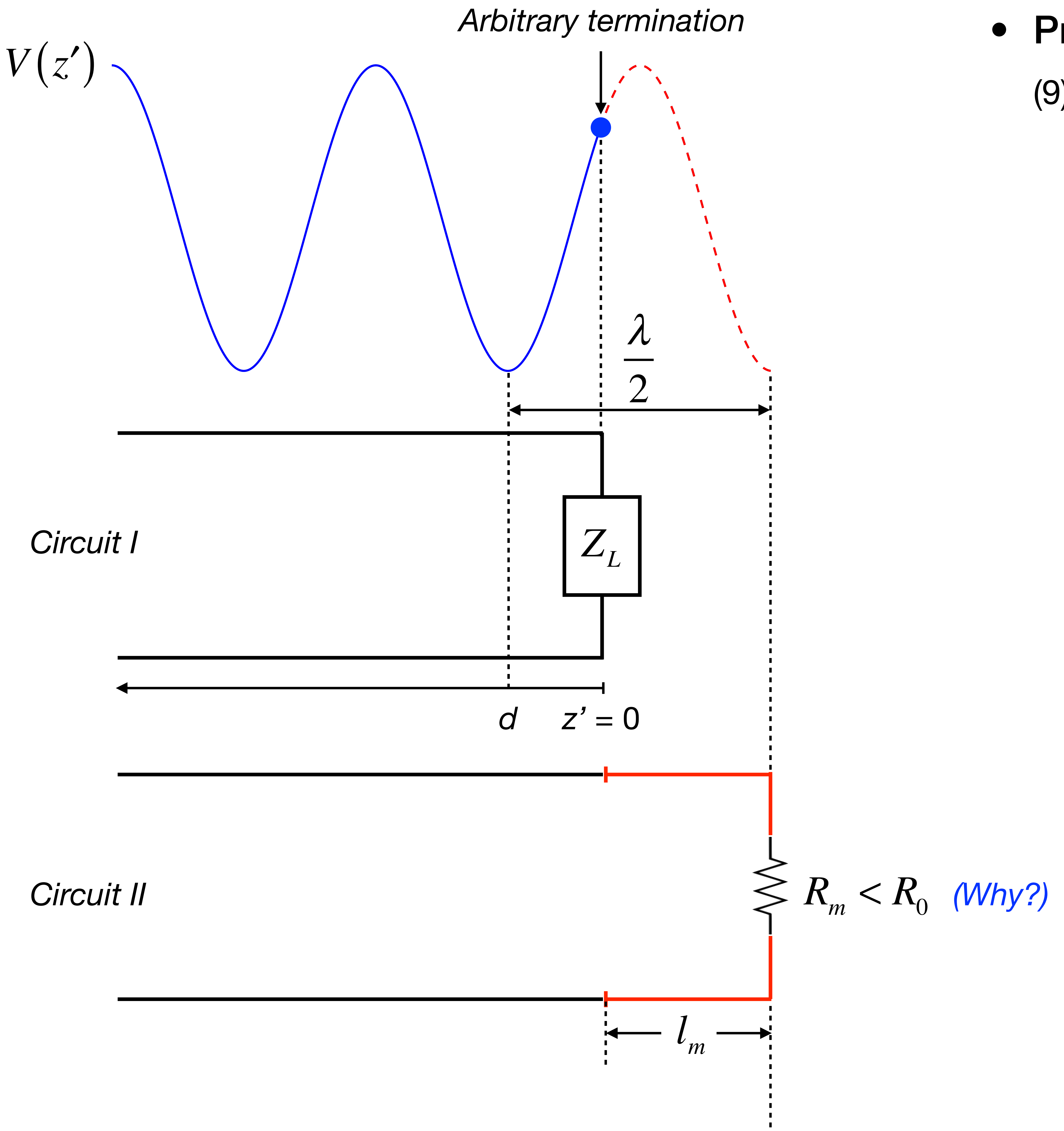
  by $z'_m/\lambda = 0.05/0.4 = 0.125$ *[in the **CCW** direction. Why?]*

  (6) **P_L** represents reflection coefficient → **$\Gamma = -j0.5$**

  (7) At **P_L**, Read $r = 0.6$, $x = 0.8$ → $z_L = 0.6 + j0.8$

  (8) Thus, **$Z_L = R_0 \cdot z_L = 30 - j40\ (\Omega)$**





- **Procedure (Cont'd)**

  (9) Equivalent length l_m and terminating resistance R_m can be found as

$$l_m = \frac{\lambda}{2} - z'_m = 0.2 - 0.05 = 0.15 \text{ (m)}$$

$$R_m = \frac{R_0}{S} = \frac{50}{3} = 16.7 \text{ } (\Omega)$$

Electromagnetics

<Chap. 9> Transmission Lines

Section 9.6 ~ 9.7

(2nd of **week 12**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Contents

Sec 7. Impedance matching

- Linear matching via quarter-wave transformer
- Parallel matching via single or double-stub approaches
- Admittance vs. impedance chart
- Examples

Chap. 9 | Impedance matching *via* Quarter-wave transformer

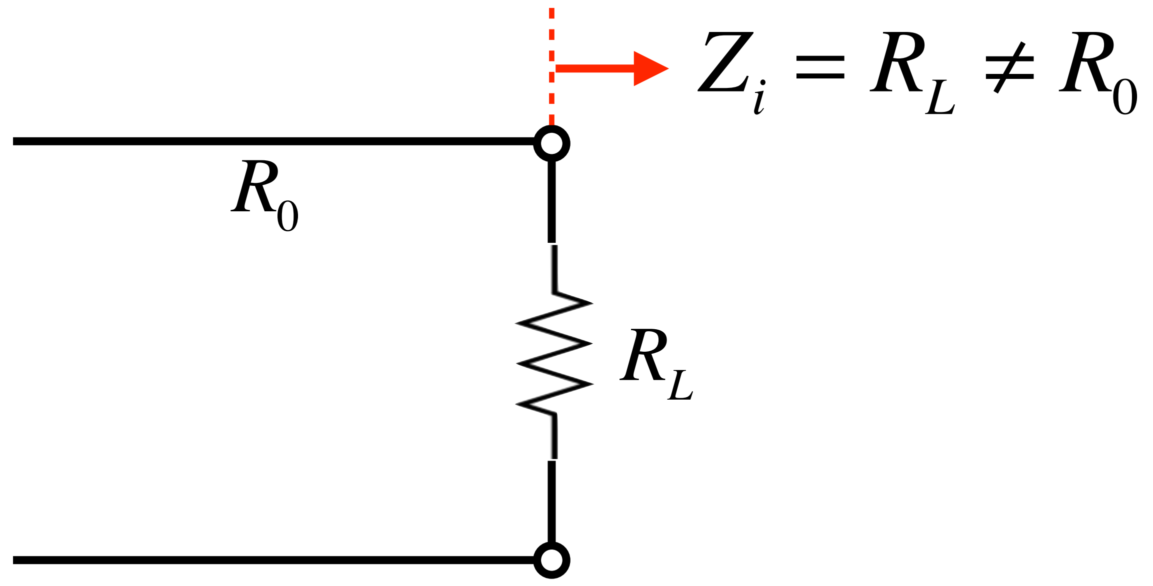
- **Maximum power transfer in TR-line**

  - Achieved under matched-load condition (i.e. $Z_L = Z_0$)

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$ *(No reflection at the load)* ⇨ $$S = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = 1$$ *(Smallest oscillation)*

- **Methods for impedance matching**

  - For resistive load ($Z_L = R_L$) → Using Quarter-wave transformer
  - For complex-valued load ($Z_L = R_L + jX_L$) → Using single-stub or double-stub matching



<Unmatched impedance>
Reflection occurs → Undesirable!

- **Quarter-wave transformer**

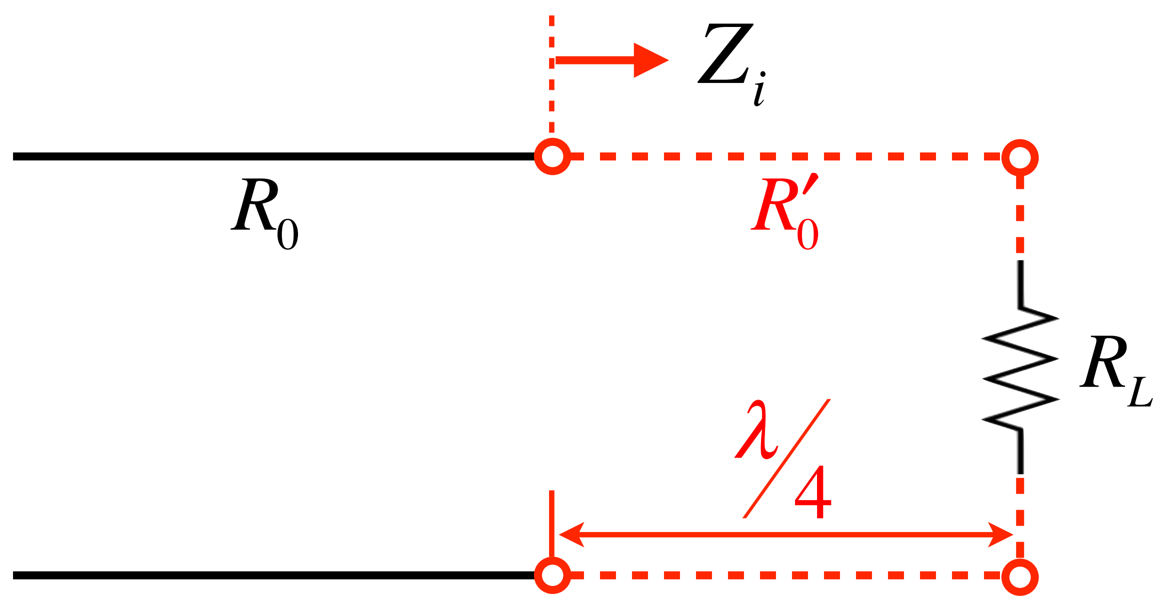
  :TR-line with ***characteristic impedance*** ***R_0'*** ***extended by λ/4*** and terminated with load R_L (unknown: R_0')

- ***Input impedance*** of quarter-wave transformer:

$$Z_i = R_0' \frac{R_L + jR' \tan\beta l}{R' + jR_L \tan\beta l} = \frac{R_0'^2}{R_L} \quad \left(\because \tan\beta l = \tan\left(\frac{2\pi}{\lambda}\cdot\frac{\lambda}{4}\right) = \tan\left(\frac{\pi}{2}\right) \to \infty\right)$$

