Electromagnetics

<Chap. 8> Plane Electromagnetic waves

Section 8.9 ~ 8.10

(1st of **week 5**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 8 | Contents for 1st class of week 5

Review of the last class

Sec 9. Normal incidence at multiple dielectric interfaces

- Wave impedance
- Impedance transformation

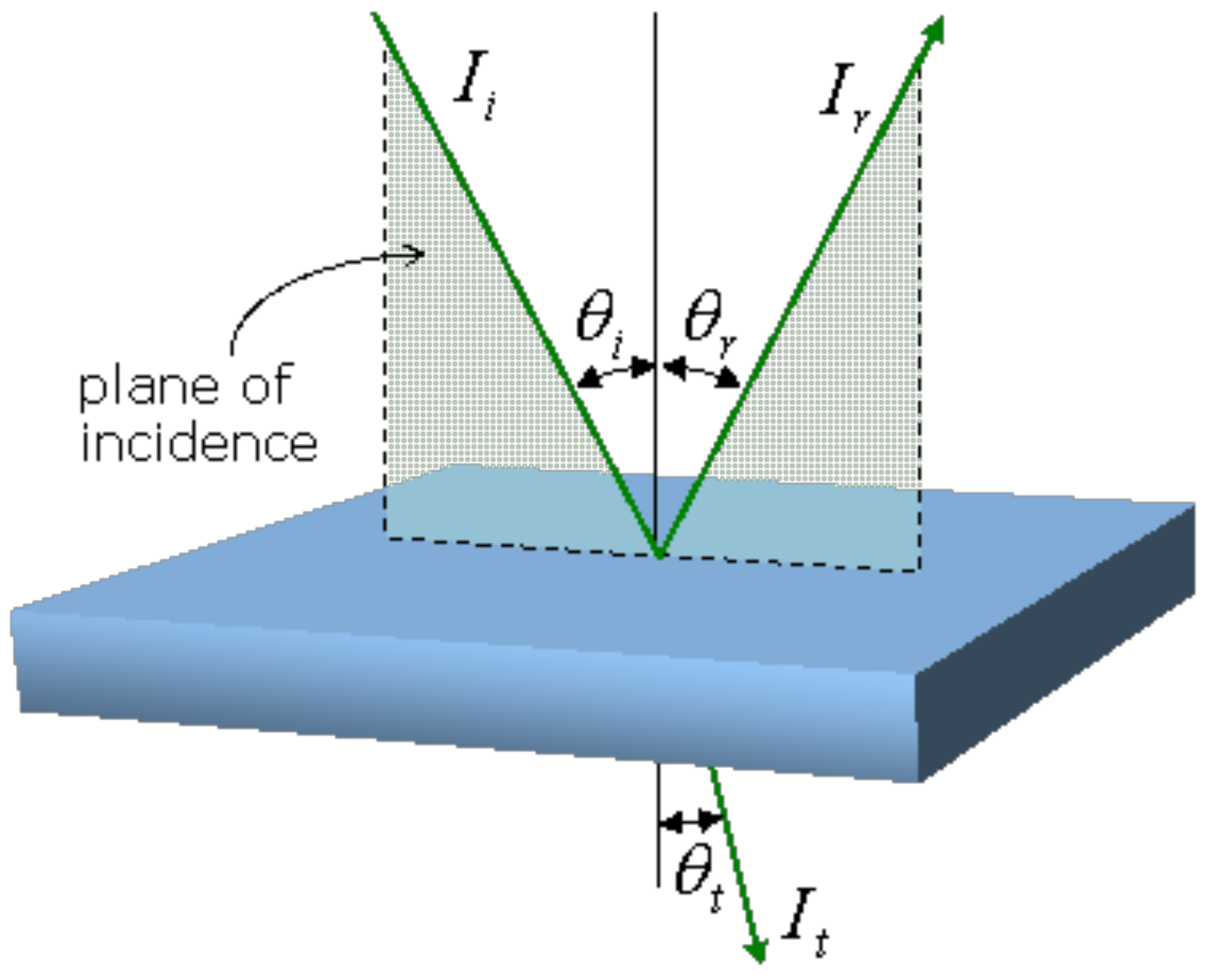
Chap. 8 | Review (Terminologies)

Definition

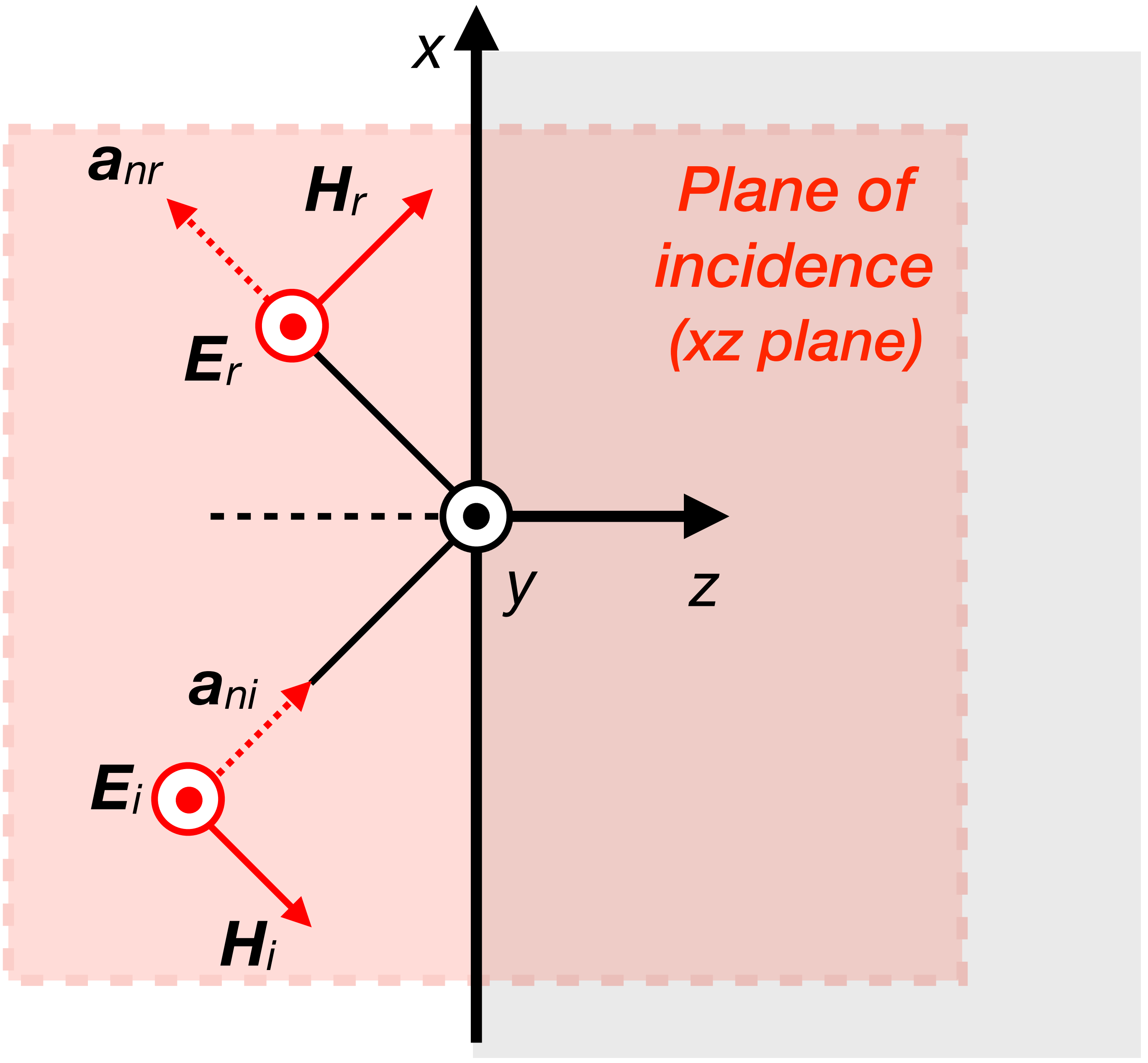
- ***"Polarization" direction of EM wave = Direction of E-field***
- ***Plane of incidence***
  - Plane that is *perpendicular to the surface*
  - Plane that *contains a vector of propagation direction*
- ***Transverse Electric (TE) wave***
  - ***E*-field ⊥ Plane of incidence**
- ***Transverse Magnetic (TM) wave***
  - ***H*-field ⊥ Plane of incidence**

Chapter. 9
TEM wave propagation
(In transmission lines)

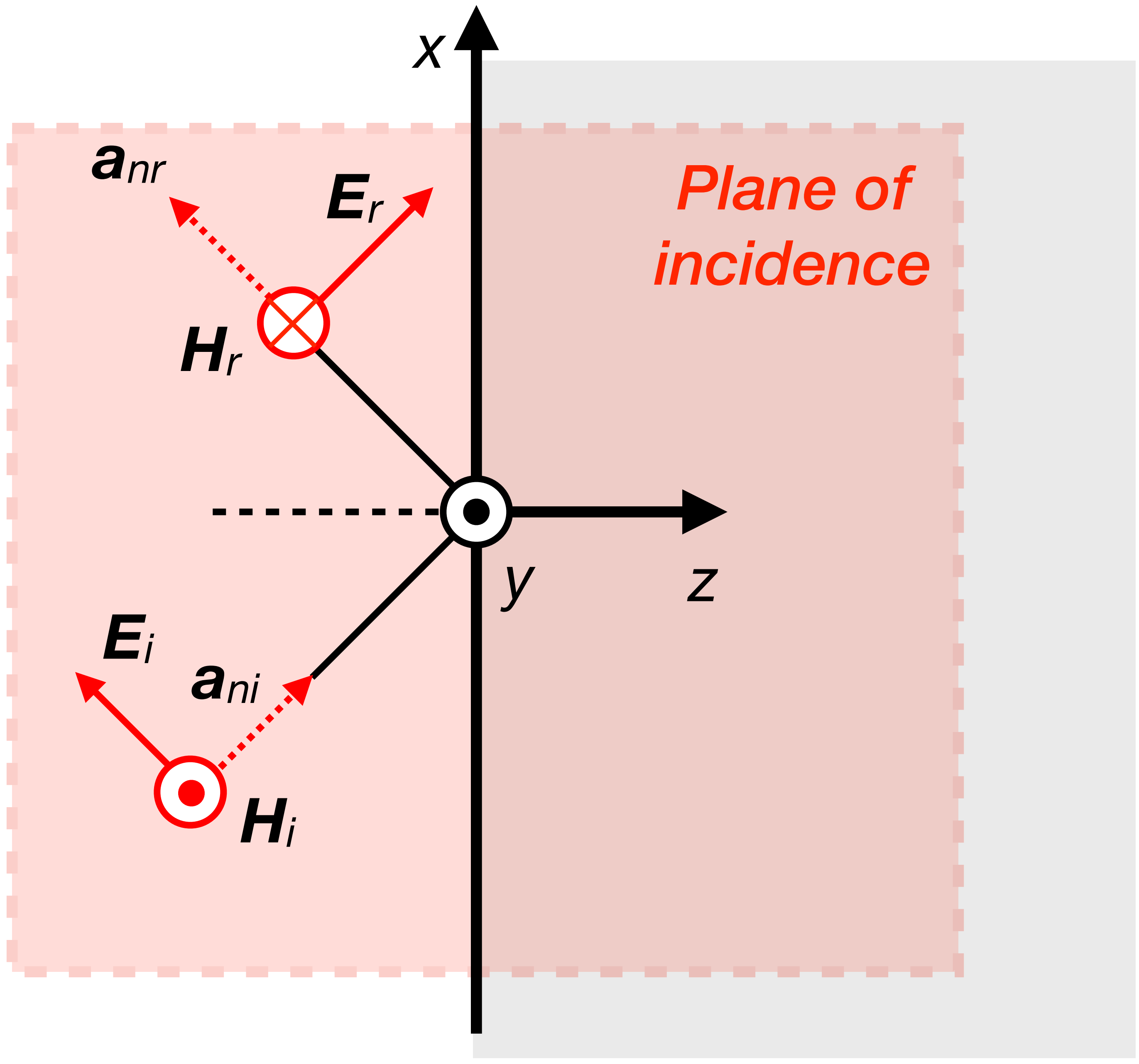
Chapter. 10
TE & TM wave propagation
(In waveguides)



Transverse Electric (TE) wave
(E ⊥ plane of incidence)



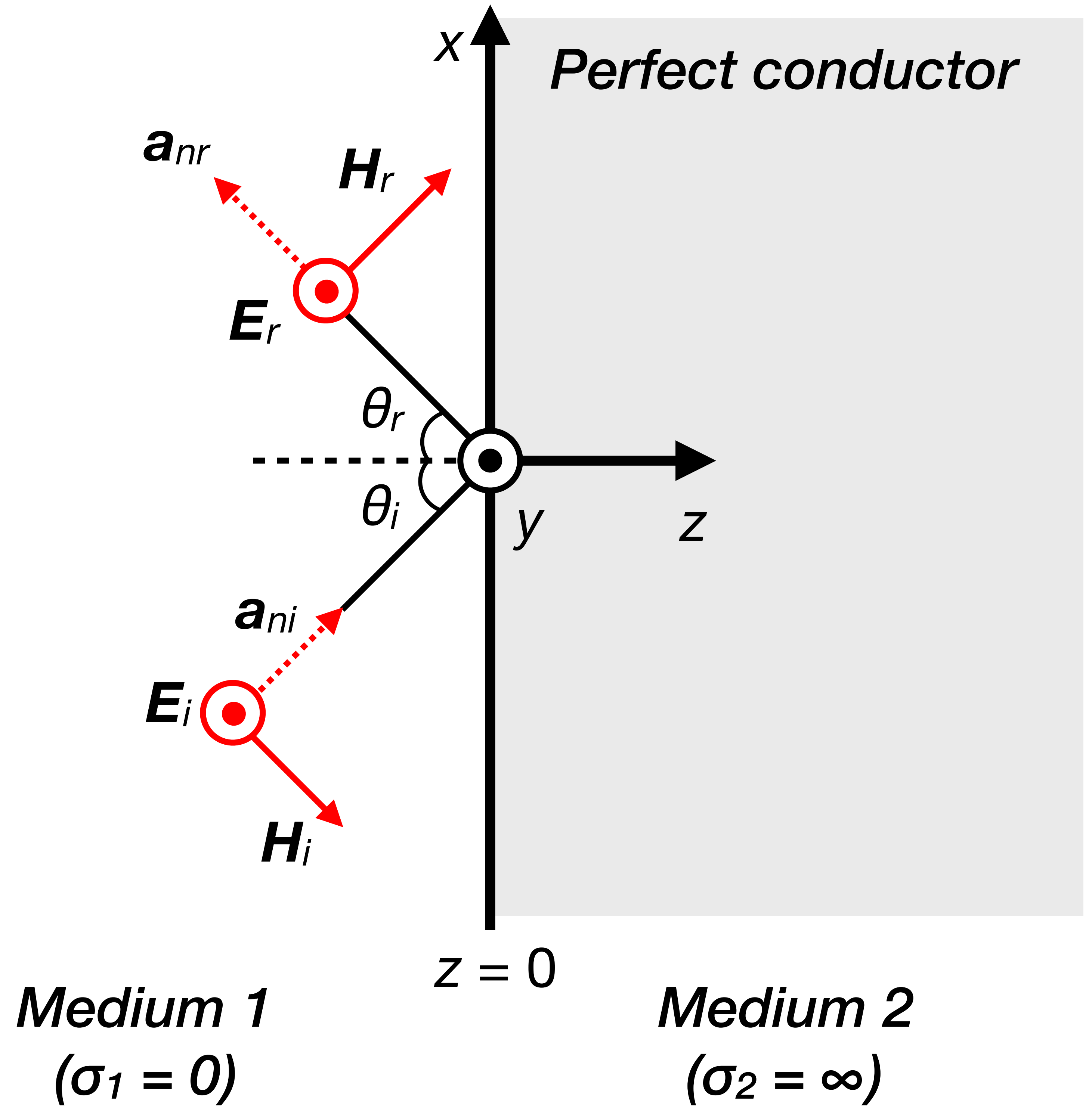
Transverse Magnetic (TM) wave
(H ⊥ plane of incidence)