- To satisfy matching condition,

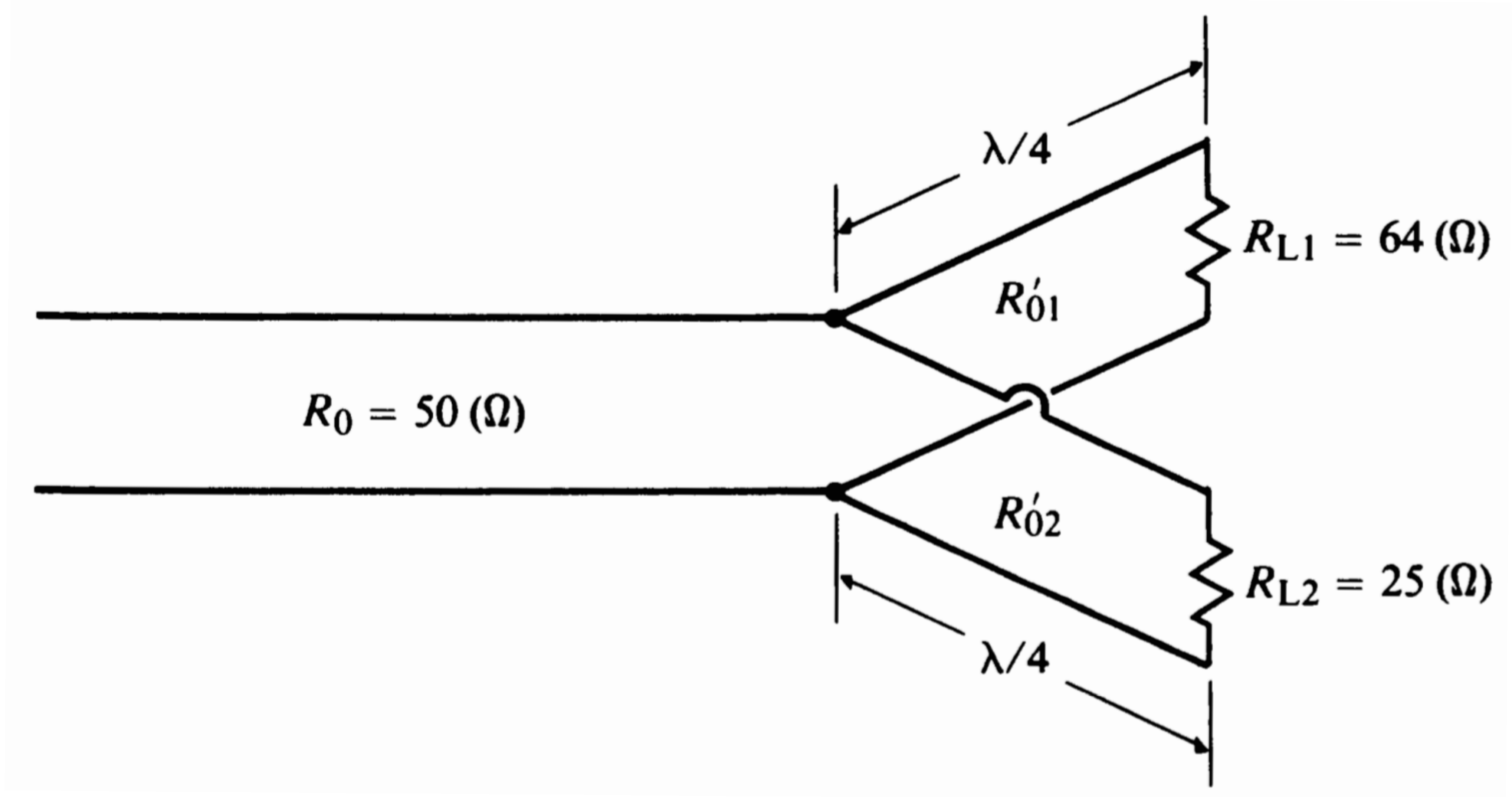
$$Z_i = R_0 \quad \rightarrow \quad \frac{R_0'^2}{R_L} = R_0 \qquad \boxed{\therefore R_0' = \sqrt{R_L R_0}}$$



<Quarter-wave transformer>

 Example: Quarter-wave transformer

- Quarter-wave transformers used for matching the loads (R_{L1} and R_{L2}) with R_0 = 50 (Ω). Power is fed "equally" to each load section. Obtain R_{01}' and R_{02}'.



- Matching condition: input impedance at junction, $Z_i = R_0$

$$Z_i = \left(\frac{1}{Z_{i1}} + \frac{1}{Z_{i2}}\right)^{-1} = R_0, \quad Z_{i1}, Z_{i2} \rightarrow \textit{input impedance of each load section}$$

- Since power ***equally*** sent to each load section,

$$Z_{i1} = Z_{i2} = 2R_0$$

- Each load connected with a quarter-wave transformer, so we have

$$Z_{i1} = \frac{R'^2_{01}}{R_{L1}} = 2R_0 \quad \rightarrow \quad R'_{01} = \sqrt{2R_0R_{L1}} = \sqrt{2\cdot 50\cdot 64} = \boxed{80(\Omega)}$$

$$Z_{i2} = \frac{R'^2_{02}}{R_{L1}} = 2R_0 \quad \rightarrow \quad R'_{02} = \sqrt{2R_0R_{L2}} = \sqrt{2\cdot 50\cdot 25} = \boxed{50(\Omega)}$$

$$\boxed{\because Z_i = R'_0 \frac{R_L + jR'\tan\beta l}{R' + jR_L\tan\beta l} = \frac{R'^2_0}{R_L}}$$

- Obtain reflection coefficient and SWR for each section.

$$\Gamma_1 = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = \frac{64-80}{64+80} = -0.11 \quad \rightarrow \quad S_1 = \frac{1+|\Gamma_1|}{1-|\Gamma_1|} = \boxed{1.25}$$

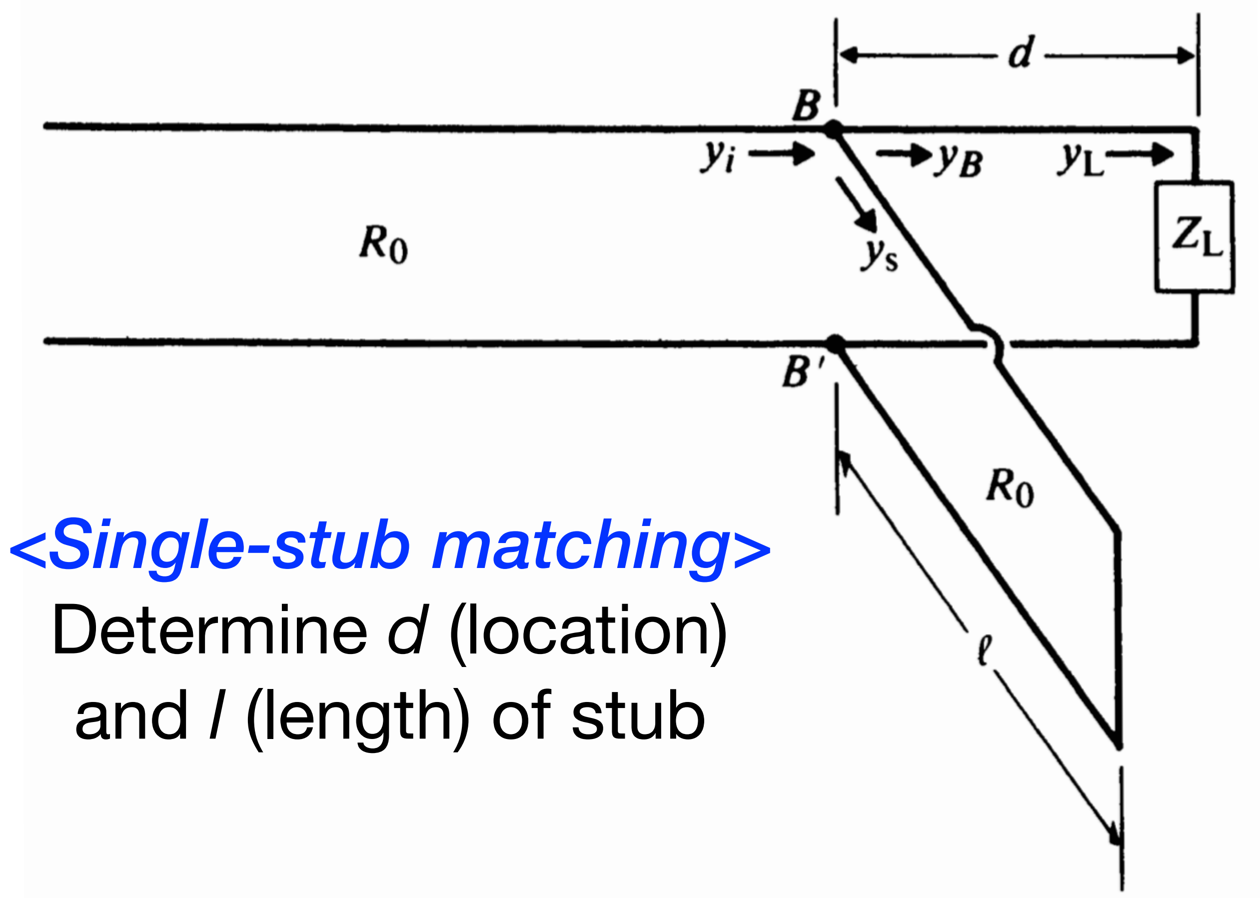
$$\Gamma_2 = \frac{R_{L2} - R'_{02}}{R_{L2} + R'_{02}} = \frac{25-50}{25+50} = -0.33 \quad \rightarrow \quad S_2 = \frac{1+|\Gamma_2|}{1-|\Gamma_2|} = \boxed{1.99}$$

Chap. 9 | Impedance matching for complex-valued load

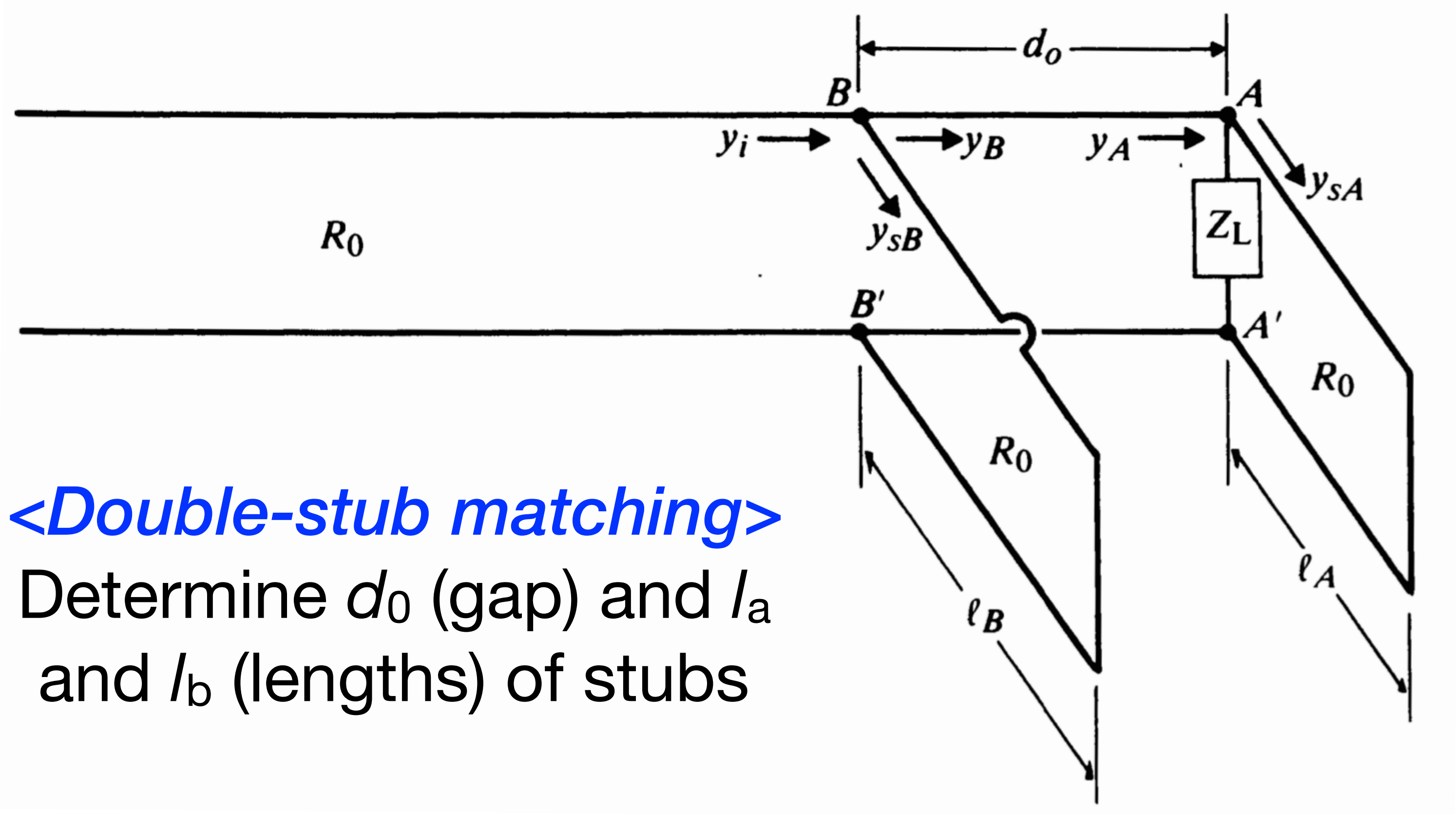
- Impedance matching for complex-valued loads ($Z_L = R_L + jX_L$)
  - Quarter-wave transformer does not work!
    - Lossless quarter-wave extension line *(real-valued R_0')* ⇔ $R_0' = \sqrt{2R_0 Z_L}$ *(Complex-valued!)*

      Contradicting

- Single / double-stub matching
  - *Open- or short-circuited* line sections *attached in parallel* with main TR-line *at an appropriate distance from the load*
  - **Purpose:** to achieve [Z_i at a joint B-B'] = R_0 (i.e. Effectively cancelling out “imaginary part of Z_L” by using parallel stubs)



<Single-stub matching>
Determine d (location) and l (length) of stub



<Double-stub matching>
Determine d_0 (gap) and l_a and l_b (lengths) of stubs

  - *Short-circuit preferable* compared to open-circuit, because
    - $Z_L \rightarrow \infty$ hard to achieve
    - Radiation from an open end
    - Coupling to nearby objects

Chap. 9 | Admittance Smith Chart

- **Admittance Smith Chart**

  - Previously, we read impedance on Smith Chart

  - Similarly, we can read admittance via ***impedance-to-admittance conversion!***

  - ***Normalized*** admittance:

$$Y_L \triangleq \frac{1}{Z_L} \quad \rightarrow \quad \boxed{y_L = \frac{1}{z_L}} \quad \text{where} \quad z_L = \frac{Z_L}{R_0}$$

$$= \frac{R_0}{Z_L} = R_0 Y_L \triangleq \frac{Y_L}{Y_0} = g + jb$$

* *Y_0*: Characteristic admittance (1/*R_0*)

* *g*: conductance

* *b*: susceptance

  - Impedance and Admittance in terms of reflection coefficient

$$z_L = \frac{1+\Gamma}{1-\Gamma} \quad \rightarrow \quad y_L = \frac{1-\Gamma}{1+\Gamma} = \frac{1+\Gamma e^{j\pi}}{1-\Gamma e^{j\pi}} = \frac{1+\Gamma'}{1-\Gamma'}$$

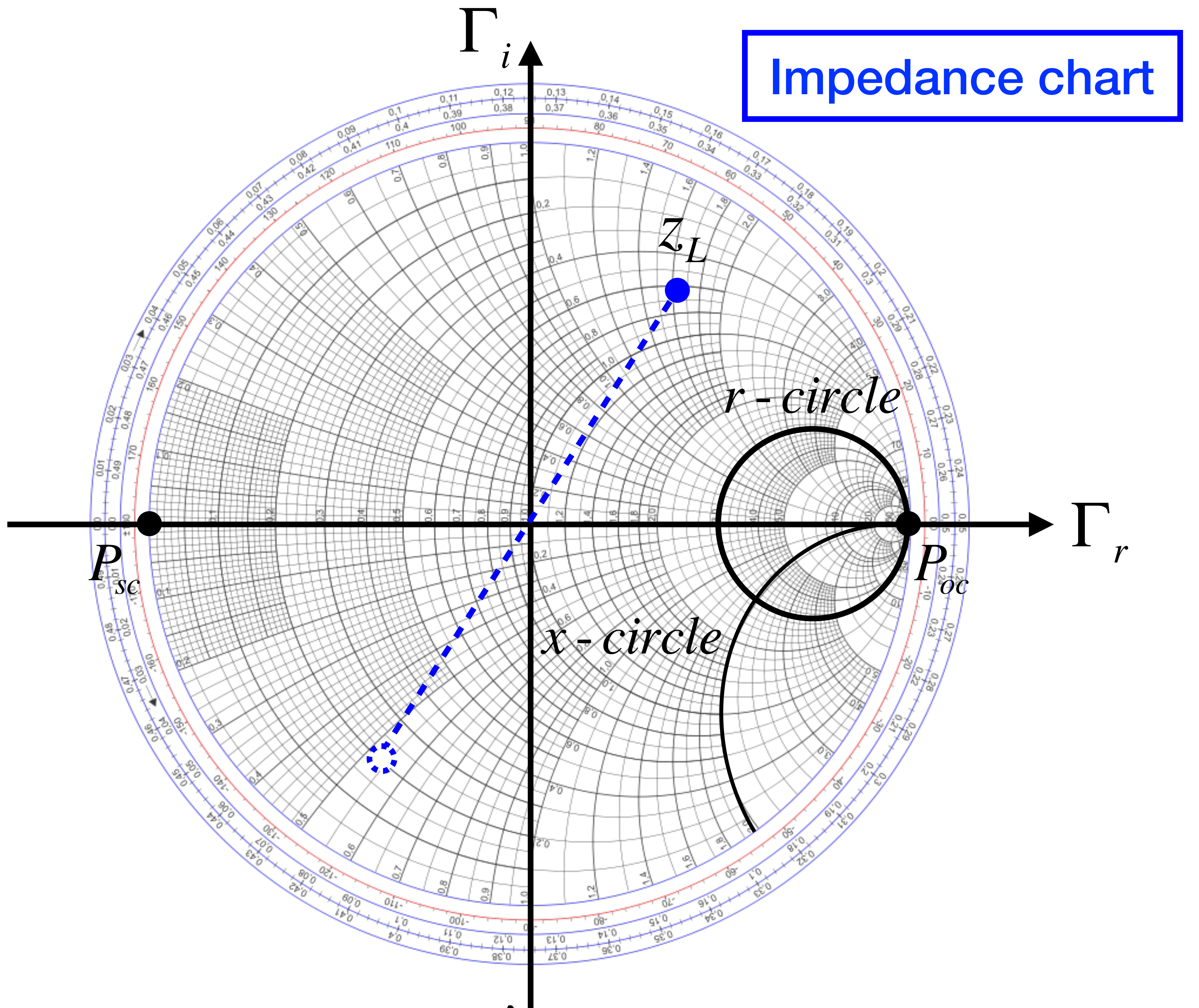
where $\Gamma' = \Gamma e^{j\pi}$

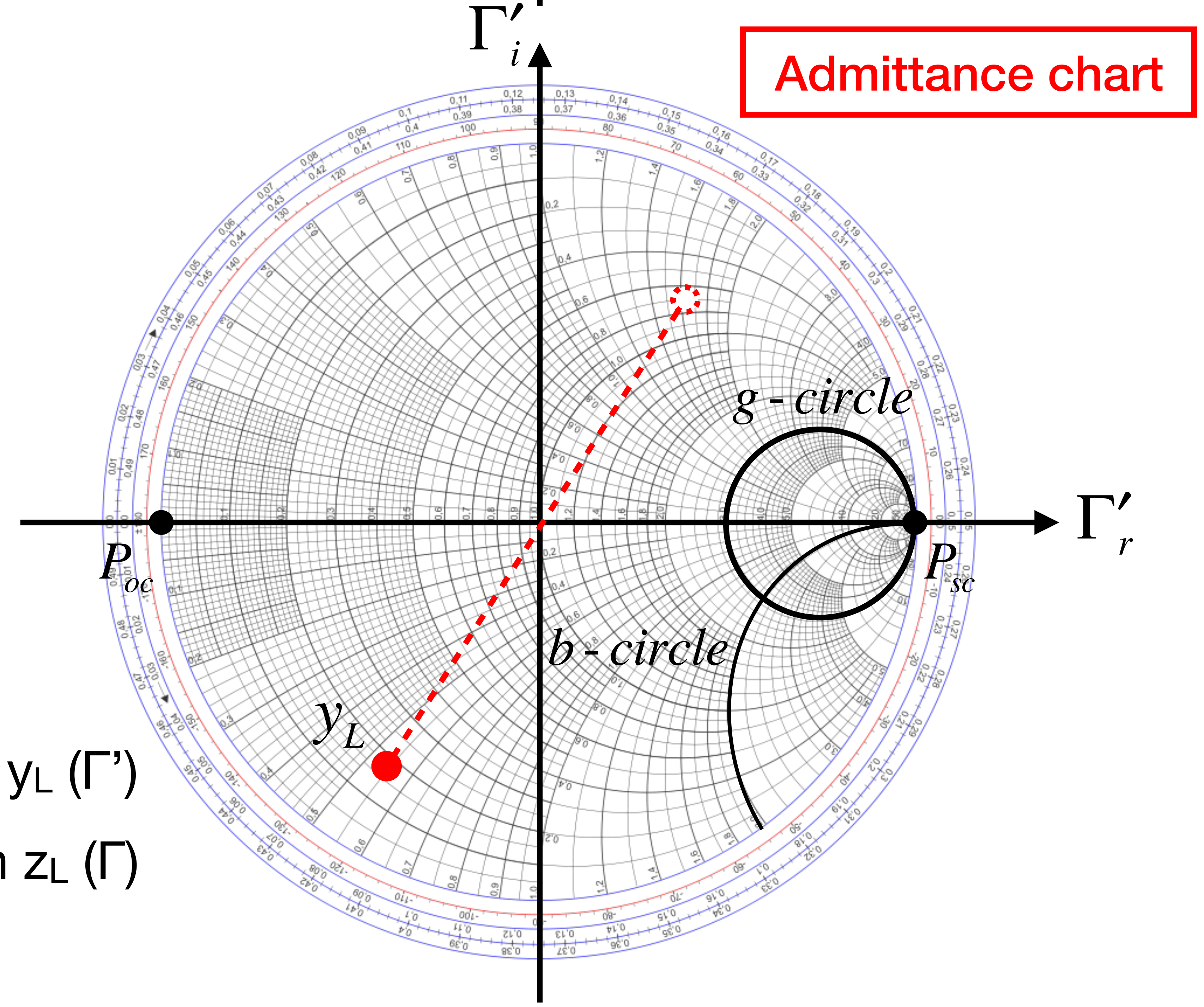
‣ *y_L* locates **“diametrically opposite”** to *z_L* on |Γ|-circle **(i.e. differed by an angle π)**