(Image source: http://www.uta.edu/optics/research/ellipsometry/ellipsometry.htm)

Review (TE wave)

Transverse Electric (TE) wave



- ***Boundary Condition***

→ [total ***E-field*** in Medium 1 at *z* = 0] = [total ***E-field*** in Medium 2 at *z* = 0]

$$\left[\mathbf{E_1}(x,0)=\mathbf{E_i}(x,0)+\mathbf{E_r}(x,0)\right]=\left[\mathbf{E_2}(x,0)=0\right]$$

$$\rightarrow \mathbf{a_y}\left(E_{i0}e^{-j\beta_1 x\sin\theta_i}+E_{r0}e^{-j\beta_1 x\sin\theta_r}\right)=0$$

- ***Snell's law of reflection***

$$\therefore E_{r0}=-E_{i0},\quad \theta_r=\theta_i$$

Snell's law of reflection

Phase is shifted by 180º

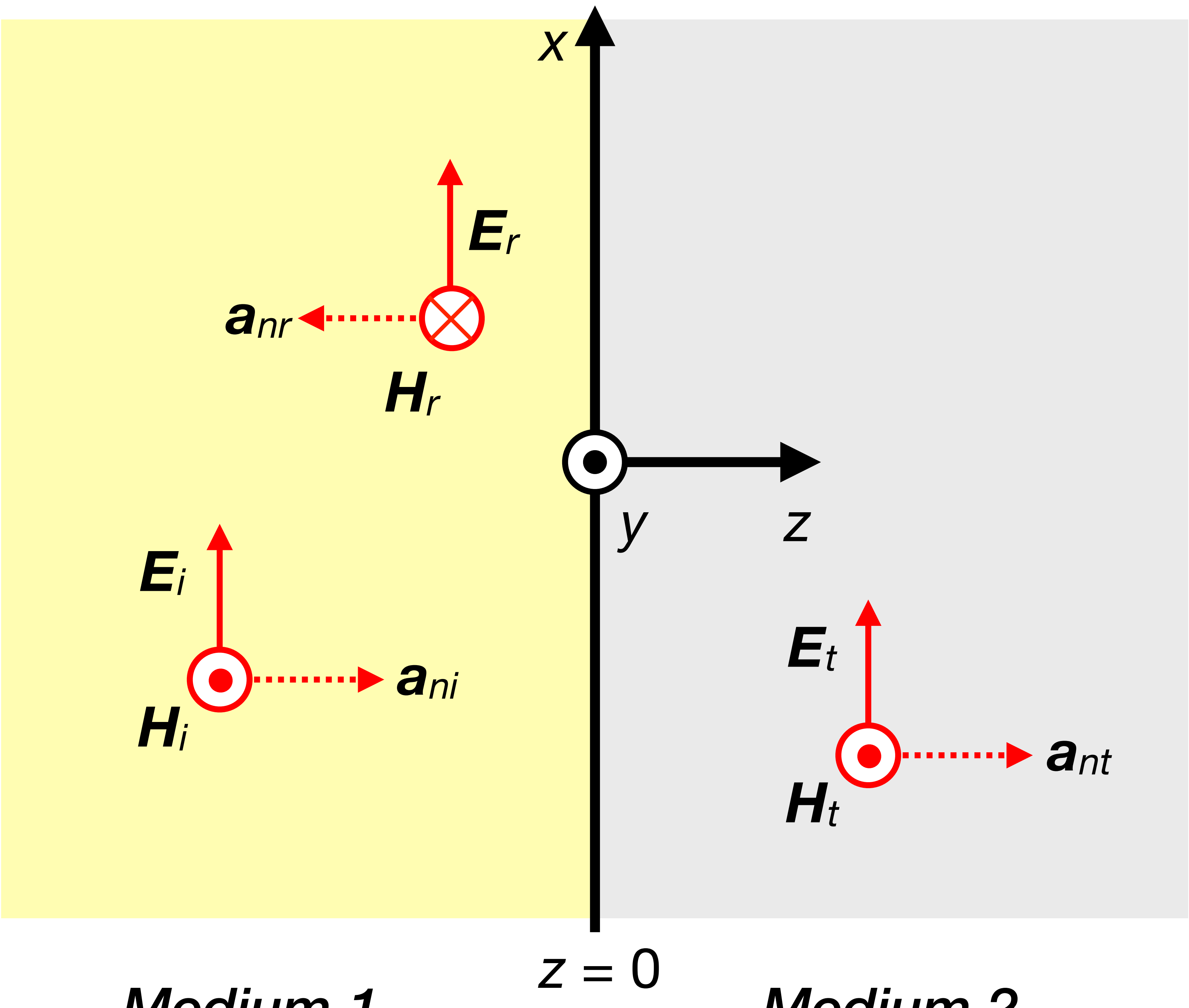
- ***Microscopic interpretation of "metallic reflection"***
  - Boundary condition for *H*-field at z = 0

$$\mathbf{a_{n2}}\times(\mathbf{H_1}-\mathbf{H_2})=\mathbf{J_s} \longrightarrow \mathbf{J_s}=\mathbf{a_y}2\frac{E_{i0}}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i}$$

  - *induced J_s* results in the *reflected wave in medium 1*
  - Microscopically, incident wave absorbed by free electrons
  - Oscillating electrons (J_s) re-radiate EM wave
  - Reflected wave ***cancels the incident wave*** in the wall

Chap. 8 | Review (Reflection and Transmission)

Two different dielectric media



Medium 1
(ε_1, μ_1)

Medium 2
(ε_2, μ_2)

- ***E_{r0} and E_{t0} in terms of E_{i0}***

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0} \rightarrow \Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$ *Reflection Coefficient*

$$E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0} \rightarrow \tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$ *Transmission Coefficient*

- ***Relationship between Γ and τ***

$$\therefore 1 + \Gamma = \tau$$

- ***Complex Γ and τ*** (i.e. complex η_1 and η_2)
  → ***phase shift*** introduced upon transmission and reflection
- If ***medium 2 is a perfect conductor*** (i.e. $\eta_2 = 0$)
  → $\Gamma = -1$ → $E_{r0} = -E_{i0}$
  → $\tau = 0$ → $E_{t0} = 0$

- ***EM energy***

$$\left[\mathbf{P}_{av1} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_1 \times \mathbf{H}_1^*\right) = \mathbf{a}_z \frac{E_{i0}^2}{2\eta_1}\left(1 - \Gamma^2\right) \right] = \left[\mathbf{P}_{av2} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_2 \times \mathbf{H}_2^*\right) = \mathbf{a}_z \frac{E_{i0}^2}{2\eta_2}\tau^2 \right]$$ (If both media are lossless)

$$\mathbf{P}_{av1} = \mathbf{P}_{av2} \rightarrow \therefore 1 - \Gamma^2 = \frac{\eta_1}{\eta_2}\tau^2$$

Chap. 8 | Examples of *Multiple Dielectric layers*

Practical use of "several layers of dielectric media"

(Image source: All about vision)

"Anti-reflective (AR)" coating on the lens

minimizes the ambient reflection

(Image source: Wikipedia)

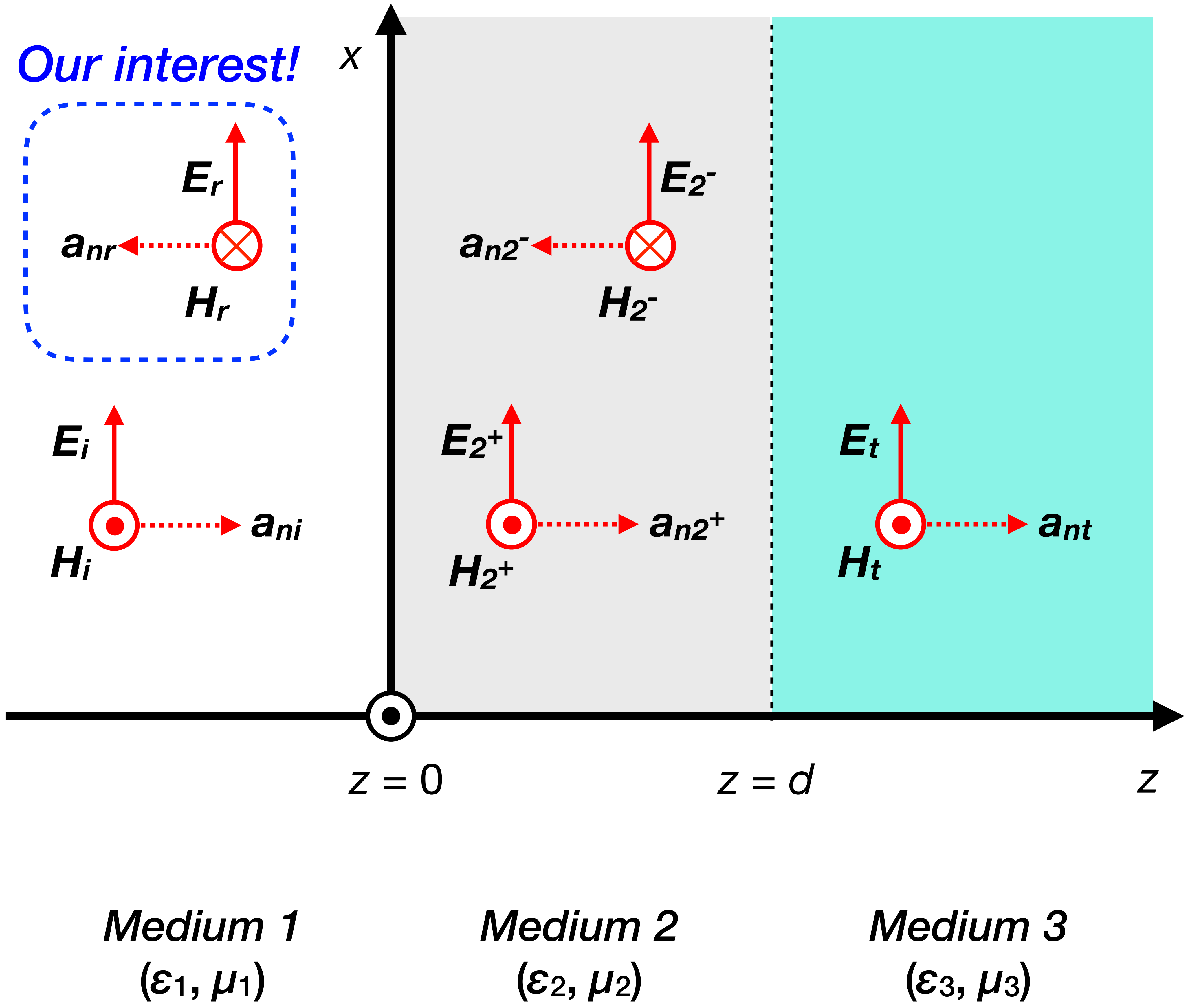
Radome (Radar + dome)

permits EM waves through enclosure with little reflection

Determining proper dielectric materials (μ, ε) and their thicknesses
→
Design & engineering problems

Total reflection in a stack of three dielectric media (1/2)

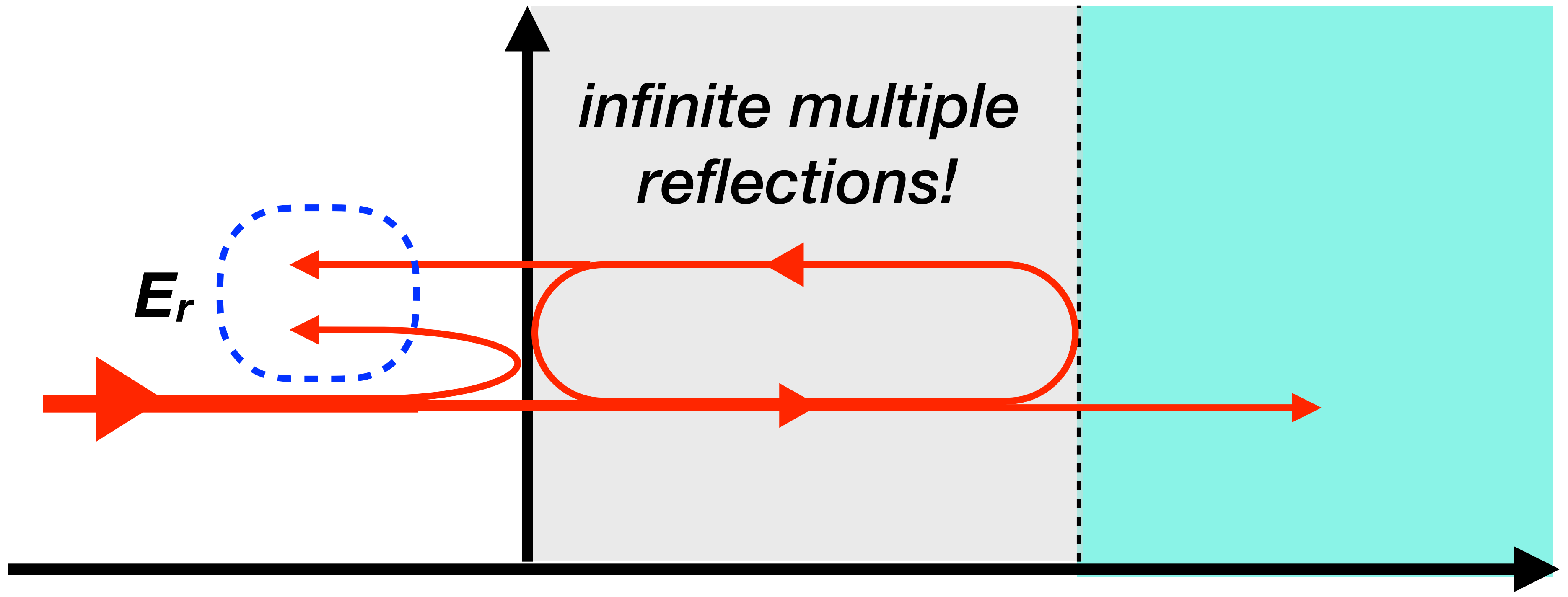
Total reflection in a stack of three dielectric media



- How do we determine E_{r0}?

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0}$$ For a *"two layer"* situation (*Last class or see Sec. 8-8*)

$$E_{r0} \neq \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0}$$ For a *"three layer"* situation ($\because$ *multiple reflections* in medium 2 at z = 0 and d)



- One option to obtain E_{r0}

→ By using *boundary conditions!*

Total reflection in a stack of three dielectric media (2/2)

- *Total electric and magnetic fields* in each medium

Medium 1

$$\begin{cases} \boldsymbol{E_1} = \boldsymbol{E_i} + \boldsymbol{E_r} = \boldsymbol{a_x}\left(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{j\beta_1 z}\right) \\ \boldsymbol{H_1} = \boldsymbol{H_i} + \boldsymbol{H_r} = \boldsymbol{a_y}\dfrac{1}{\eta_1}\left(E_{i0}e^{-j\beta_1 z} - E_{r0}e^{j\beta_1 z}\right) \end{cases}$$

$$\because \boldsymbol{H} = \frac{\boldsymbol{a_n} \times \boldsymbol{E}}{\eta}$$

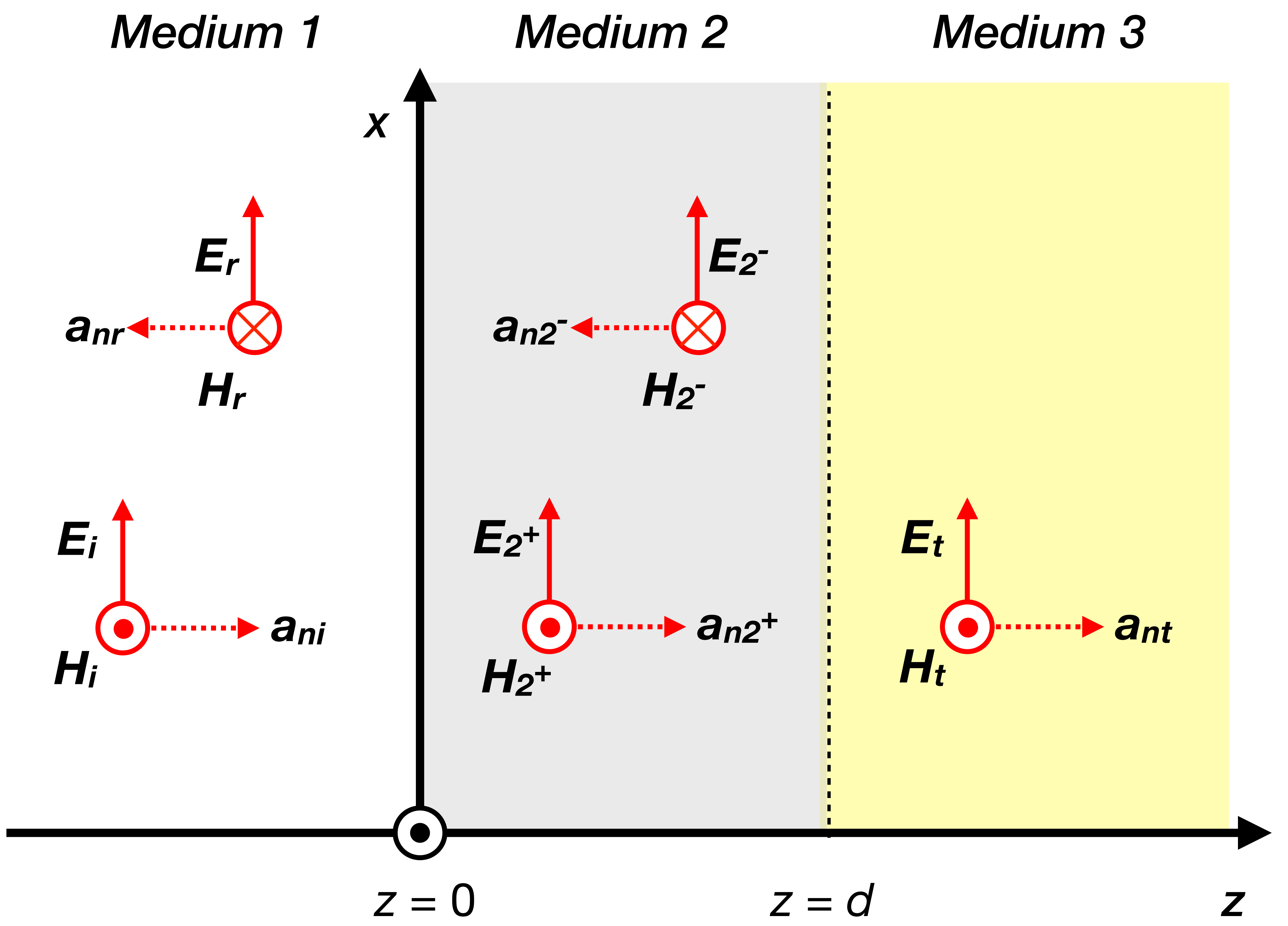
Medium 2

$$\begin{cases} \boldsymbol{E_2} = \boldsymbol{E_2}^+ + \boldsymbol{E_2}^- = \boldsymbol{a_x}\left(E_2^+ e^{-j\beta_2 z} + E_2^- e^{j\beta_2 z}\right) \\ \boldsymbol{H_2} = \boldsymbol{H_2}^+ + \boldsymbol{H_2}^- = \boldsymbol{a_y}\dfrac{1}{\eta_1}\left(E_2^+ e^{-j\beta_2 z} - E_2^- e^{j\beta_2 z}\right) \end{cases}$$

Medium 3

$$\begin{cases} \boldsymbol{E_3} = \boldsymbol{a_x} E_3 e^{-j\beta_3 z} \\ \boldsymbol{H_3} = \boldsymbol{a_y}\dfrac{E_3}{\eta_3} e^{-j\beta_3 z} \end{cases}$$

4 unknowns (E_{r0}, E_2^+, E_2^-, E_3) ⇨ *4 B.C. equations*



$$\text{At } z = 0: \begin{cases} \boldsymbol{E_1}(0) = \boldsymbol{E_2}(0) \\ \boldsymbol{H_1}(0) = \boldsymbol{H_2}(0) \end{cases}$$

$$\text{At } z = d: \begin{cases} \boldsymbol{E_2}(d) = \boldsymbol{E_3}(d) \\ \boldsymbol{H_2}(d) = \boldsymbol{H_3}(d) \end{cases}$$

Chap. 8 | Wave impedance (1/2)

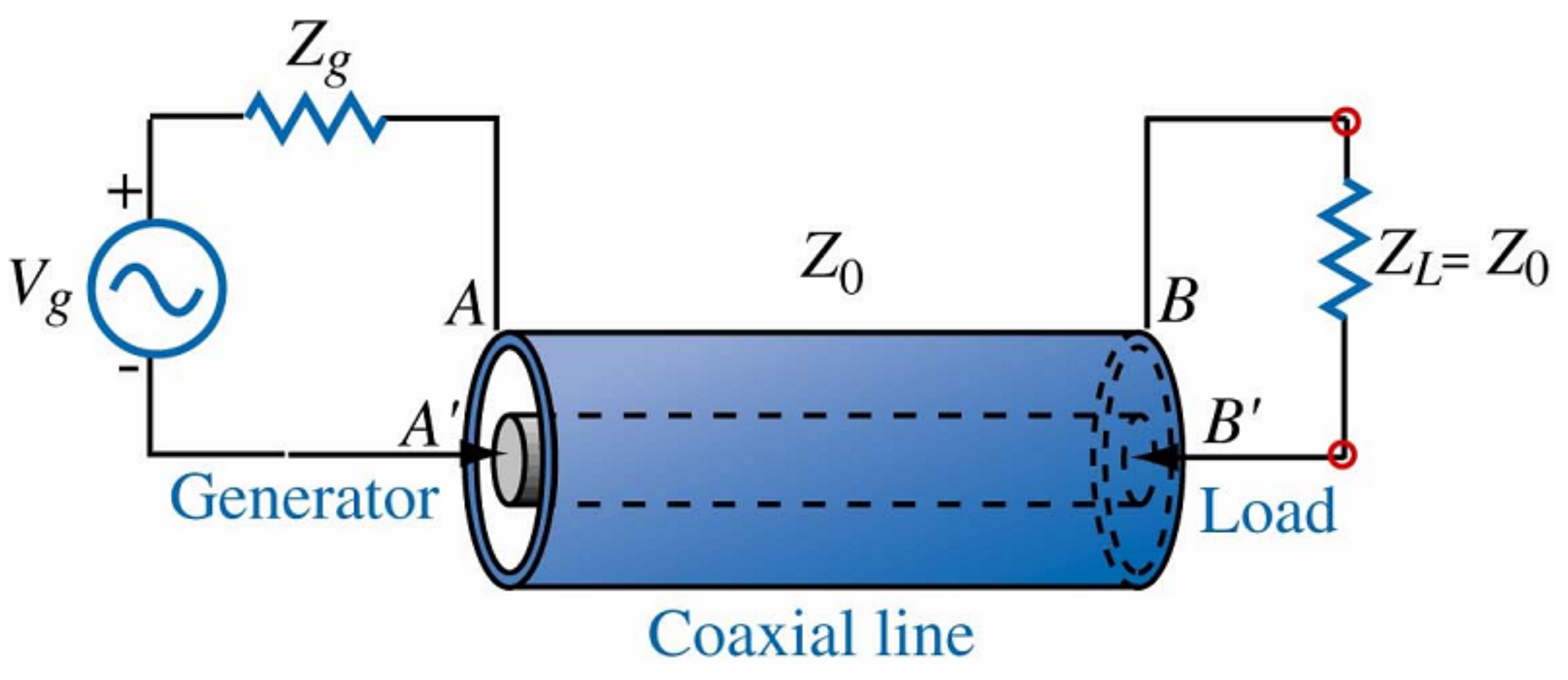
Wave impedance of the total field

: Ratio of the total electric field intensity to the total magnetic field intensity

$$Z(z) \triangleq \frac{\text{Total } E_x(z)}{\text{Total } H_y(z)} \quad (\Omega)$$

Why do we care about the impedance?

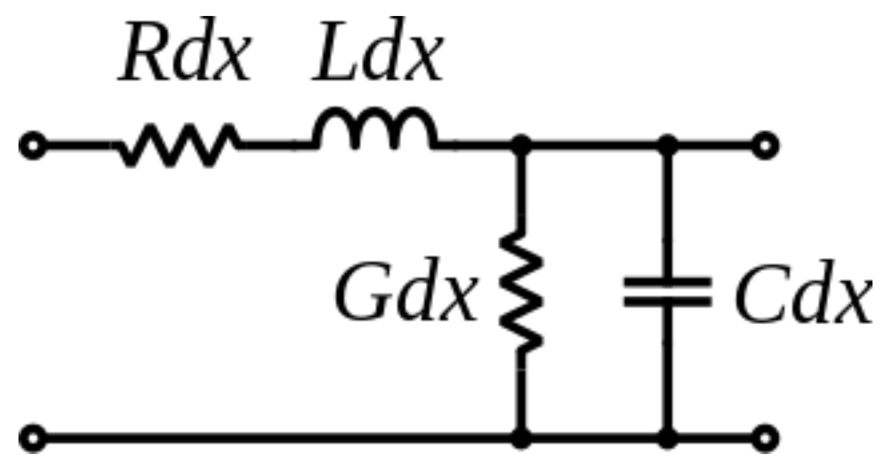
- *In Chapter 9 <Theory and Applications of Transmission Lines>*



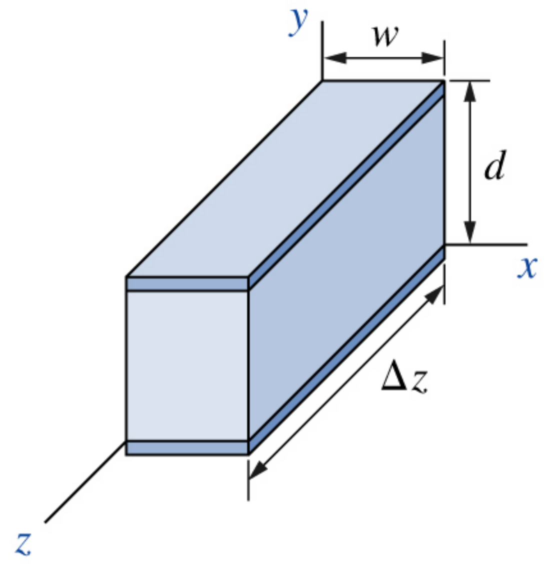
Transmission Line:

A passage with a certain geometry and components (*μ*, ε, σ) where EM wave travels through

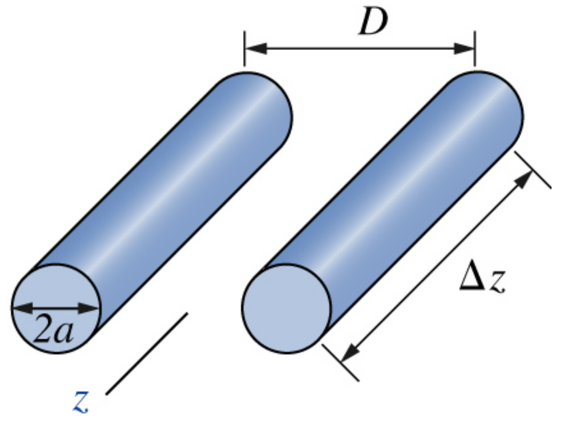
Transmission line can be modeled as *circuit elements*



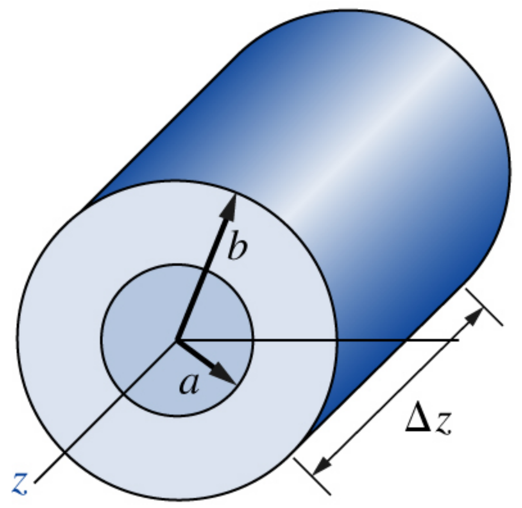
"Transmission line lumped element equivalent circuit"