  - ***Impedance-to-Admittance conversion*** and vice versa

‣ Rotate *z_L* by 180° in Impedance Chart (Γ) → Chart becomes Admittance Chart with y_L (Γ’)

‣ Rotate *y_L* by 180° in Admittance Chart (Γ’) → Chart becomes Impedance Chart with z_L (Γ)





Single-stub matching (1/2)

- **Parallel connection of short-circuited stub**

  - Admittance more useful than impedance for ***"parallel" connection***

    - Admittance (Y): a measure of how well a circuit will allow a current to flow $\left(Y \triangleq \dfrac{1}{Z} \right)$

  - ***Impedance-matching*** condition

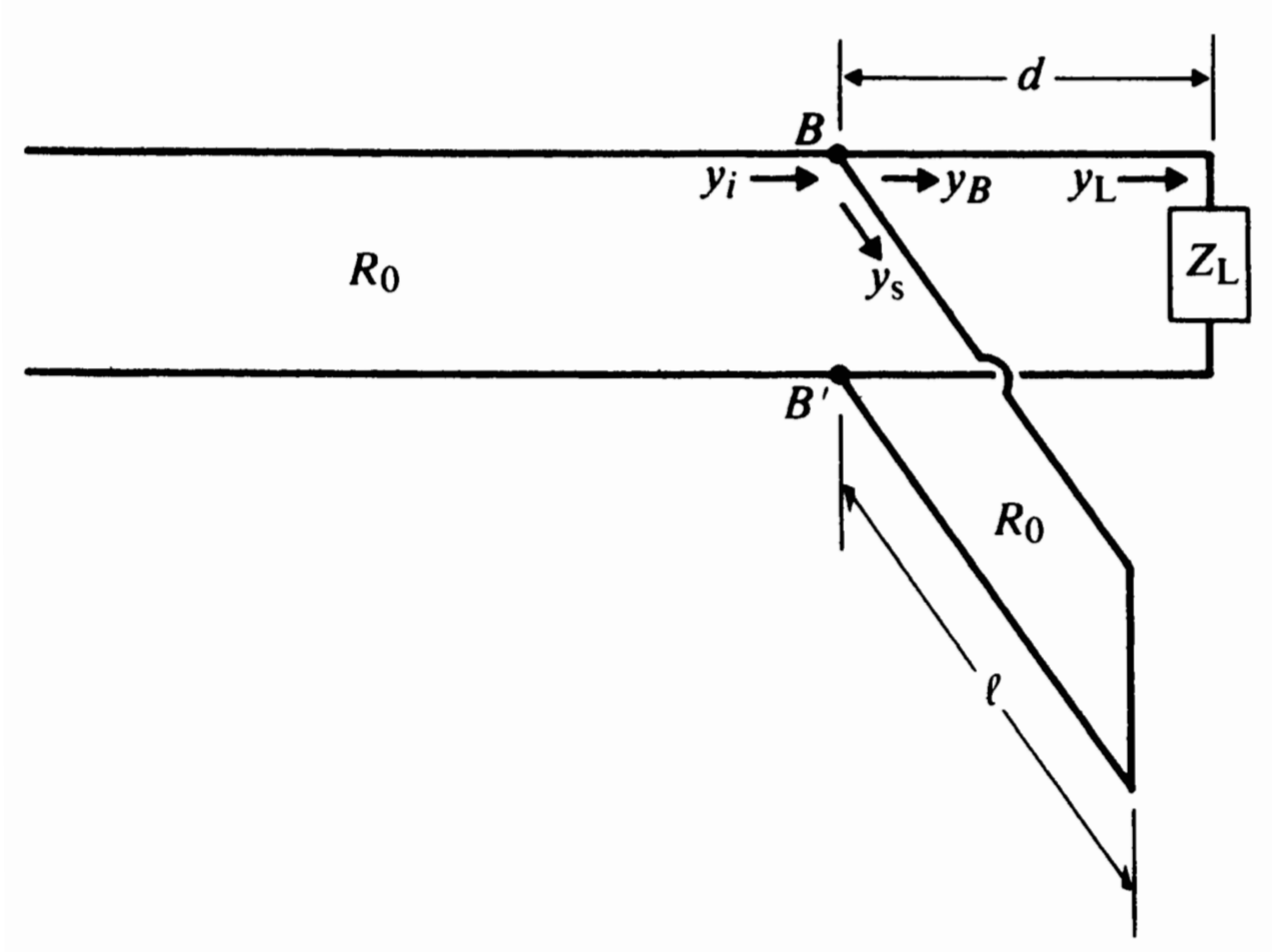
$$\left[Y_i = Y_L + Y_s \right] = Y_0$$

Y_i: Total input admittance at *B-B'* terminals toward load

Y_B: admittance of load section

Y_s: admittance of short-circuited stub section

Y_0: Characteristic admittance of main TR-line ($1/R_0$)



  - ***Normalized*** admittance

$$Y_0 = Y_B + Y_s \quad \rightarrow \quad \boxed{1 = \frac{Y_B}{Y_0} + \frac{Y_s}{Y_0} \triangleq y_B + y_s}$$

    - y_s should be purely resistive ***(Why?)***

$$y_s = \frac{Y_s}{Y_0} = \frac{R_0}{R_s} = \frac{R_0}{jR_0 \tan \beta l} = -\frac{j}{\tan \beta l} \quad \rightarrow \quad \boxed{y_s \triangleq -jb_B \quad \cdots (1)}$$

    - From normalized admittance equation,

$$\boxed{y_B = 1 - y_s = 1 + jb_B \quad \cdots (2)}$$

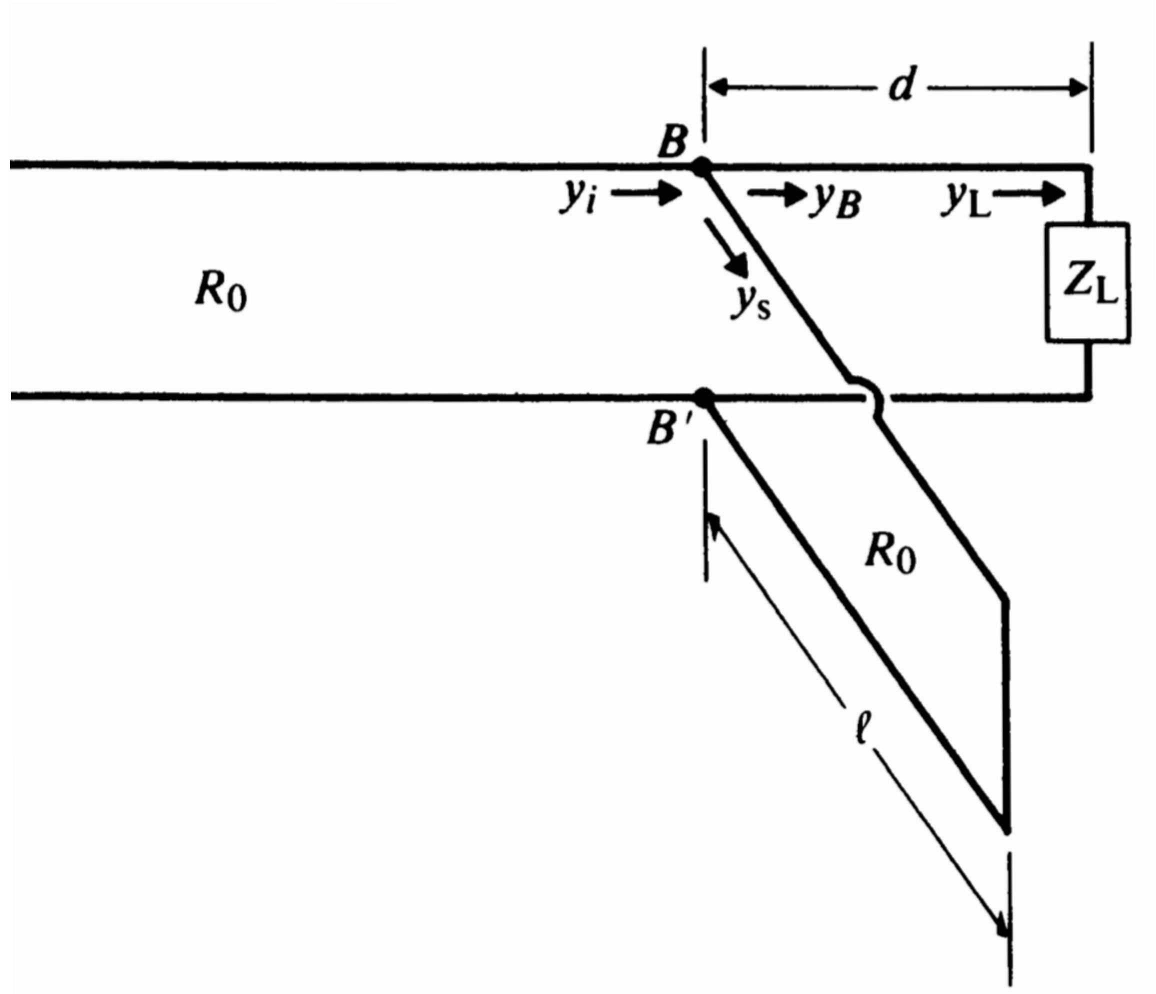
- ***What to do next?***

  - From eqn. (1), we define length (l) of the stub
  - From eqn. (2), we define distance (d) of the stub

> - Note that y_s (admittance of short-circuit stub) cancels imaginary part of y_B (admittance of load section)
>
> → ***Our original purpose!***

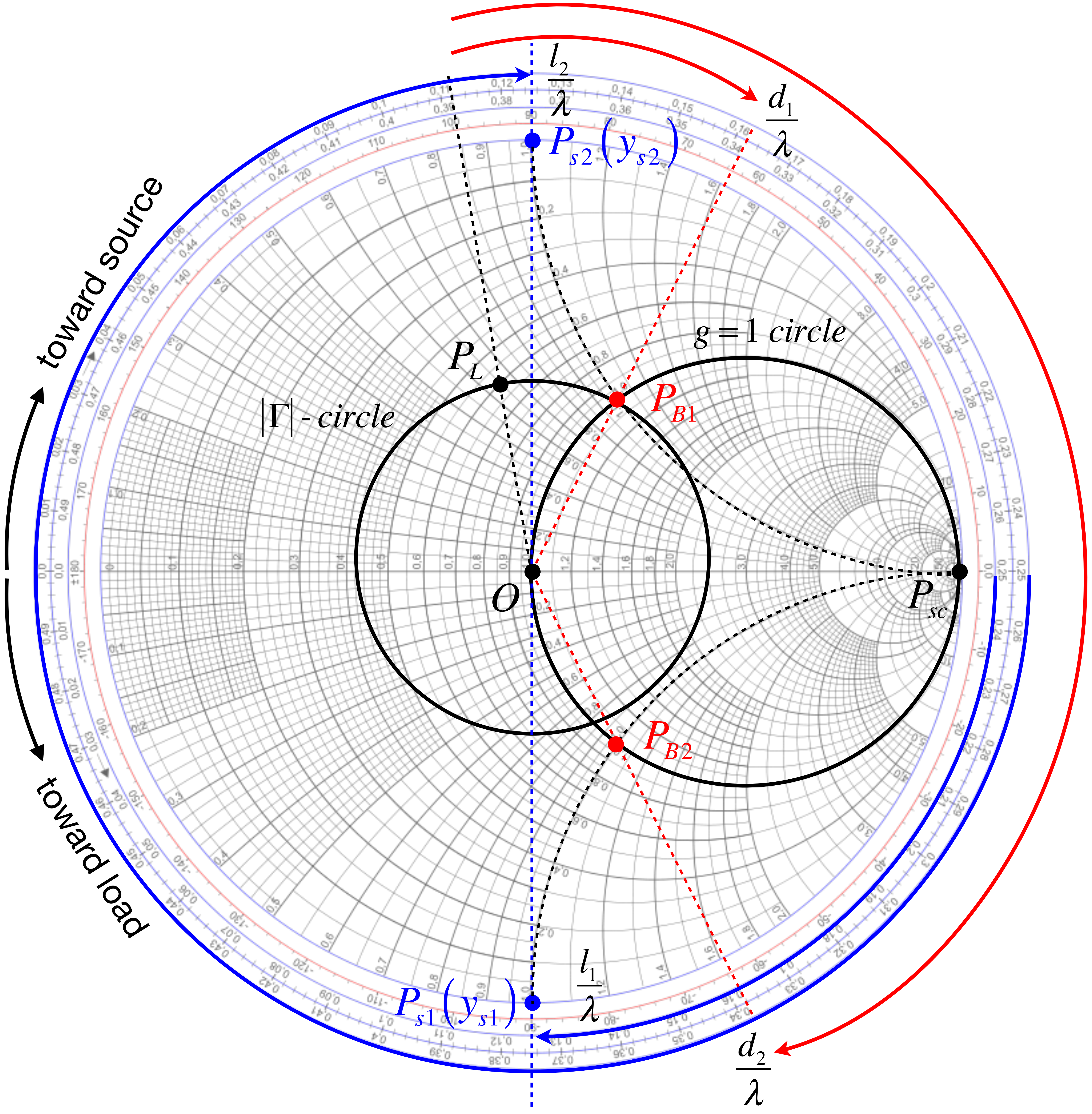
Single-stub matching (2/2)

- **Procedures**
  1. Find point **P_L** for load admittance $y_L = g + jb$ in **admittance Chart**.
  2. Draw **|Γ|-circle** passing through P_L *(* Any points on |Γ|-circle represent load section of arbitrary length d).*
  3. Find **intersections** between **|Γ|-circle** and **(g = 1) circle**. These are denoted as points P_{B1} and P_{B2}, representing **two solutions for y_B** satisfying *condition (2)*. (i.e. $y_{B1} = 1 + jb_{B1}$, $y_{B2} = 1 + jb_{B2}$)
  4. Find distances d_1 and d_2 for P_{B1} and P_{B2} from **angles between [OP_L and OP_{B1}]** and **between [OP_L and OP_{B2}]** in <u>CW direction</u> (*why?*)
  5. Find angle values for $y_{s1} = -jb_{B1}$ (point P_{s1}) and $y_{s2} = -jb_{B2}$ (point P_{s2}) that **cancel out imaginary part of y_B** *(Condition (1))*. Choose lengths l_1 and l_2 from **angles between [OP_{sc} and OP_{s1}]** and **between [OP_{sc} and OP_{s2}]** in <u>CW direction</u>. (*Why* OP_{sc}?)



Conditions for matching

$$\begin{cases} y_s \triangleq -jb_B & \cdots(1) \\ y_B = 1 + jb_B & \cdots(2) \end{cases}$$



<Admittance Chart>

Example: Single-stub matching

- **Characteristic impedance of TR-line R_0 = 50 (Ω) and terminated by $Z_L = 35 - j47.5$ (Ω). Find position (d) and length (l) of short-circuited stub for impedance matching.**

- Find normalized z_L ($=Z_L/R_0 = 0.7 - j0.95$) on ***impedance chart***.

(1) Rotate z_L by 180° to convert it to y_L (Point P_L). The chart now becomes ***admittance chart***. We start from here.

(2) Draw **|Γ|-circle** passing through P_L.

(3) Find **two intersecting points** between |Γ|-circle and (g = 1)-circle (P_{B1} and P_{B2}) that yield $\boldsymbol{y_{B1} = 1 + j1.2}$ and $\boldsymbol{y_{B2} = 1 - j1.2}$ *(satisfying condition 1)*

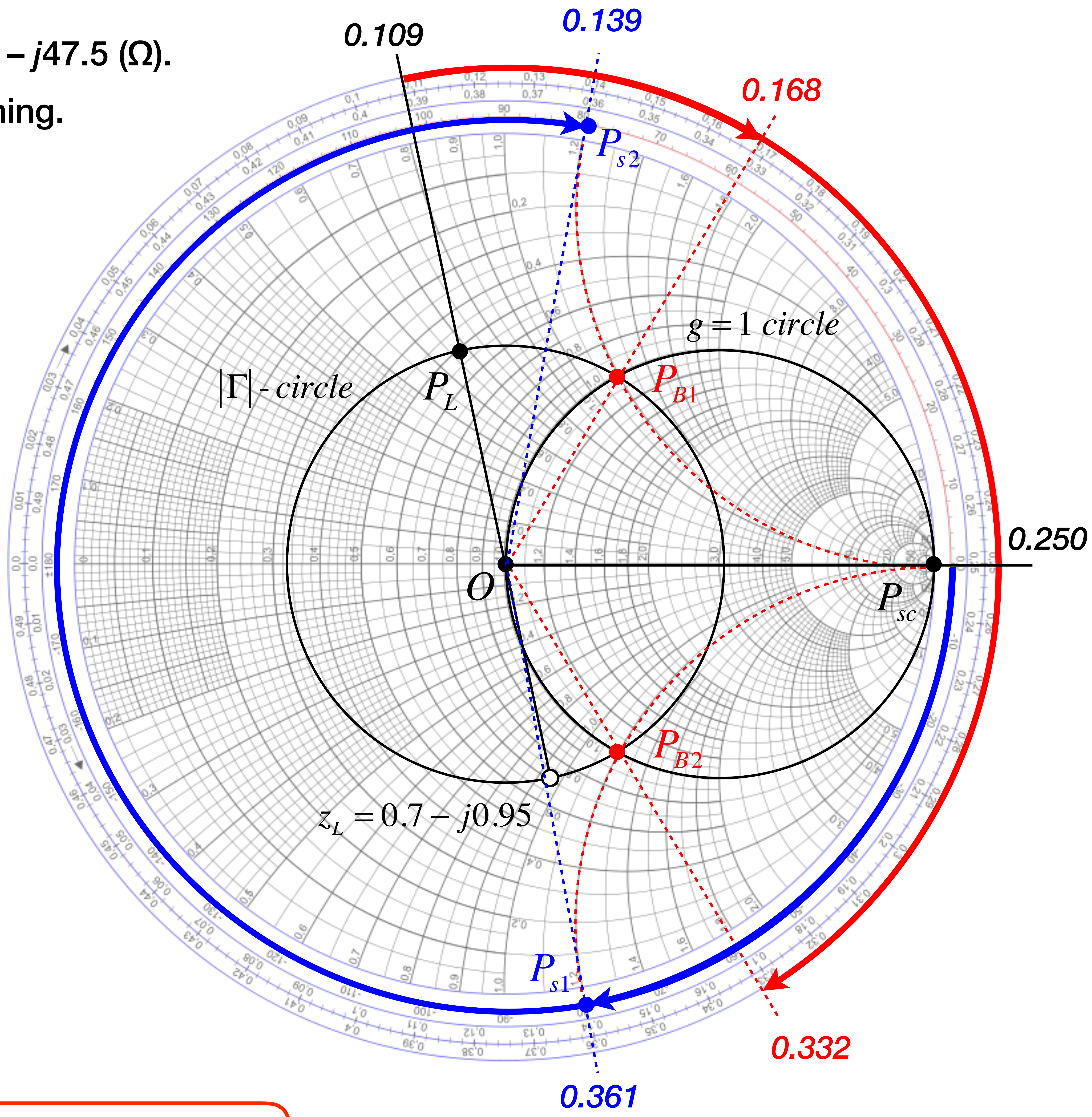
(4) Determine d_1 and d_2 from **angles between [OP_L and OP_{B1}]** and **between [OP_L and OP_{B2}]** in <u>CW direction</u>.

$$\begin{cases} d_1 = (0.168 - 0.109)\lambda = 0.059\lambda \\ d_2 = (0.332 - 0.109)\lambda = 0.223\lambda \end{cases}$$

(5) Read the **angle values** for $\boldsymbol{y_{s1} = -j1.2}$ and $\boldsymbol{y_{s2} = j1.2}$ (at points P_{s1} and P_{s2}) *(Satisfying condition 2)*. Determine l_{B1} and l_{B2} from **angles between [OP_{sc} and OP_{s1}]** and **between [OP_{sc} and OP_{s2}]**.

$$\begin{cases} l_{B1} = (0.361 - 0.250)\lambda = 0.111\lambda \\ l_{B2} = (0.139 + 0.250)\lambda = 0.389\lambda \end{cases}$$

∴***Shorter length preferred*** *unless there is mechanical constraint! Thus, choose d_1 and l_{B1}.*