Parallel plate



Two-wire

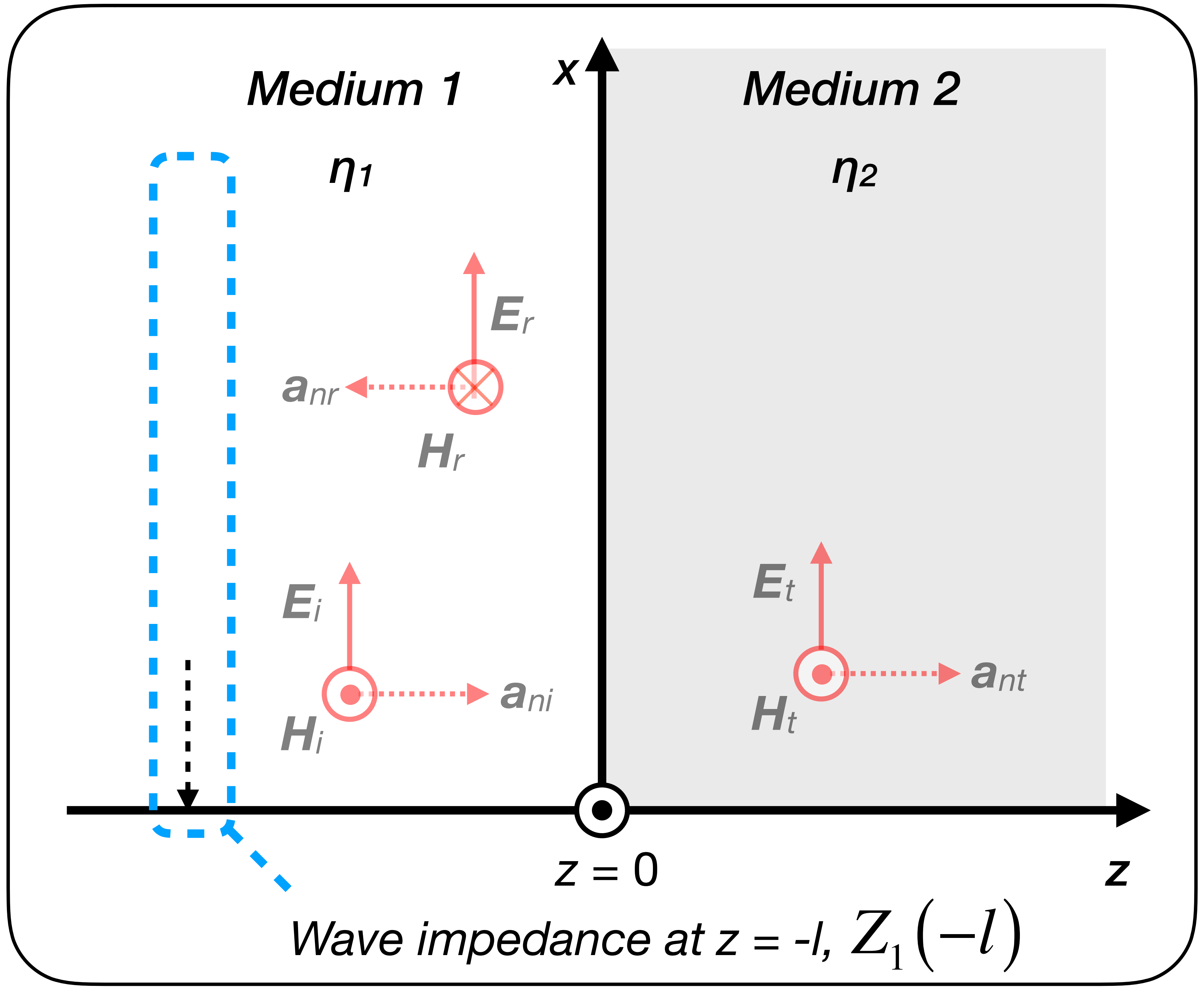


Coaxial

Chap. 8 | Wave impedance (2/2)

Wave impedance vs. intrinsic (characteristic) impedance

- ***Intrinsic (characteristic) impedance (η or Z_0)***
  - Used for a single wave propagating *in an "unbounded" medium*
  - *i.e. there is no reflected wave*
  - equivalent to *infinitely long transmission line*
- ***Wave impedance (Z)***
  - Used for waves propagating *across many different media where reflection occurs*



Example: Two dielectrics situation

$\because \dfrac{E_{r0}}{E_{i0}} = \Gamma$

$$\begin{cases} \boldsymbol{E_1}(z) = \boldsymbol{a_x} E_{1x}(z) = \boldsymbol{a_x}\left(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{j\beta_1 z}\right) = \boldsymbol{a_x} E_{i0}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right) \\ \boldsymbol{H_1}(z) = \boldsymbol{a_y} H_{1y}(z) = \boldsymbol{a_y}\left(\dfrac{E_{i0}}{\eta_1}e^{-j\beta_1 z} - \dfrac{E_{r0}}{\eta_1}e^{j\beta_1 z}\right) = \boldsymbol{a_y}\dfrac{E_{i0}}{\eta_1}\left(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}\right) \end{cases}$$

$\because \Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

- *Wave impedance*

$$Z_1(z) = \frac{E_{1x}(z)}{H_{1y}(z)} = \eta_1 \frac{e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}}{e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}}$$

- *Wave impedance at z = -l*

$$Z_1(-l) = \eta_1 \frac{e^{j\beta_1 l} + \Gamma e^{-j\beta_1 l}}{e^{j\beta_1 l} - \Gamma e^{-j\beta_1 l}} = \eta_1 \frac{\eta_2 \cos\beta_1 l + j\eta_1 \sin\beta_1 l}{\eta_1 \cos\beta_1 l + j\eta_2 \sin\beta_1 l}$$

- *If $\eta_2 = \eta_1$*

$$Z_1(-l) = \eta_1$$

∵ No reflected wave involved

→ Wave impedance of total field = intrinsic impedance

Chap. 8 | Impedance transformation (1/2)

Wave impedance in medium 2

- ***Total field in medium 2***
  - Result of ***multiple reflections*** at z = 0 and $z = d$
  - Grouped into a wave traveling in +z and other in -z directions
- ***Wave impedance in medium 2 at z = 0***

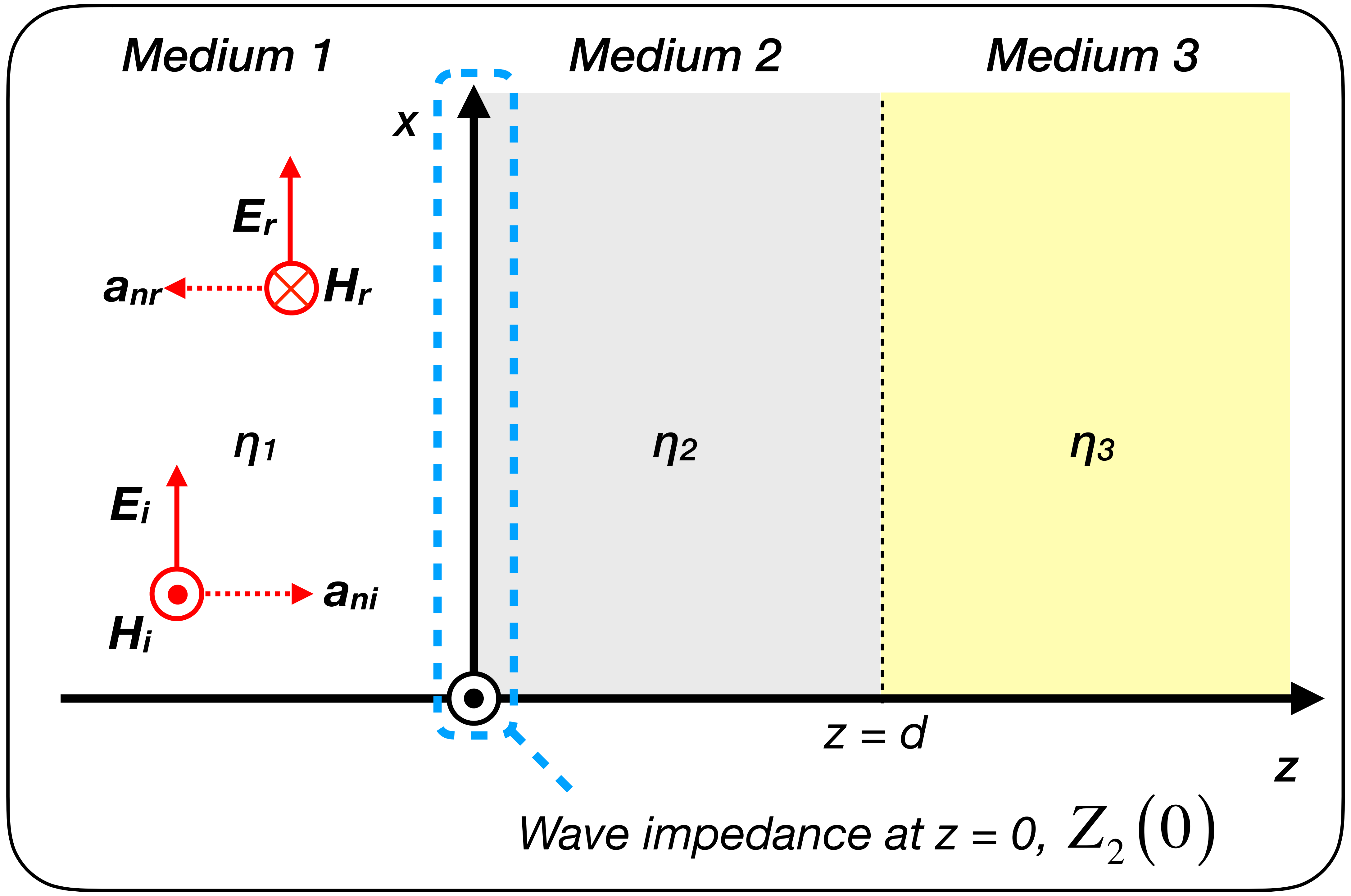
$$Z_2(0) = \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d}$$

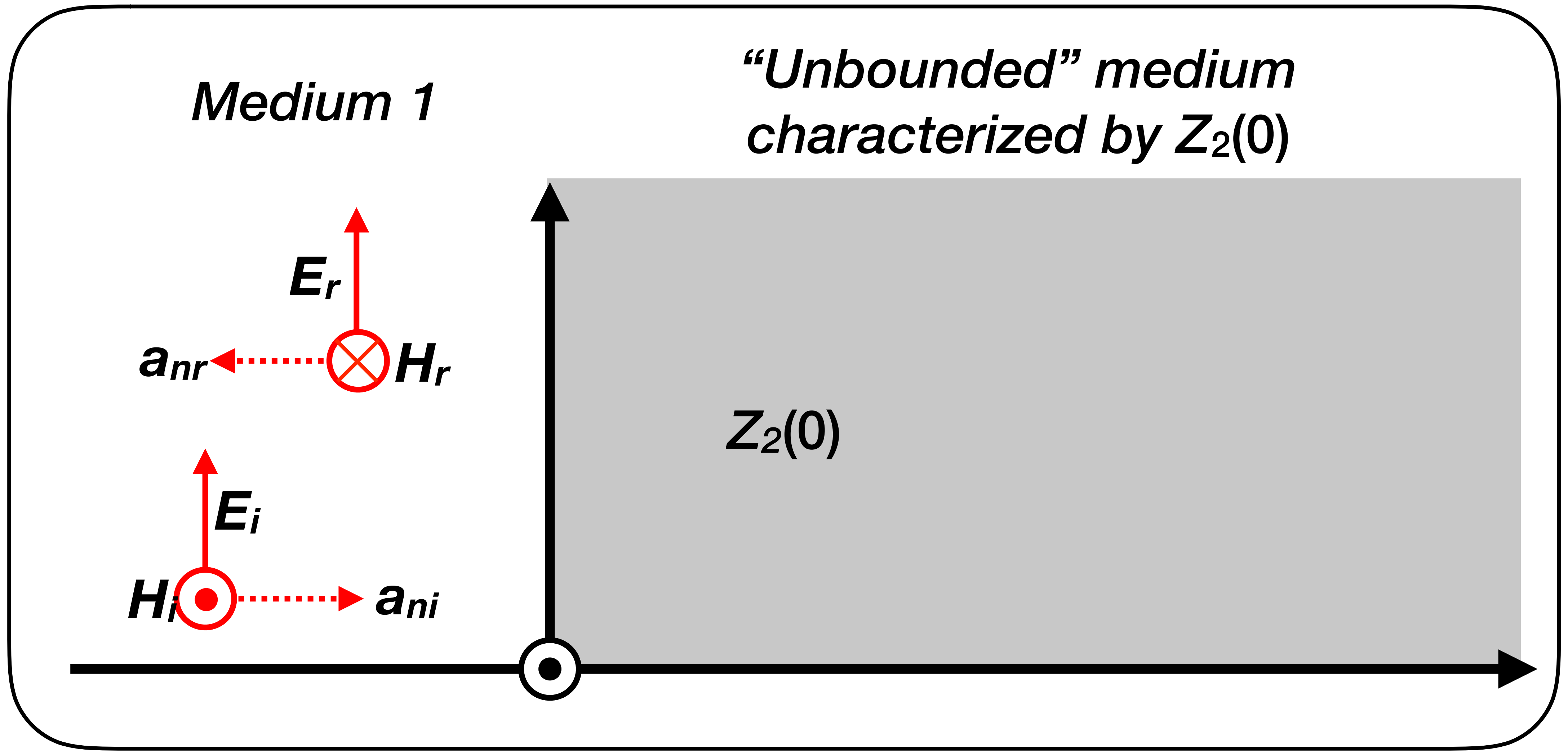
$$\eta_2 \to \eta_3,\ \eta_1 \to \eta_2$$
$$\beta_1 \to \beta_2,\ l \to d$$

$$Z_1(-l) = \eta_1 \frac{\eta_2 \cos\beta_1 l + j\eta_1 \sin\beta_1 l}{\eta_1 \cos\beta_1 l + j\eta_2 \sin\beta_1 l}$$

(Wave impedance in medium 1 for ***two dielectrics***)

- ***<u>Equivalent situation</u> for the wave traveling in medium 1:***
  - It encounters a discontinuity at z = 0
  - Discontinuity → ***Unbounded medium*** with ***an intrinsic impedance $Z_2(0)$***





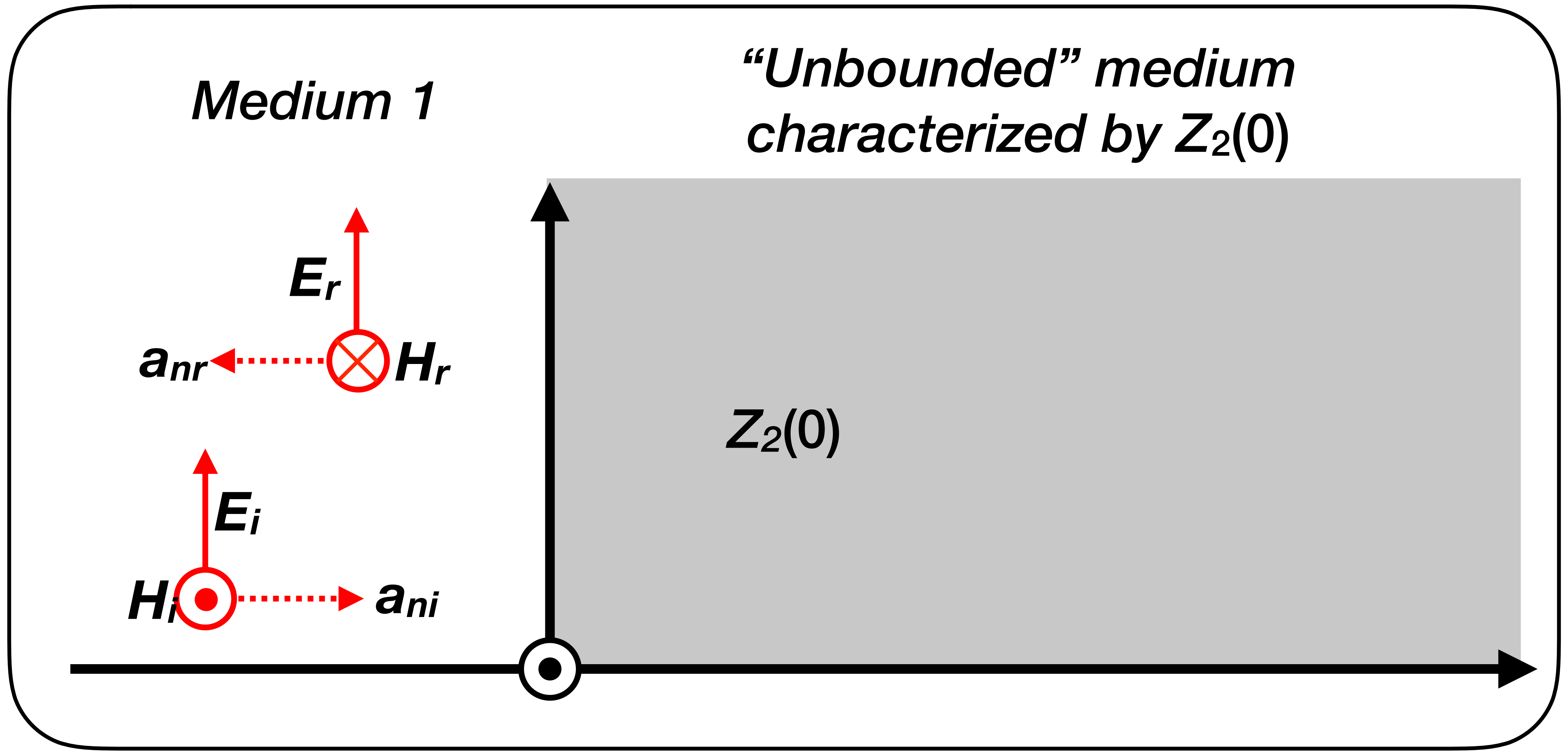
Chap. 8 | Impedance transformation (2/2)