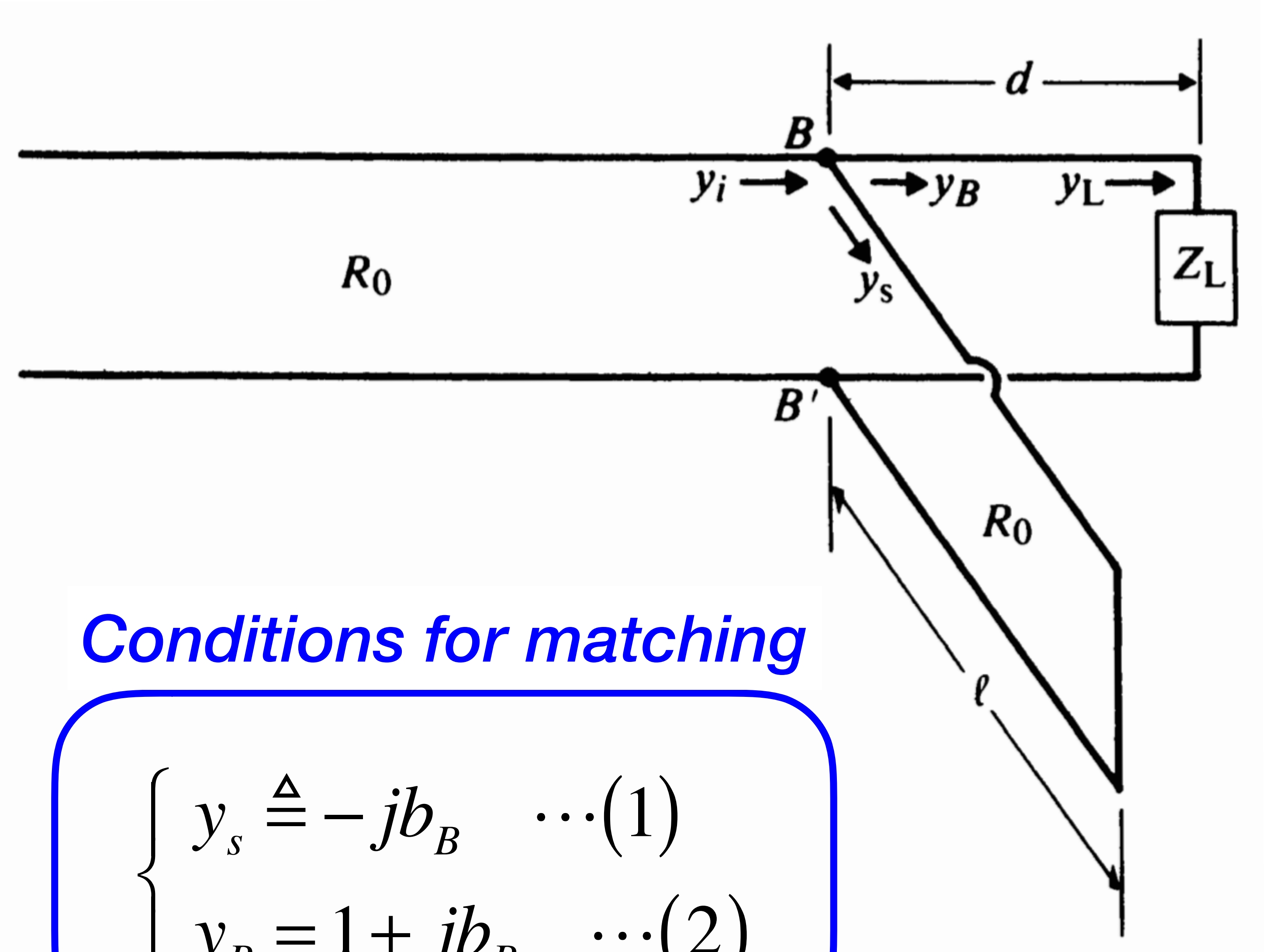
Chap. 9 | Analytical solution for single-stub matching (1/2)

- Problem of Smith Chart approach

  - In actual case, Smith chart leads to ***error due to graphical approximation (i.e. inter- or extrapolation)***

    → needs fine-adjustment of lengths

  - Instead, we can ***analytically*** obtain the solutions (***d*** and ***l***)!



Conditions for matching

$$\begin{cases} y_s \triangleq -jb_B & \cdots(1) \\ y_B = 1 + jb_B & \cdots(2) \end{cases}$$

- Analytical approach

  - Input impedance of "load" section (i.e. toward Z_L) at B-B' junction:

$$Z_{L,B-B'} = R_0 \frac{Z_L + jR_0 \tan\beta d}{R_0 + jZ_L \tan\beta d} \quad \rightarrow \quad z_{L,B-B'} = \frac{Z_{L,B-B'}}{R_0} = \frac{(Z_L + jR_0 \tan\beta d)/R_0}{(R_0 + jZ_L \tan\beta d)/R_0} = \frac{z_L + j\tan\beta d}{1 + jz_L \tan\beta d}$$

$$z_{L,B-B'} = \frac{z_L + j\tan\beta d}{1 + jz_L \tan\beta d} = \frac{(r_L + jx_L) + jt}{1 + j(r_L + jx_L)t} \quad \text{where } z_L \triangleq r_L + jx_L \text{ and } t \triangleq \tan\beta d$$

  - Normalized Admittance is then given as

$$y_B = \frac{1}{z_{L,B-B'}} = \frac{1 + j(r_L + jx_L)t}{(r_L + jx_L) + jt} = g_B + jb_B$$

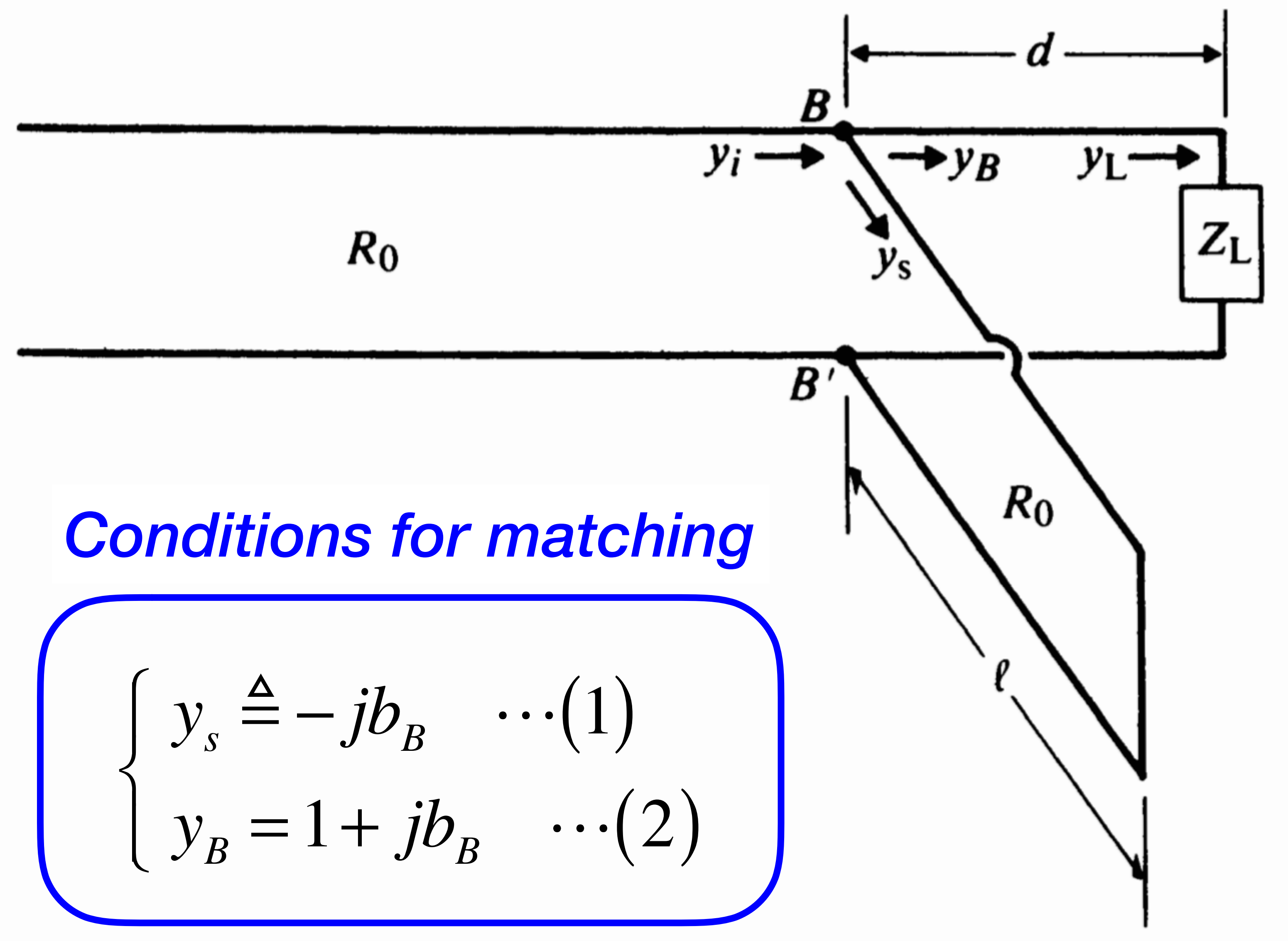
where $g_B = \dfrac{r_L(1 - x_L t) + r_L t(x_L + t)}{r_L^2 + (x_L + t)^2}$, $b_B = \dfrac{r_L^2 t - (1 - x_L t)(x_L + t)}{r_L^2 + (x_L + t)^2}$

  - y_B should satisfy *condition (2)* as

$$g_B = \frac{r_L(1 - x_L t) + r_L t(x_L + t)}{r_L^2 + (x_L + t)^2} = 1$$

$$\rightarrow \quad (r_L - 1)t^2 - 2x_L t + (r_L - r_L^2 - x_L^2) = 0 \quad \cdots(3)$$

Analytical solution for single-stub matching (2/2)



Conditions for matching

$$\begin{cases} y_s \triangleq -jb_B & \cdots(1) \\ y_B = 1 + jb_B & \cdots(2) \end{cases}$$

- Analytical approach

  - Solutions to eqn. (3) can be divided into two cases:

$$(r_L - 1)t^2 - 2x_L t + (r_L - r_L^2 - x_L^2) = 0 \quad \cdots(3)$$

▸ If $r_L = 1$ $\quad t = \tan\beta d = \tan\frac{2\pi d}{\lambda} = -\frac{x_L}{2} \quad \rightarrow \quad \boxed{\frac{d}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{x_L}{2}\right)}$

▸ If $r_L \neq 1$ $\quad t = \frac{x_L \pm \sqrt{x_L^2 - (r_L - 1)(r_L - r_L^2 - x_L^2)}}{r_L - 1} = \tan\beta d = \tan\frac{2\pi d}{\lambda}$

depending on x_L and r_L → t: ***either negative or positive***

$$\boxed{\frac{d}{\lambda} = \frac{1}{2\pi}\tan^{-1} t \quad (t \geq 0), \quad \frac{d}{\lambda} = \frac{1}{2\pi}(\pi + \tan^{-1} t) \quad (t < 0)}$$

- Now, let's obtain ***l***. input impedance of stub at B-B' given as:

$$Z_{s,B-B'} = R_0 \frac{Z_L + jR_0 \tan\beta l}{R_0 + jZ_L \tan\beta l} = jR_0 \tan\beta l \quad \rightarrow \quad z_{s,B-B'} = \frac{Z_{s,B-B'}}{R_0} = j\tan\beta l$$

- Admittance then given as:

$$y_s = \frac{1}{z_{s,B-B'}} = \frac{1}{j\tan\beta l} = -jb_B \quad \rightarrow \quad \tan\beta l = \tan\frac{2\pi l}{\lambda} = \frac{1}{b_B}$$

(∵ *Condition 2*)