"Effective" reflection coefficient at z = 0

$$\Gamma_0 = \frac{E_{r0}}{E_{i0}} = \frac{Z_2(0) - \eta_1}{Z_2(0) + \eta_1}$$

c.f.) $\Gamma = \frac{\eta_a - \eta_b}{\eta_a + \eta_b}$ for *two dielectrics a, b*

where $Z_2(0) = \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d}$



Impedance transformation

$$\Gamma = \frac{\eta_3 - \eta_1}{\eta_3 + \eta_1}$$

Without medium 2

→

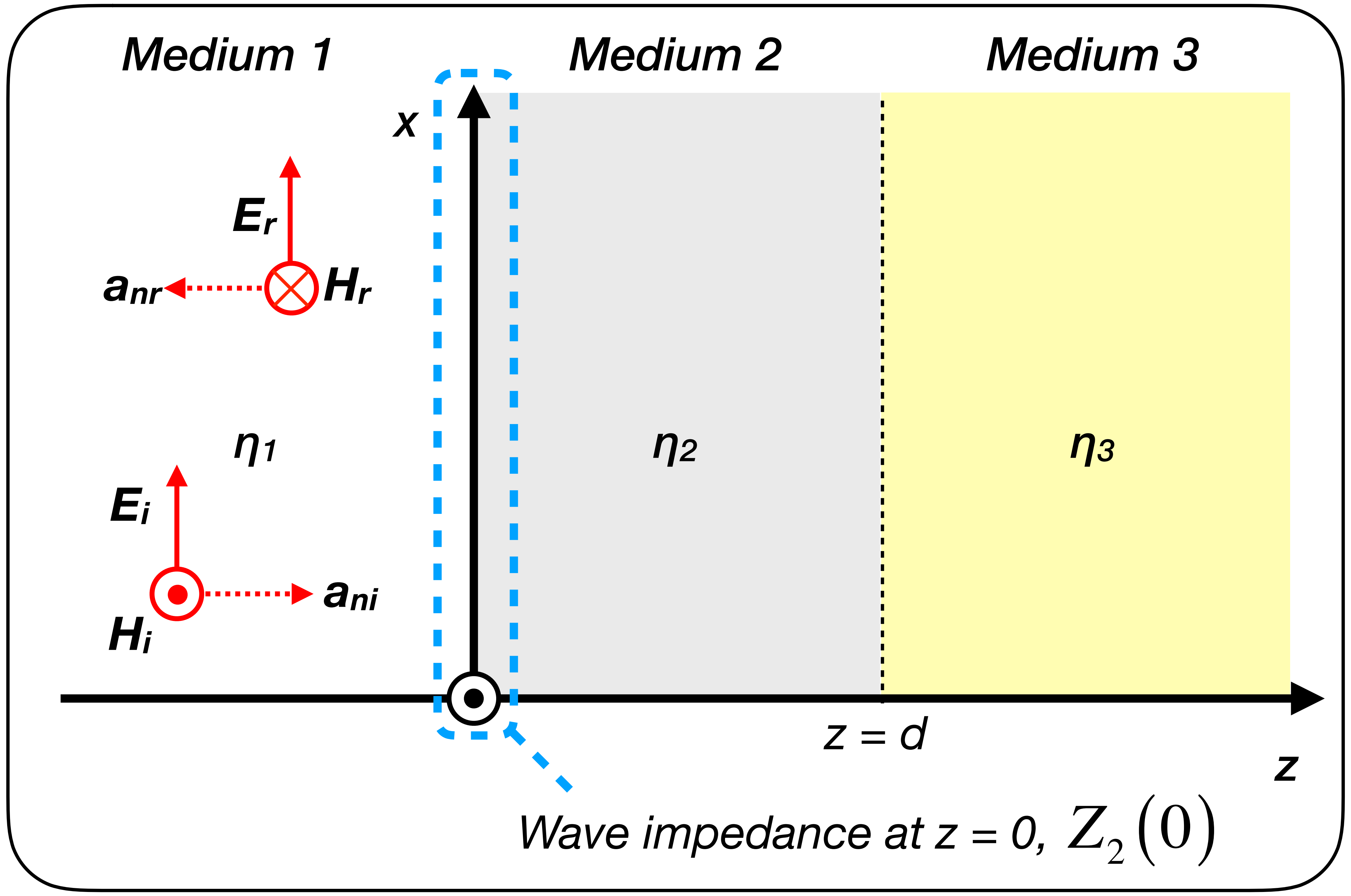
$$\Gamma_0 = \frac{Z_2(0) - \eta_1}{Z_2(0) + \eta_1}$$

With medium 2

- *Transforming η_3 into $Z_2(0)$*
  - With suitable choices of d and η_2

- *In many applications, Γ_0 and E_{r0} are the only quantities of interest*

$$E_{r0} = \Gamma_0 E_{i0} \quad \text{where} \quad \Gamma_0 = \frac{Z_2(0) - \eta_1}{Z_2(0) + \eta_1}$$



Impedance transformation Example (1/2)

Q. 8-12: To obtain zero "effective" reflection coefficient at z = 0, What should be d and η_2?

$$\Gamma_0 = \frac{E_{r0}}{E_{i0}} = \frac{Z_2(0)-\eta_1}{Z_2(0)+\eta_1} = 0 \longrightarrow Z_2(0) = \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d} = \eta_1$$

Equating real and imaginary parts separately,

$$\eta_2\left(\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d\right) = \eta_1\left(\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d\right)$$

$$\rightarrow \eta_3 \cos\beta_2 d = \eta_1 \cos\beta_2 d \quad \cdots(1)$$

$$\rightarrow \eta_2^{\,2} \sin\beta_2 d = \eta_1\eta_3 \sin\beta_2 d \quad \cdots(2)$$

For Equation (a) to be satisfied,

$$\eta_3 \cos\beta_2 d = \eta_1 \cos\beta_2 d$$

$$\rightarrow \eta_1 = \eta_3 \quad \cdots\text{case A}$$

$$\rightarrow \cos\beta_2 d = 0 \quad \cdots\text{case B}$$

Impedance transformation Example (2/2)

Q. 8-12: To obtain zero "effective" reflection coefficient at *z* = 0, What should be *d* and η_2?

$$\eta_2\left(\eta_3\cos\beta_2 d + j\eta_2\sin\beta_2 d\right) = \eta_1\left(\eta_2\cos\beta_2 d + j\eta_3\sin\beta_2 d\right)$$

→ $\eta_3\cos\beta_2 d = \eta_1\cos\beta_2 d \quad \cdots(1)$

→ $\eta_2^{\,2}\sin\beta_2 d = \eta_1\eta_3\sin\beta_2 d \quad \cdots(2)$

$$\eta_3\cos\beta_2 d = \eta_1\cos\beta_2 d$$

→ $\eta_1 = \eta_3 \quad \cdots\text{case A}$

→ $\cos\beta_2 d = 0 \quad \cdots\text{case B}$

1. Case A ($\eta_1 = \eta_3$)

Equation (2) becomes $\eta_2^{\,2}\sin\beta_2 d = \eta_1^{\,2}\sin\beta_2 d$

→ $\eta_1 = \eta_2 = \eta_3$: Trivial case (Not interesting)

→ $\sin\beta_2 d = 0 \quad \rightarrow \quad d = \dfrac{n\pi}{\beta_2} = n\dfrac{\lambda_2}{2}, \qquad n = 0,1,2\ldots$

Dielectric 2 is called ***half-wave dielectric window***

Narrow-band device??

2. Case B ($\eta_1 \neq \eta_3$)

$$\cos\beta_2 d = 0 \quad \rightarrow \quad d = \frac{(2n+1)\pi}{\beta_2} = (2n+1)\frac{\lambda_2}{4}, \qquad n = 0,1,2\ldots$$

Dielectric 2 is called ***quarter-wave impedance transformer***

And $\sin\beta_2 d \neq 0$. ∴ from Equation (2), we get

$$\eta_2^{\,2} = \eta_1\eta_3 \quad \text{or} \quad \eta_2 = \sqrt{\eta_1\eta_3}$$

Electromagnetics

<Chap. 8> Plane Electromagnetic waves

Section 8.9 ~ 8.10

(2nd of **week 5**)

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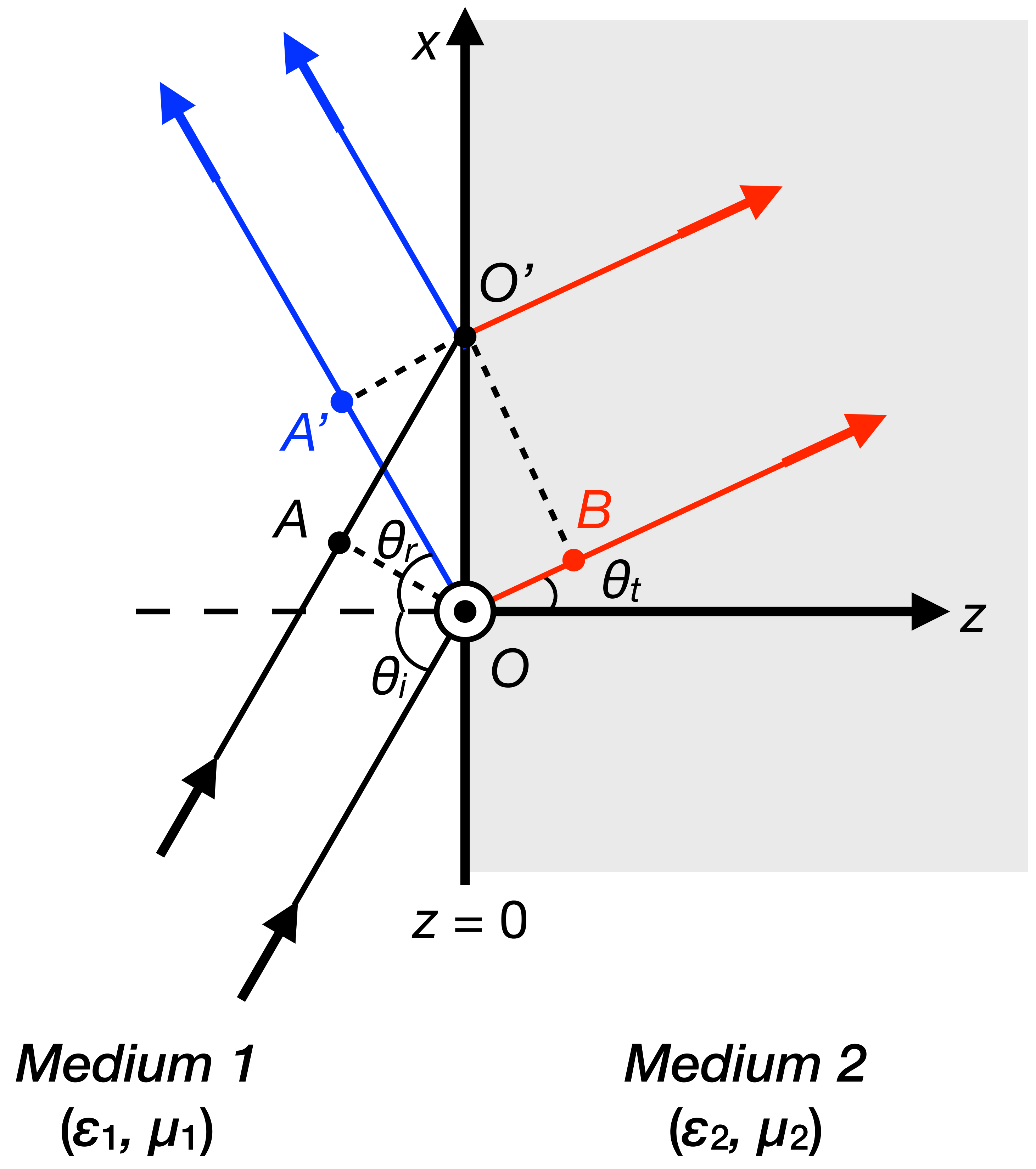
Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 8 | Contents for 2nd week of week 5

Sec 10. Oblique incidence at a plane dielectric interfaces

- Snell’s law of reflection & refraction
- Total internal reflection
- Brewster’s angle: TM vs. TE

Snell's law



$\overline{AO}$, $\overline{A'O'}$, $\overline{BO'}$: Intersections of the ***wave fronts (equi-phase surface)*** of ***incident***, ***reflected***, ***transmitted*** waves

- ***Snell's Law of Reflection***

$\overline{AO'} = \overline{OA'}$: Length of the trajectory of ***incident wave*** = ***reflected wave***

(∵Same phase velocity)

$$\rightarrow \left[\overline{AO'} = \overline{OO'}\cos\left(90^\circ - \theta_r\right)\right] = \left[\overline{OA'} = \overline{OO'}\cos\left(90^\circ - \theta_i\right)\right] \qquad \therefore \theta_i = \theta_r$$

- ***Snell's Law of Refraction***

$t_{\overrightarrow{OB}} = t_{\overrightarrow{AO'}}$: Time for ***transmitted wave*** to travel from *O* to *B*

= Time for ***incident wave*** to travel from *A* to *O'*

$$\rightarrow \left[t_{\overrightarrow{OB}} = \frac{\overline{OB}}{u_{p2}}\right] = \left[t_{\overrightarrow{OB}} = \frac{\overline{AO'}}{u_{p1}}\right] \rightarrow \frac{\overline{OB}}{\overline{AO'}} = \frac{\overline{OO'}\sin\theta_t}{\overline{OO'}\sin\theta_i} = \frac{u_{p2}}{u_{p1}}$$

$$\therefore \frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2}$$

$$\because u_p = \frac{dz}{dt} = \frac{\omega}{\beta}$$

$$\because n \triangleq \frac{c}{u_p}$$

Refractive index

: Ratio of speed of light to that in the medium

Chap. 8 | Snell's Law of Refraction

Formula for Snell's law of refraction

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2}$$

- Nonmagnetic media ($\mu_1 = \mu_2 = \mu_0$)

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\sqrt{\mu_1\varepsilon_1}}{\sqrt{\mu_2\varepsilon_2}} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}} = \frac{n_1}{n_2} = \frac{\eta_2}{\eta_1}$$