Length solution

$$\boxed{\frac{l}{\lambda} = \frac{1}{2\pi}\tan^{-1}\frac{1}{b_B} \quad (b_B \geq 0), \quad \frac{l}{\lambda} = \frac{1}{2\pi}\left(\pi + \tan^{-1}\frac{1}{b_B}\right) \quad (b_B < 0)}$$

Chap. 9 | Impedance matching *via* Double-stub matching

- **Problem of single-stub matching**
  - Frequency-dependence of location of the stub, d = Cλ (i.e. distance from load)
  - As frequency of signal varies, location of the stub should change! → Practically hard from mechanical point of view

- **Double-stub matching**
  - Two short-circuited stubs ***attached at fixed locations*** and apart by d_0 (arbitrarily chosen)
  - Only need to adjust their lengths l_A and l_B for matching with Z_L

- Matching condition:

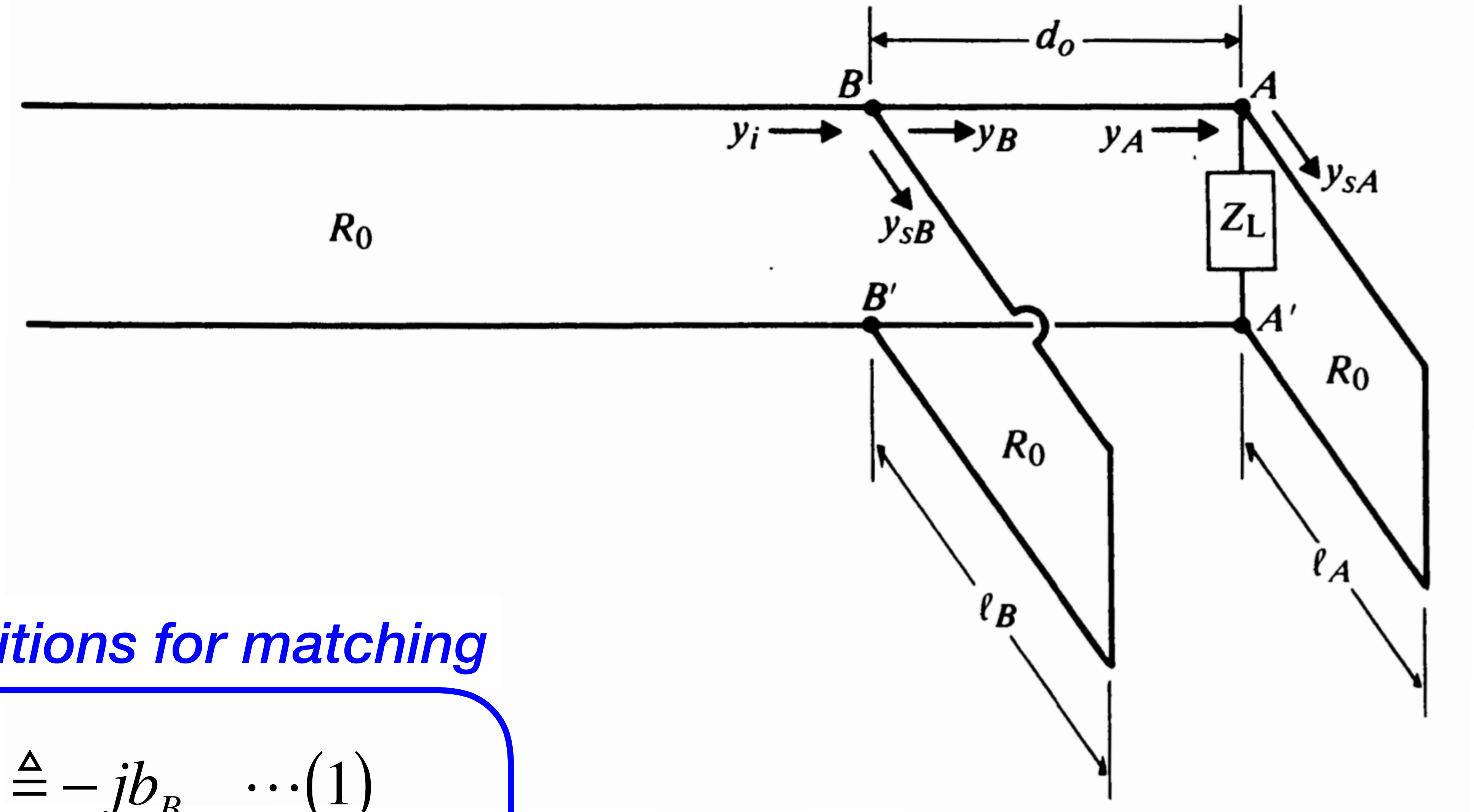
$$[Y_i = Y_B + Y_s] = Y_0$$

Y_i: Total input admittance at *B-B'*

Y_B: admittance of load section at B-B'

Y_s: admittance of short-circuited stub at B-B'

Y_0: Characteristic admittance of main TR-line ($1/R_0$)



- Normalized admittance:

$$\frac{Y_i}{Y_0} = 1 = \frac{Y_B}{Y_0} + \frac{Y_s}{Y_0} \triangleq y_B + y_s \quad \text{where} \quad y_s = -jb_B \ \textbf{\textit{(Why?)}}$$

Thus,

$$y_B = 1 - y_s = 1 + jb_B$$

⇨

Conditions for matching

$$\begin{cases} y_s \triangleq -jb_B & \cdots(1) \\ y_B = 1 + jb_B & \cdots(2) \end{cases}$$

∴ *Conditions for double-stub matching same as those for single-stub matching!*

Chap. 9 | Example: double-stub matching (1/2)

- **Characteristic impedance of TR-line R_0 = 50 (Ω) and terminated by Z_L = 60 + j80 (Ω). $d_0 = \lambda/8$. Find l_A and l_B.**

- First, locate $y_L = g_L + jb_L$ [$= 1/z_L = R_0/Z_L =$ **0.3** $- j0.4$] on Admittance chart.

(1) Draw **(g = 1)-circle** for $y_B = 1 + jb_B$ (admittance of load section at B-B').

(2) Rotate **(g = 1)-circle** by [$\lambda/d_0 = 1/8 = 0.125$] in ***CCW direction (toward load)***.

$$\frac{d_0}{\lambda} = 0.125 \quad \rightarrow \quad 4\pi\frac{d_0}{\lambda} = \frac{\pi}{2} \quad (\text{rad})$$

(3) **"Rotated" circle** representing total admittance at A-A', y_A such that

$$y_A = y_{sA} + y_L = (-jb_{sA}) + (g_L + jb_L) = g_L - j(b_{sA} + b_L) = 0.3 - j(b_{sA} - 0.4)$$

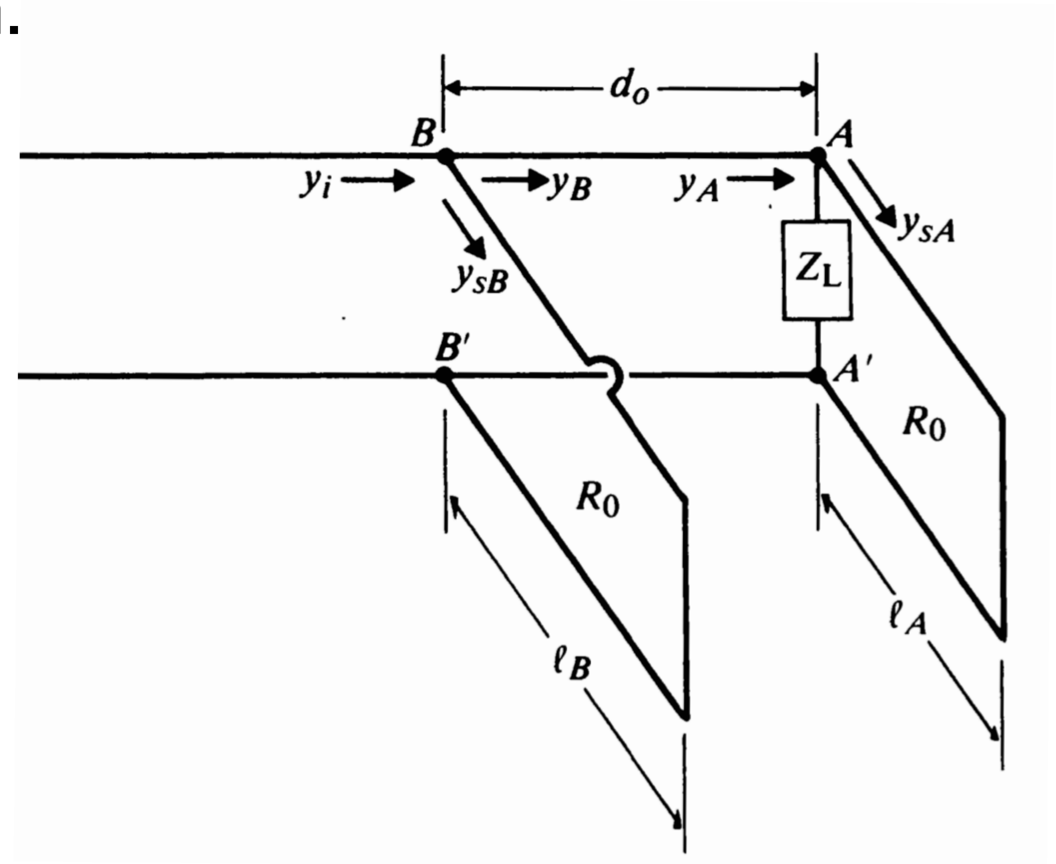
($\because$ short-circuit)