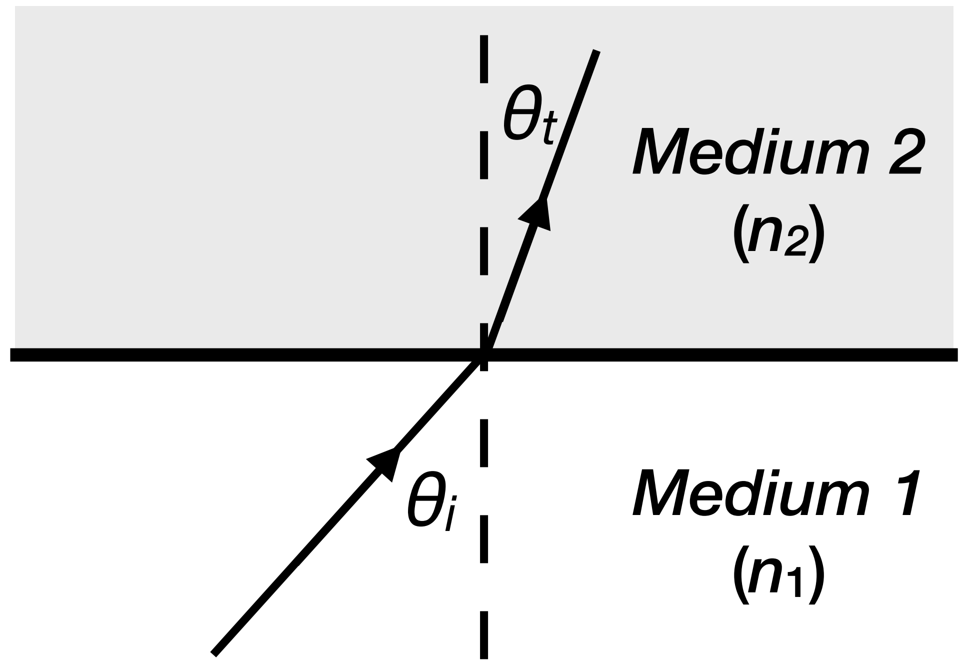
$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$n = \frac{c}{u_p} = \frac{\sqrt{\mu\varepsilon}}{\sqrt{\mu_e\varepsilon_0}} = \sqrt{\mu_r\varepsilon_r}$$

- Medium 1 is free space ($\varepsilon_{r1} = 1$, $n_1 = 1$)

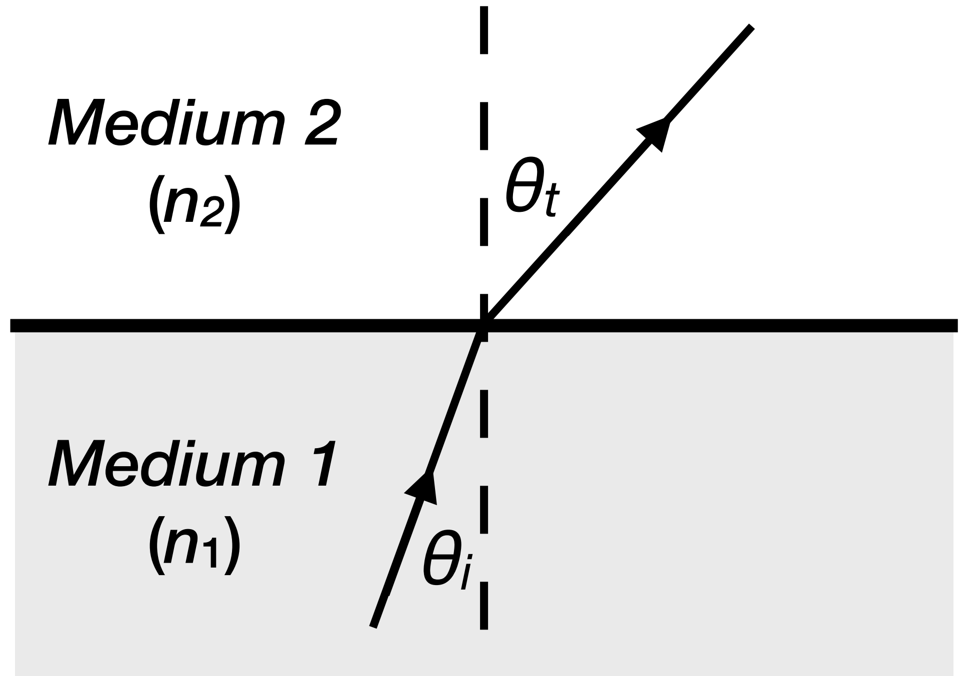
$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{1}{\varepsilon_{r2}}} = \frac{1}{n_2} = \frac{\eta_2}{120\pi}$$

- ***Relationship between angles and refractive indices***



Propagation to *"denser" medium*
($n_2 > n_1 \rightarrow \theta_t < \theta_i$)

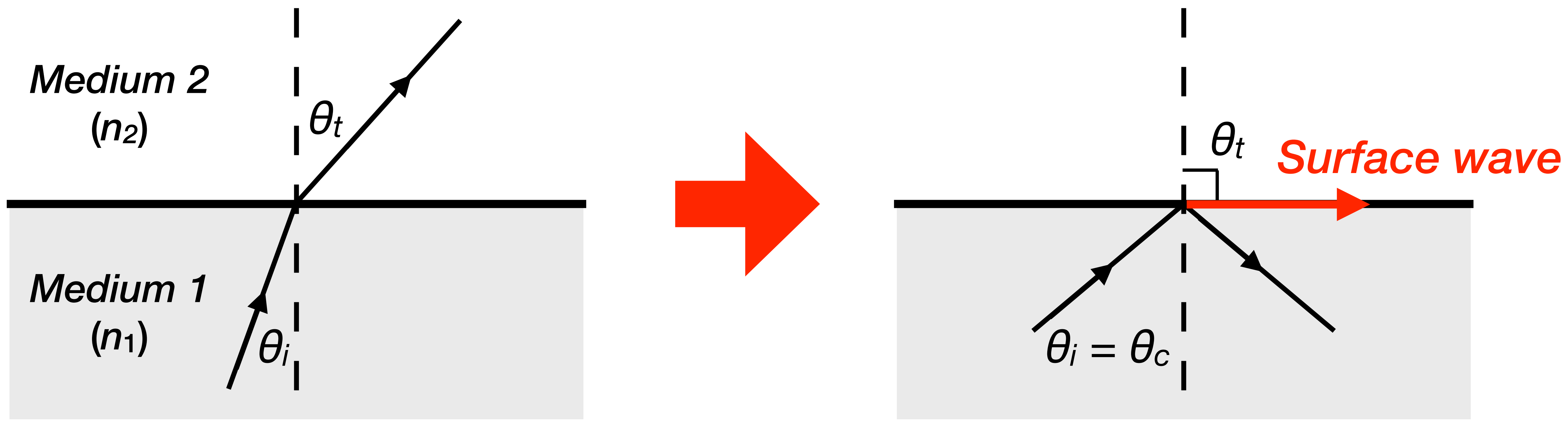
→ Ray ***bent toward*** the normal



Propagation to *"less dense" medium*
($n_2 < n_1 \rightarrow \theta_t > \theta_i$)

→ Ray ***bent away from*** the normal

Total reflection



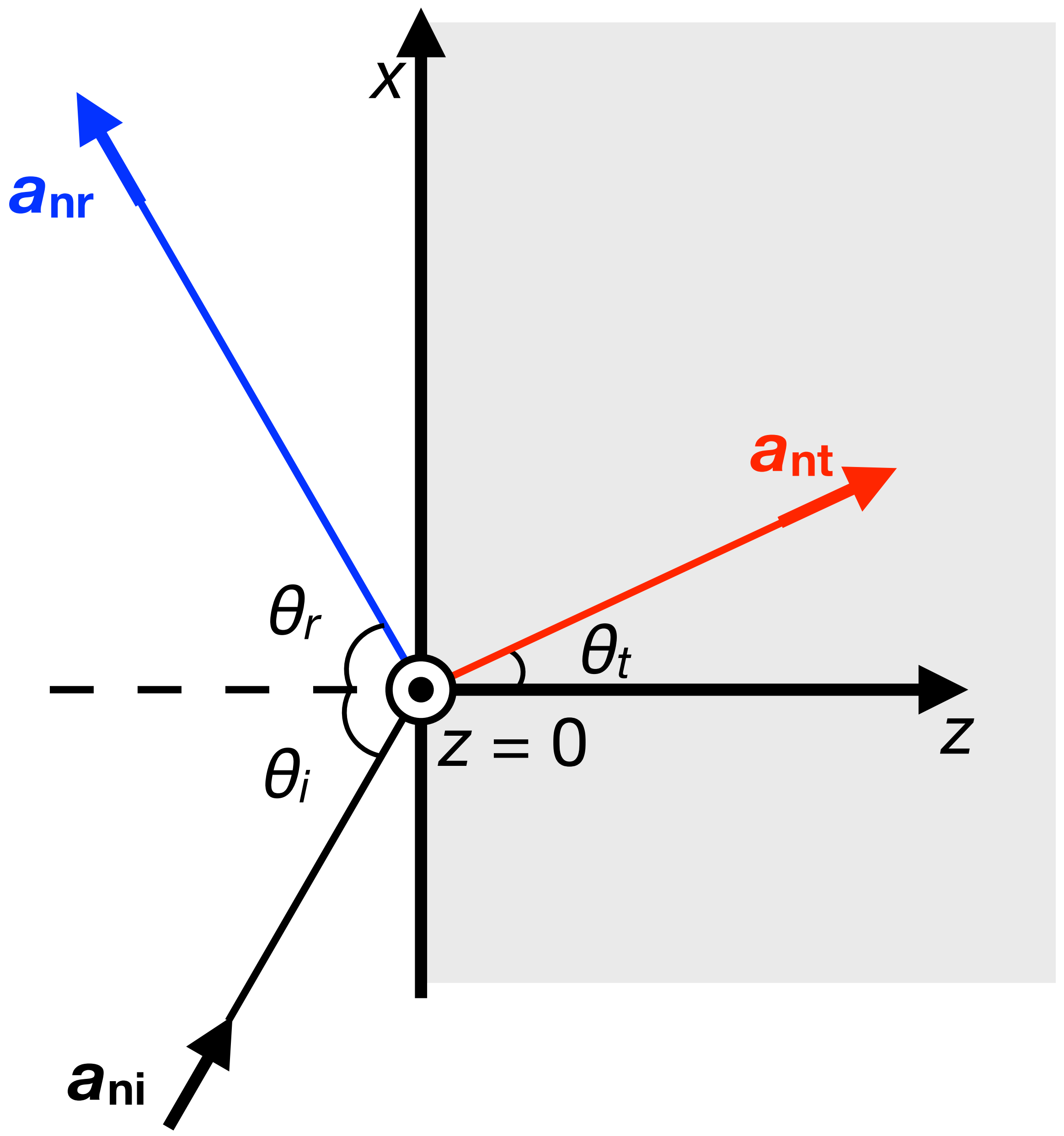
θ_c: *critical angle* when θ_t = 90°

- Wave is incident on a less dense medium 2 from medium 1 ($\varepsilon_1 > \varepsilon_2 \rightarrow n_1 > n_2$)
  - $\theta_i < \theta_c$: both reflection + refraction
  - $\theta_i = \theta_c$: Reflection + refraction along the surface
  - $\theta_i > \theta_c$: *Total reflection*

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \rightarrow \boxed{\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

$$\theta_i = \theta_c,\ \theta_t = \frac{\pi}{2}$$

What will happen to refracted wave when $\theta_i > \theta_c$?



- Mathematical interpretation

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i = \frac{\sin\theta_i}{\sin\theta_c} > 1 \rightarrow \cos\theta_t = \sqrt{1-\sin^2\theta_t} = -j\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$$

- Refracted wave has a direction vector $\boldsymbol{a}_{nt}$ such that

$$\boldsymbol{a}_{nt} = \boldsymbol{a}_x \sin\theta_t + \boldsymbol{a}_z \cos\theta_t \rightarrow e^{-j\beta_2 \boldsymbol{a}_{nt}\cdot\boldsymbol{R}} = e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$= e^{-j\left(\beta_2 \sin\theta_i \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\right)x} e^{-\left(\beta_2\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}\right)z} = \boxed{e^{-j\beta_R x} e^{-\alpha_R z}}$$

Evanescent wave

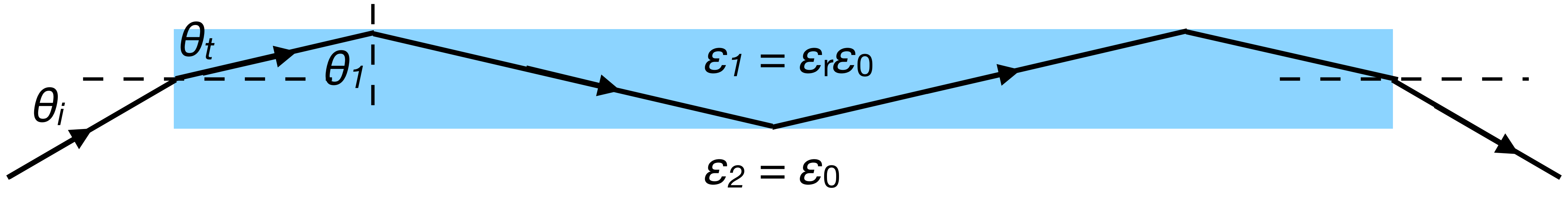
- *Evanescent wave*
  - Propagating along the surface (i.e. along *x direction*) → *Surface wave*
  - Decaying exponentially away from the surface (i.e. in z direction)

Example 8-14: Optical fiber

- ***A dielectric rod*** or ***fiber of a transparent material*** that can be used to ***guide light or electromagnetic wave*** under the conditions of ***total internal reflection***

Question

: What is minimum dielectric constant of guiding medium (ε_1) so that incident wave ***at any angles*** can be confined within the rod?