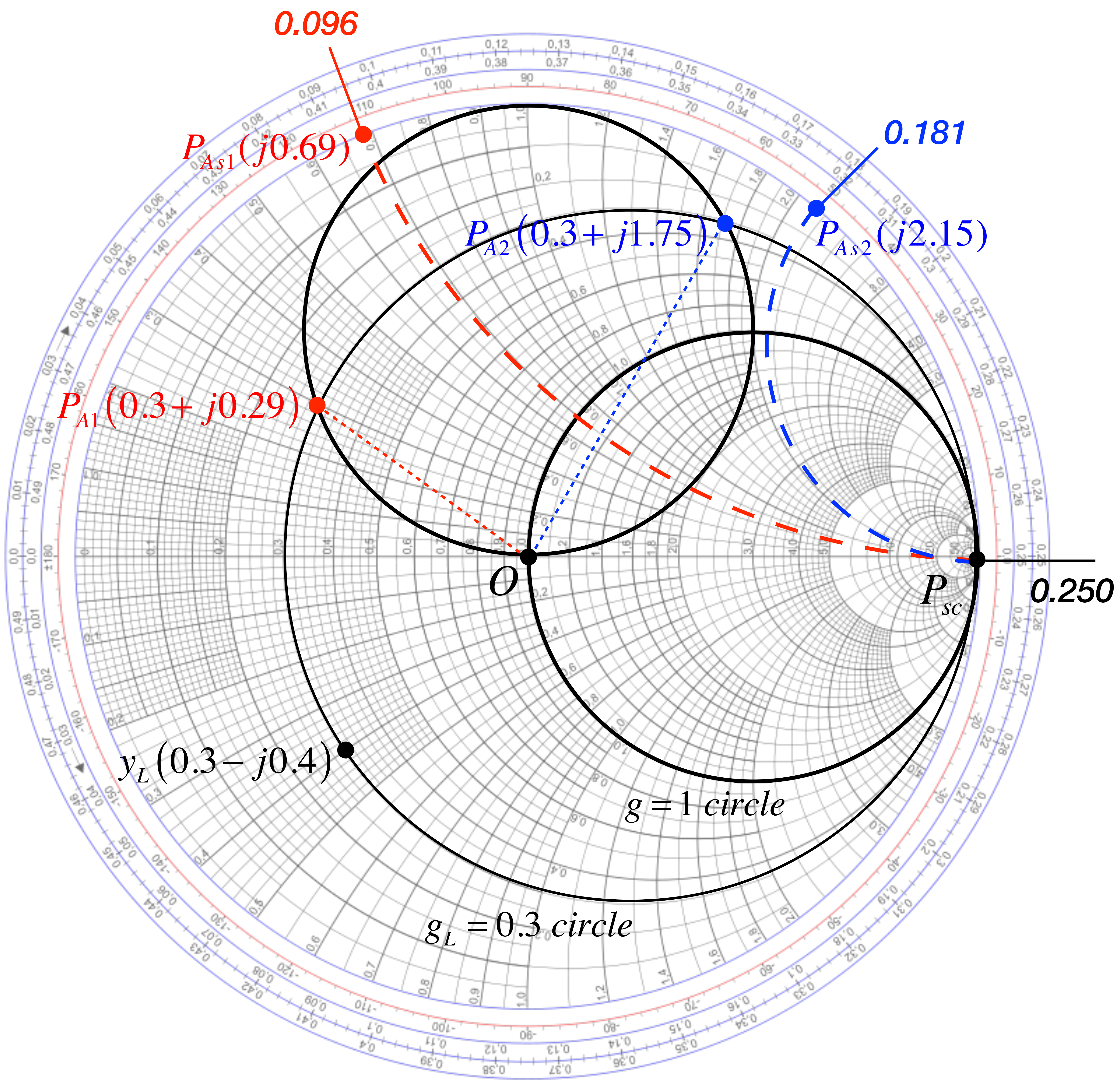
(4) Thus, **intersections [between "rotated" circle and $g_L = 0.3$ circle]** are two solutions with $y_{A1} = 0.3 + j0.29$ and $y_{A2} = 0.3 + j1.75$ (P_{A1} and P_{A2}).

(5) Since $y_{sA} = y_L - y_A$, we get $y_{sA1} = j0.69$, $y_{sA2} = j2.15$ (P_{As1} and P_{As2}).

(6) Determine length l_{A1} and l_{A2} from **angles between [OP_{sc} and OP_{As1}]** and **between [OP_{sc} and OP_{As2}] in CW direction.**

$$\begin{cases} l_{A1} = (0.096 + 0.250)\lambda = 0.346\lambda \\ l_{A2} = (0.181 + 0.250)\lambda = 0.431\lambda \end{cases}$$





Example: double-stub matching (2/2)

- **Characteristic impedance of TR-line R_0 = 50 (Ω) and terminated by Z_L = 60 + j80 (Ω). d_0 = λ/8. Find l_A and l_B.**

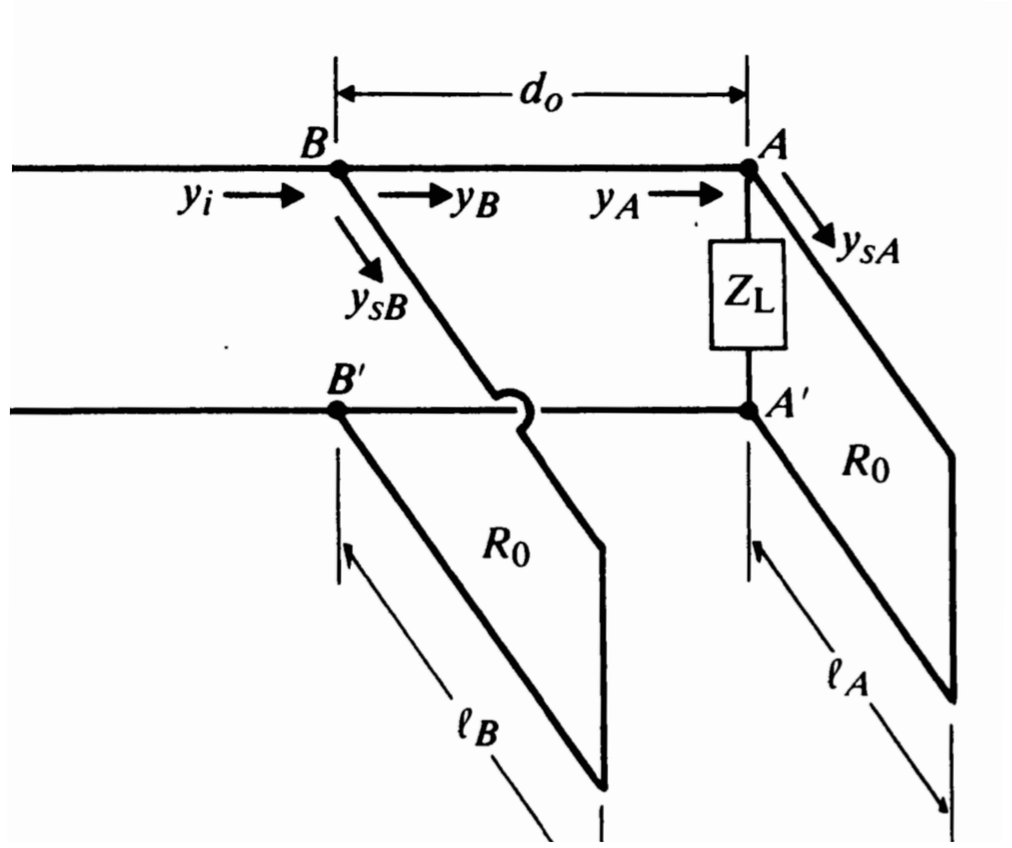
(7) Rotate **OP_{A1}** and **OP_{A2}** back in **CW direction** by d_0/λ (=0.125) and find corresponding points on (g = 1)-circle. These are solutions y_B (**P_{B1}** and **P_{B2}**)

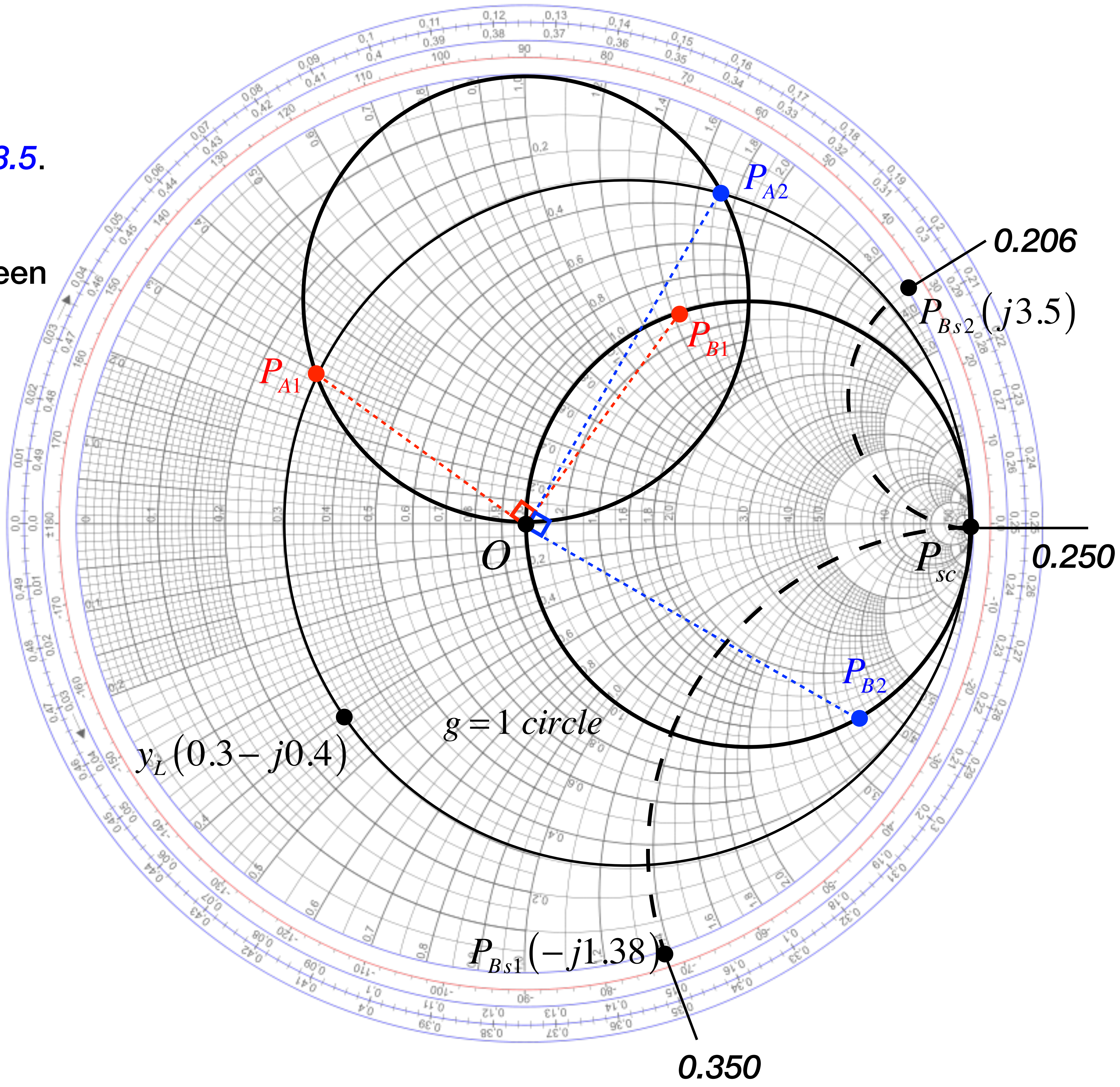
(8) Read points **P_{B1}** and **P_{B2}** yielding ***y_{B1} = 1 + j1.38*** and ***y_{B2} = 1 – j3.5***.

(9) Thus, ***y_{sB}*** should cancel imaginary part of ***y_B*** such that ***y_{sB1} = –j1.38*** and ***y_{sB2} = j3.5***. These are denoted as points **P_{Bs1}** and **P_{Bs2}** on chart.

(10) Determine lengths l_{B1} and l_{B2} from **angles between [OP_{sc} and OP_{Bs1}]** and **between [OP_{sc} and OP_{Bs2}]**.

$$\begin{cases} l_{B1} = (0.350 - 0.250)\lambda = 0.100\lambda \\ l_{B2} = (0.206 + 0.250)\lambda = 0.456\lambda \end{cases}$$





- **Special case**
  - If y_L lies within (g = 2)-circle, no solution exists! (No overlap with rotated circle)
  - In this case, solution given as left