- For total internal reflection,

$$\sin\theta_c < \sin\theta_1 = \sin\left(\frac{\pi}{2} - \theta_t\right) = \cos\theta_t \quad \cdots(1)$$

and $$\frac{1}{\sin\theta_c} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \sqrt{\varepsilon_r} \quad \cdots(2)$$

- On the other hand, from Snell' law of refraction,

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{1}{\varepsilon_r}} \quad \rightarrow \quad \sin\theta_t = \sin\theta_i \sqrt{\frac{1}{\varepsilon_r}} \quad \cdots(3)$$

- By plugging (3) and (2) into (1),

$$\sin\theta_c < \cos\theta_t \quad \rightarrow \quad \frac{1}{\sqrt{\varepsilon_r}} < \sqrt{1 - \frac{\sin^2\theta_i}{\varepsilon_r}} \quad \rightarrow \quad \varepsilon_r > 1 + \sin^2\theta_i \le 2$$

$$\therefore \varepsilon_r \ge 2 \quad \rightarrow \quad n \ge 1.4$$

Glass or Quartz

Chap. 8 | Oblique incidence of TE wave at dielectric boundary (1/4)

Polarization dependence

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2}$$

Snell's law of refraction

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Critical angle

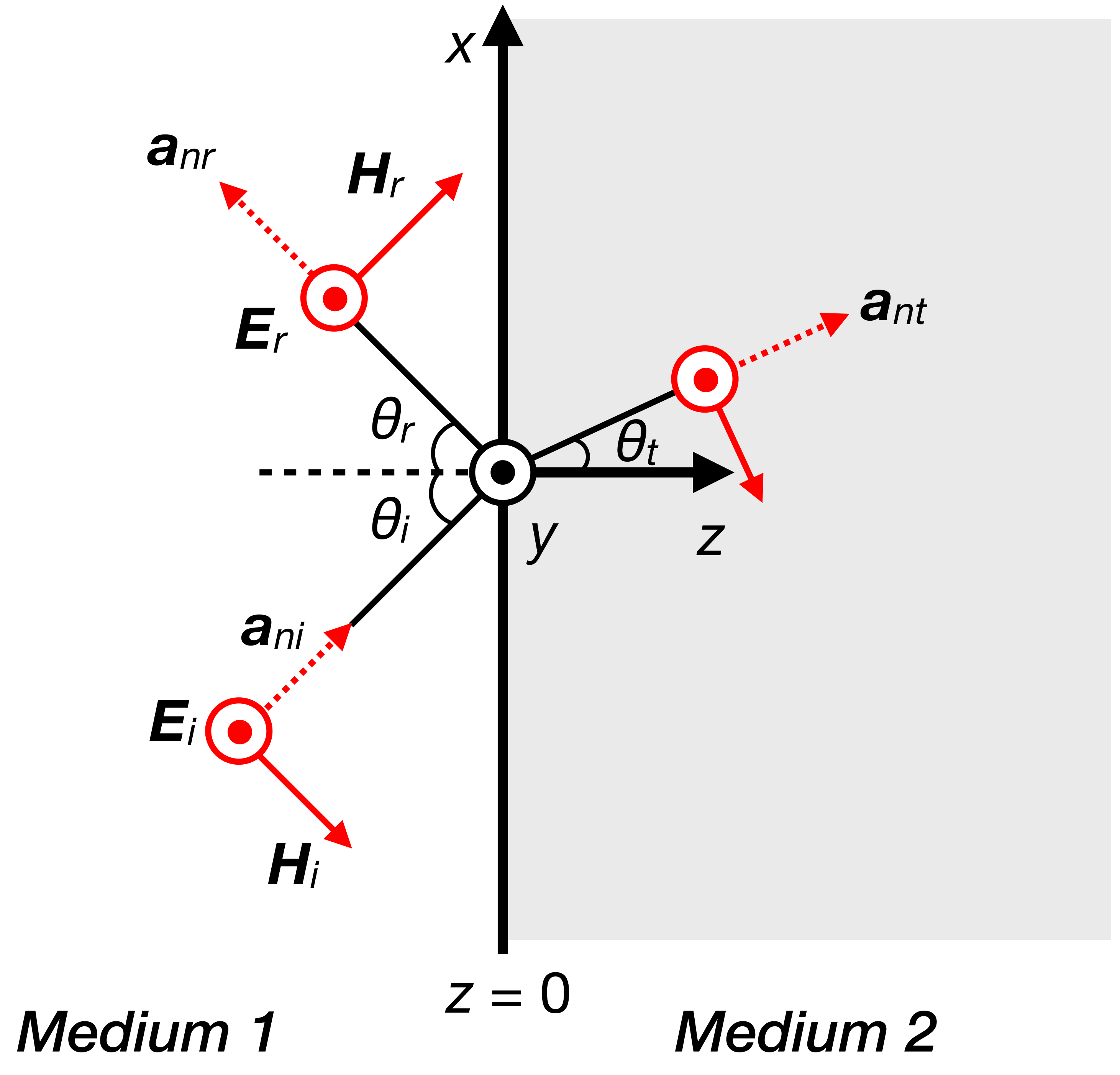
Snell's law of refraction, critical angle → Polarization-*independent*

Reflection and transmission coefficients → Polarization-*dependent*

4 Unknown variables

$$E_{r0},\ E_{t0},\ \theta_r,\ \theta_t$$

Transverse Electric wave



- Incident wave

$$\begin{cases} \boldsymbol{E_i}(x,z) = \boldsymbol{a_y} E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \\ \boldsymbol{H_i}(x,z) = \dfrac{E_{i0}}{\eta_1}\left(-\boldsymbol{a_x}\cos\theta_i + \boldsymbol{a_z}\sin\theta_i\right) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \end{cases}$$

- Reflected wave

$$\begin{cases} \boldsymbol{E_r}(x,z) = \boldsymbol{a_y} E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\ \boldsymbol{H_r}(x,z) = \dfrac{E_{r0}}{\eta_1}\left(\boldsymbol{a_x}\cos\theta_r + \boldsymbol{a_z}\sin\theta_r\right) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \end{cases}$$

- Transmitted wave

$$\begin{cases} \boldsymbol{E_t}(x,z) = \boldsymbol{a_y} E_{t0} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \\ \boldsymbol{H_t}(x,z) = \dfrac{E_{t0}}{\eta_2}\left(-\boldsymbol{a_x}\cos\theta_t + \boldsymbol{a_z}\sin\theta_t\right) e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \end{cases}$$

 Oblique incidence of TE wave at dielectric boundary (2/4)

Boundary condition

- At z = 0, Tangential components of ***E*** and ***H*** should be continuous

$$\begin{cases} \mathbf{E}_i(x,z) = \mathbf{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \\ \mathbf{E}_r(x,z) = \mathbf{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\ \mathbf{E}_t(x,z) = \mathbf{a}_y E_{t0} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \end{cases}$$

➡

$$E_{iy}(x,0) + E_{ry}(x,0) = E_{ty}(x,0)$$

$$\rightarrow E_{i0} e^{-j\beta_1 x\sin\theta_i} + E_{r0} e^{-j\beta_1 x\sin\theta_r} = E_{t0} e^{-j\beta_2 x\sin\theta_t}$$

This should satisfy for all x →

all three exponential factors should be equal

$$\begin{cases} \beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t & \cdots(1) \\ E_{i0} + E_{r0} = E_{t0} & \cdots(2) \end{cases}$$

→ Phase matching

$$\begin{cases} \mathbf{H}_i(x,z) = \dfrac{E_{i0}}{\eta_1}\left(-\mathbf{a}_x\cos\theta_i + \mathbf{a}_z\sin\theta_i\right) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \\ \mathbf{H}_r(x,z) = \dfrac{E_{r0}}{\eta_1}\left(\mathbf{a}_x\cos\theta_r + \mathbf{a}_z\sin\theta_r\right) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\ \mathbf{H}_t(x,z) = \dfrac{E_{t0}}{\eta_2}\left(-\mathbf{a}_x\cos\theta_t + \mathbf{a}_z\sin\theta_t\right) e^{-j\beta_2(x\sin\theta_t - z\cos\theta_t)} \end{cases}$$

➡

$$H_{ix}(x,0) + H_{rx}(x,0) = H_{tx}(x,0)$$

$$\frac{1}{\eta_1}\left(-E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}\right) = \frac{E_{r0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

$$\begin{cases} \beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t & \cdots(1) \\ \dfrac{1}{\eta_1}\left(E_{i0} - E_{r0}\right)\cos\theta_i = \dfrac{E_{t0}}{\eta_2}\cos\theta_t & \cdots(3) \end{cases}$$

➡ *Try solving equation for a_z components!*

Oblique incidence of TE wave at dielectric boundary (3/4)

Boundary condition

$$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t \quad \cdots(1)$$

→ $\sin\theta_i = \sin\theta_r \quad \rightarrow \quad \theta_i = \theta_r$ **: Snell's law of reflection**

→ $\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{\beta_1}{\beta_2} = \dfrac{u_{p2}}{u_{p1}} = \dfrac{n_1}{n_2}$ **: Snell's law of refraction**

$$\begin{cases} E_{i0} + E_{r0} = E_{t0} \quad \cdots(2) \\ \dfrac{1}{\eta_1}\left(E_{i0} - E_{r0}\right)\cos\theta_i = \dfrac{E_{t0}}{\eta_2}\cos\theta_t \quad \cdots(3) \end{cases}$$

→ $\Gamma_{TE} = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \dfrac{\left(\eta_2/\cos\theta_t\right) - \left(\eta_1/\cos\theta_i\right)}{\left(\eta_2/\cos\theta_t\right) + \left(\eta_1/\cos\theta_i\right)}$ c.f.) $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

→ $\tau_{TE} = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \dfrac{2\left(\eta_2/\cos\theta_t\right)}{\left(\eta_2/\cos\theta_t\right) + \left(\eta_1/\cos\theta_i\right)}$ c.f.) $\tau = \dfrac{2\eta_2}{\eta_2 + \eta_1}$

$$\therefore \eta_1 \rightarrow \frac{\eta_1}{\cos\theta_i}, \quad \eta_2 \rightarrow \frac{\eta_2}{\cos\theta_t}$$

- If $\theta_i = \theta_t = 0$, expressions Γ_{TE} and τ_{TE} reduce to those for normal incidence
- Relationship between Γ_{TE} and τ_{TE}

$$1 + \Gamma_{TE} = \tau_{TE}$$

- If medium 2 is a perfect conductor ($\eta_2 = 0$),
  - $\Gamma_{TE} = -1$ ($E_{r0} = -E_{i0}$), $\tau_{TE} = 0$ ($E_{t0} = 0$)

Oblique incidence of TE wave at dielectric boundary (4/4)

Brewster angle

- Incident angle (θ_{TE}) that makes reflection coefficient zero

$$\Gamma_{TE} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\left(\eta_2/\cos\theta_t\right) - \left(\eta_1/\cos\theta_i\right)}{\left(\eta_2/\cos\theta_t\right) + \left(\eta_1/\cos\theta_i\right)} = 0 \quad \rightarrow \quad \frac{\eta_1}{\cos\theta_{TE}} = \frac{\eta_2}{\cos\theta_t} \quad \cdots(1)$$

According to Snell's law of refraction,

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{n_1}{n_2}\sin^2\theta_{TE}} \quad \cdots(2) \quad \leftarrow \quad \because \frac{\sin\theta_t}{\sin\theta_{TE}} = \frac{n_1}{n_2}$$

By plugging (2) into (1), we get

$$\cos\theta_{TE} = \frac{\eta_1}{\eta_2}\cos\theta_t = \frac{\eta_1}{\eta_2}\sqrt{1-\frac{n_1}{n_2}\sin^2\theta_{TE}} \quad \rightarrow \quad 1-\sin^2\theta_{TE} = \frac{\eta_1^{\,2}}{\eta_2^{\,2}}\left(1-\frac{n_1}{n_2}\sin^2\theta_{TE}\right)$$

$$\therefore \sin^2\theta_{TE} = \frac{1-\eta_1^{\,2}/\eta_2^{\,2}}{1-\eta_1^{\,2}n_1/\eta_2^{\,2}n_2} = \frac{1-\mu_1\varepsilon_2/\mu_2\varepsilon_1}{1-\left(\mu_1/\mu_2\right)^2}$$

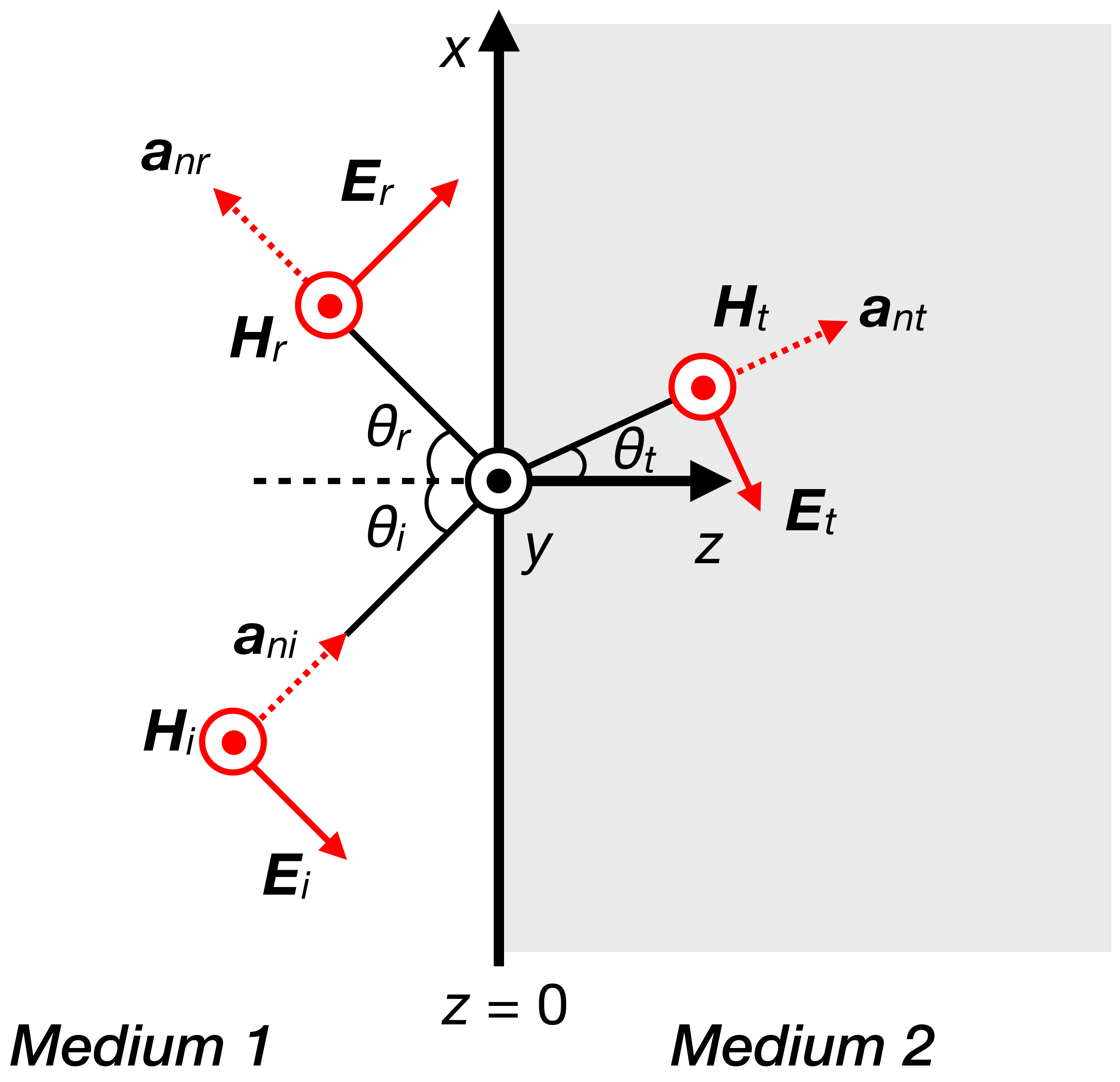
θ_{TE}: Brewster's angle of no reflection for TE wave

✳For nonmagnetic media ($\mu_1 = \mu_2 = \mu_0$)

- Right side of the equation diverges to infinity
- ***Brewster's angle for TE wave DOES NOT exist for such case***

Chap. 8 | Oblique incidence of TM wave at dielectric boundary (1/3)

Transverse Magnetic wave



- Incident wave

$$\begin{cases} \boldsymbol{E_i}(x,z) = E_{i0}\left(\boldsymbol{a_x}\cos\theta_i - \boldsymbol{a_z}\sin\theta_i\right)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \\ \boldsymbol{H_i}(x,z) = \boldsymbol{a_y}\dfrac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \end{cases}$$

- Reflected wave

$$\begin{cases} \boldsymbol{E_r}(x,z) = E_{r0}\left(\boldsymbol{a_x}\cos\theta_r + \boldsymbol{a_z}\sin\theta_r\right)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\ \boldsymbol{H_r}(x,z) = -\boldsymbol{a_y}\dfrac{E_{r0}}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \end{cases}$$

- Transmitted wave

$$\begin{cases} \boldsymbol{E_t}(x,z) = E_{t0}\left(\boldsymbol{a_x}\cos\theta_t - \boldsymbol{a_z}\sin\theta_t\right)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \\ \boldsymbol{H_t}(x,z) = \boldsymbol{a_y}\dfrac{E_{t0}}{\eta_2}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \end{cases}$$

- Boundary condition
- At $z = 0$, tangential components of ***E*** and ***H*** should be equal:

$$\begin{cases} E_{ix}(x,0) + E_{ix}(x,0) = E_{ix}(x,0) \\ H_{iy}(x,0) + H_{iy}(x,0) = H_{iy}(x,0) \end{cases} \rightarrow \begin{cases} (E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t \\ \dfrac{1}{\eta_1}(E_{i0} - E_{r0}) = \dfrac{1}{\eta_2}E_{t0} \end{cases}$$

 Oblique incidence of TM wave at dielectric boundary (2/3)

Reflection and transmission coefficients for TM wave

$$\begin{cases}(E_{i0}+E_{r0})\cos\theta_i = E_{t0}\cos\theta_t \\ \dfrac{1}{\eta_1}(E_{i0}-E_{r0}) = \dfrac{1}{\eta_2}E_{t0}\end{cases} \rightarrow \begin{cases}\Gamma_{TM} = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \\ \tau_{TM} = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}\end{cases}$$

$$\therefore 1+\Gamma_{TM} = \tau_{TM}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)$$

Brewster's angle for TM wave

$$\Gamma_{TM} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_{TM}}{\eta_2\cos\theta_t + \eta_1\cos\theta_{TM}} = 0 \rightarrow \eta_1\cos\theta_{TM} = \eta_2\cos\theta_t = \eta_2\sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_{TM}}$$

$$\therefore \sin^2\theta_{TM} = \frac{1-\mu_2\varepsilon_1/\mu_1\varepsilon_2}{1-(\varepsilon_1/\varepsilon_2)^2}$$

- For nonmagnetic media ($\mu_1 = \mu_2 = \mu_0$),

$$\sin\theta_{TM} = \frac{1}{1+\varepsilon_1/\varepsilon_2}$$: ***a solution always exists!***

c.f.) solution ***does not exist for TE wave*** since $\sin^2\theta_{TE}\Big|_{\mu_1=\mu_2=\mu_0} = \dfrac{1-\mu_1\varepsilon_2/\mu_2\varepsilon_1}{1-(\mu_1/\mu_2)^2} \rightarrow \infty$

✳ When an unpolarized wave is incident on a boundary at the ***Brewster angle θ_{TM}***, ***only the TE wave (= perpendicular polarization)*** is reflected

→ ***Brewster's angle = Polarizing angle***

Chap. 8 | Oblique incidence of TM wave at dielectric boundary (3/3)

TE vs. TM

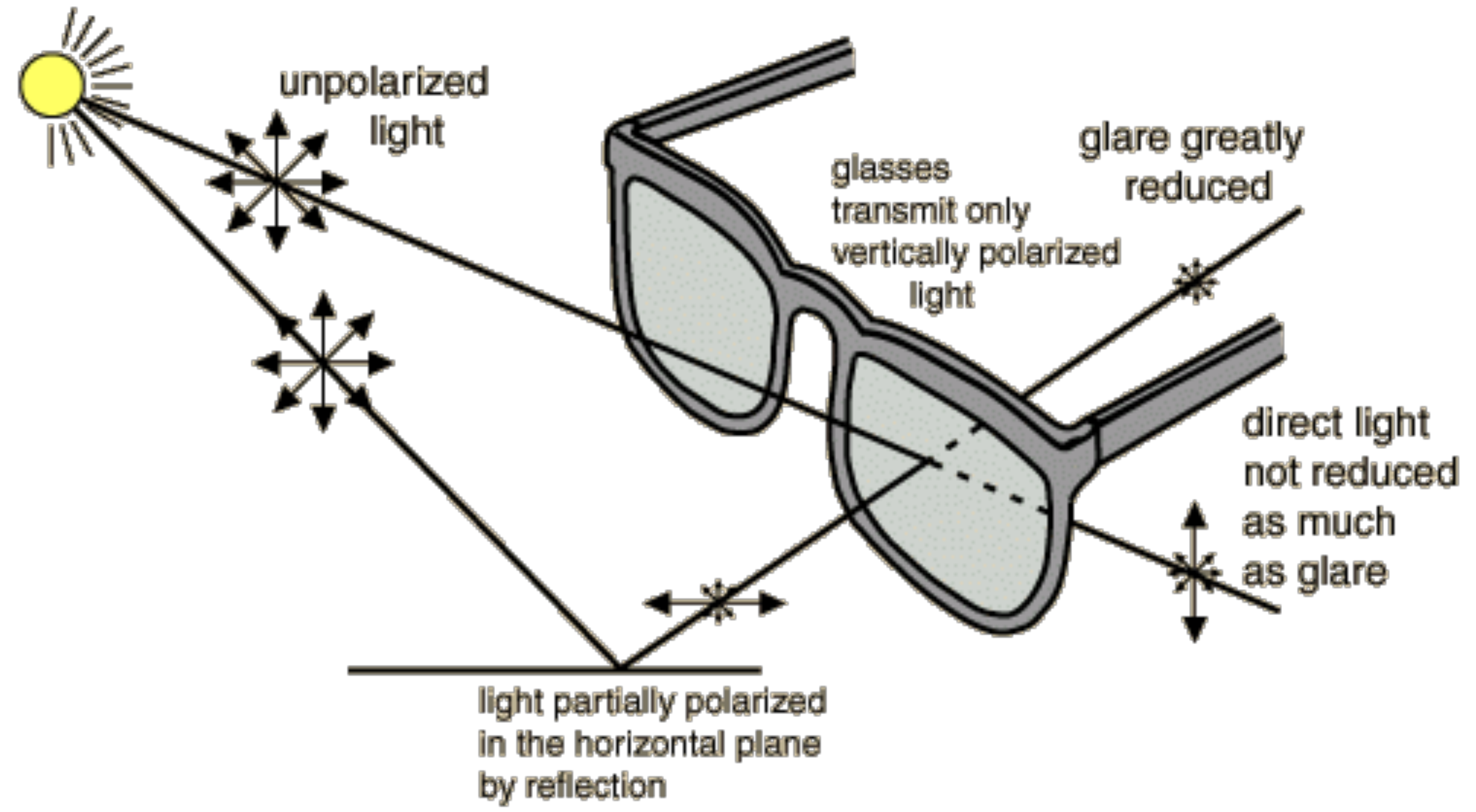
$$|\Gamma_{TE}|^2 > |\Gamma_{TM}|^2 \quad \textit{for all angle } \theta_i$$

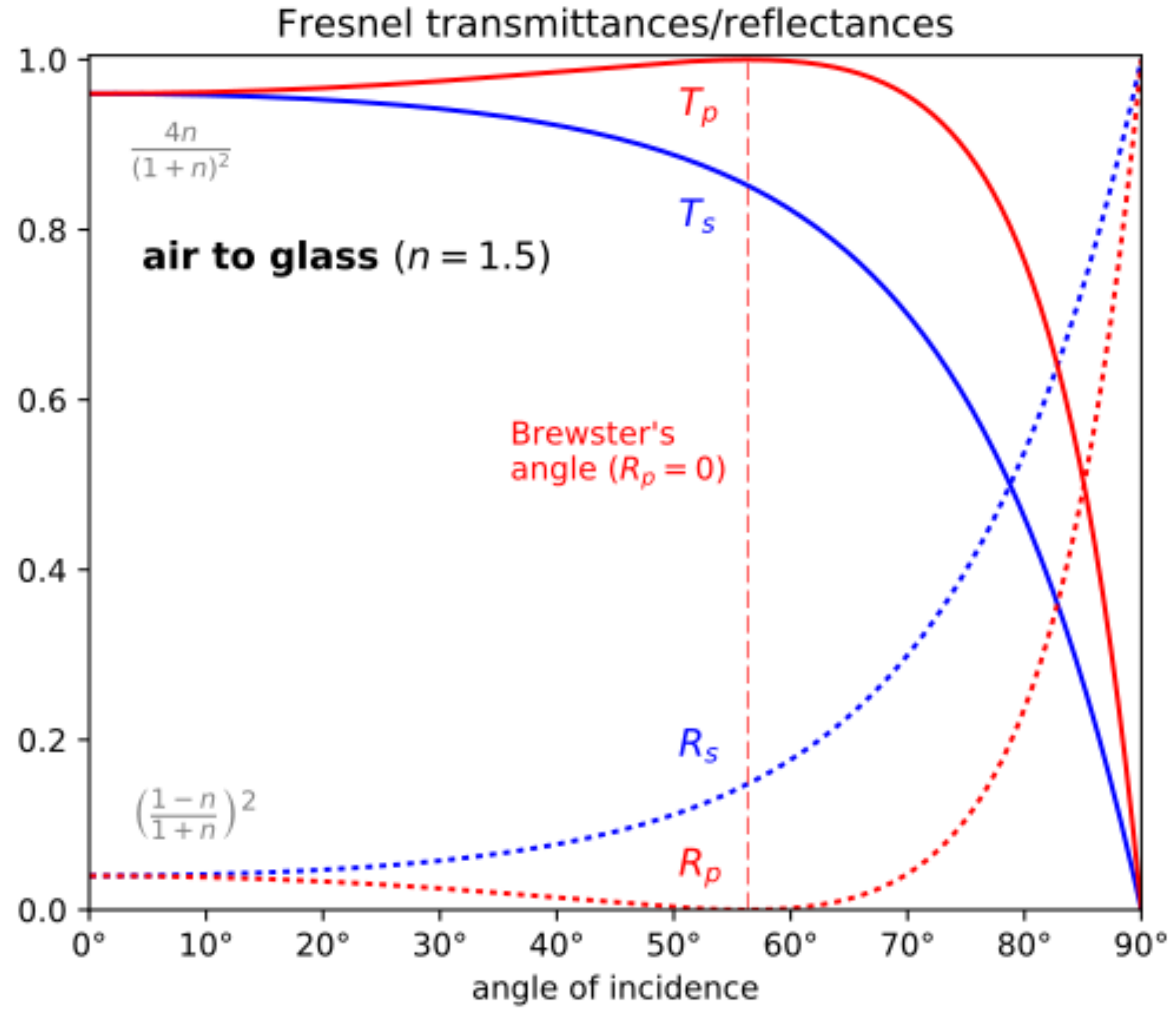
- When an "unpolarized" light strikes a plane dielectric surface,
- Reflected wave has ***more TE than TM components***

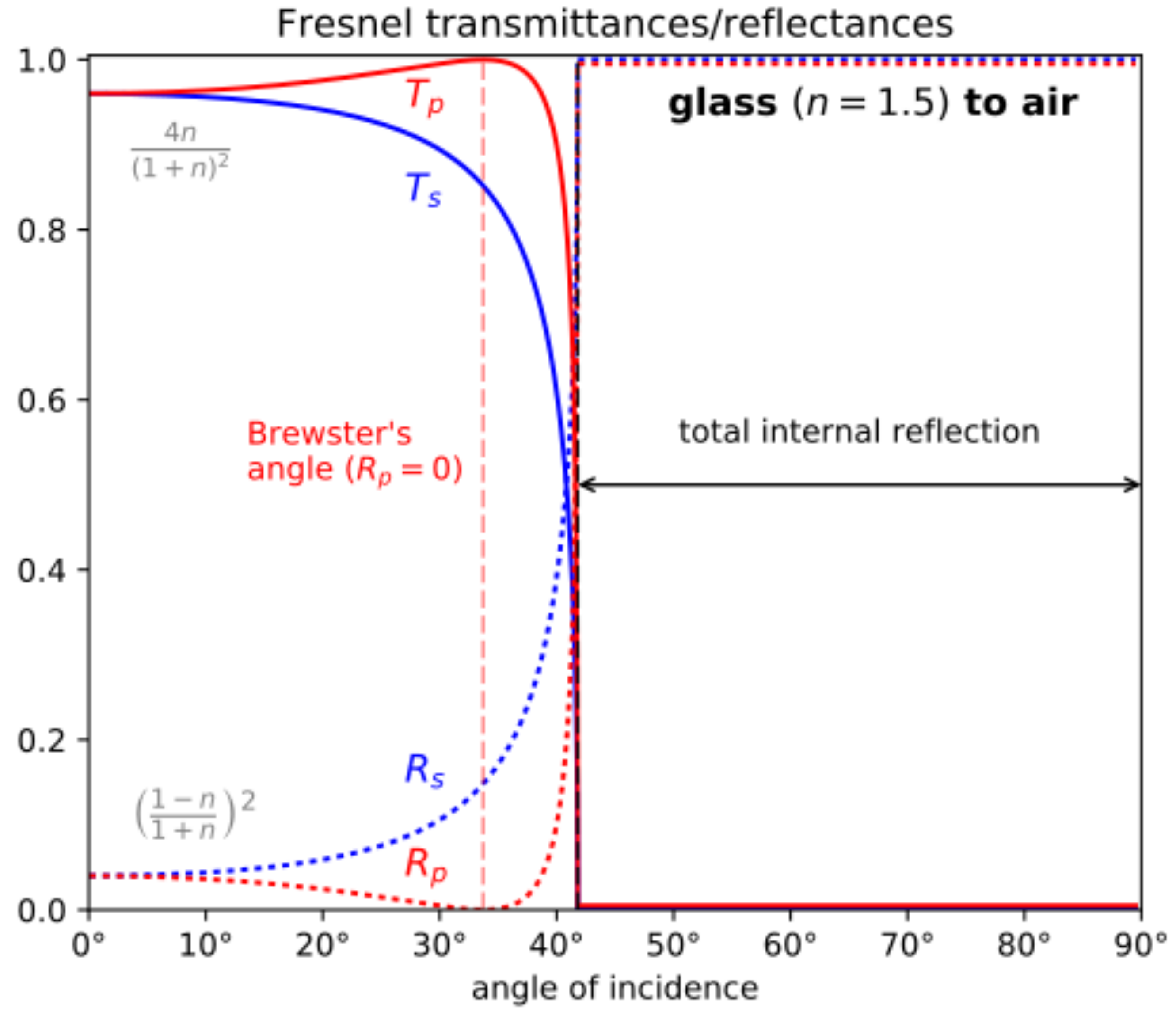
	Reference: Plane of incidence (P.o.I)	***Reference: Electric field***
E ⊥ *P.o.I* H // *P.o.I*	Transverse Electric (***TE***) Wave	Perpendicular Polarization (***s-polarization***)
H ⊥ *P.o.I* E // *P.o.I*	Transverse Magnetic (***TM***) Wave	Parallel Polarization (***p-polarization***)

Practical application:

- Polarized sunglasses are designed to reduce sun glare
  - Reflected sunlight from horizontal surfaces has predominantly ***TE (perpendicular polarization)*** components ($\because |\Gamma_{TE}|^2 > |\Gamma_{TM}|^2$)
    ∴ *E-field* is mostly ***parallel*** to the Earth surface
  - Polarizer on glass only allows ***TM (parallel polarization)***







Img src: Woodies, Wikipedia