Electromagnetics

<Chap. 10> Waveguides and Cavity Resonators

Section 10.7

(1st of **week 9**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 10 | Contents for 1st class of week 9

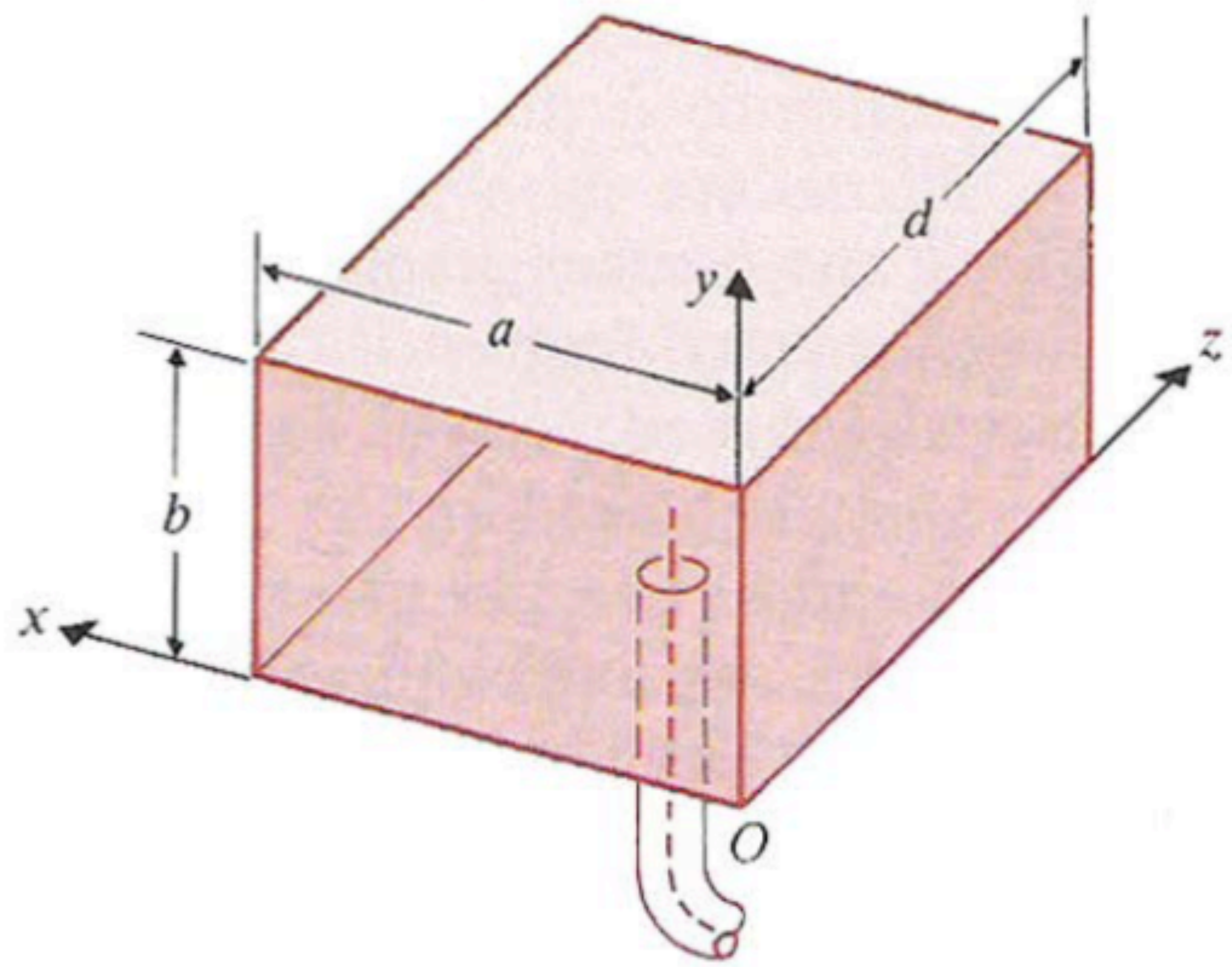
Sec 7. Rectangular cavity resonator

- Derivation of general solutions for E and H-fields
- TE & TM wave characteristics
- Quality factor (*Q*)

Chap. 10 | Rectangular cavity resonators

- ***Rectangular cavity resonators***
  - Hollow, rectangular metal box with sides of *a*, *b*, *d*
  - Conducting walls
    - Leading to multiple reflections → ***Standing waves***

      (∵Linear superposition of two EM waves of same frequency in opposite directions)
    - ***No wave propagation***, but ***confinement in an enclosed cavity***
    - Standing waves formed in all directions (*x*, *y*, and *z*) → ***Strong resonance!***
  - Both TM and TE waves can be supported



- ***EM waves within the cavity***

  : Solutions of wave equations with ***given boundary condition***

$$
\begin{cases}
\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right) \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \\
\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right) \boldsymbol{H} + k^2 \boldsymbol{H} = 0
\end{cases}
$$

where
$$
\begin{cases}
\boldsymbol{E} = \boldsymbol{a_x} E_x + \boldsymbol{a_y} E_y + \boldsymbol{a_z} E_z \\
\boldsymbol{H} = \boldsymbol{a_x} H_x + \boldsymbol{a_y} H_y + \boldsymbol{a_z} H_z
\end{cases}
$$

Tangential **E-fields, E_t = 0!** ←

Normal **H-fields, H_n = 0!** ←

Medium 1 (dielectric)	*Medium 2* (Conductor)
$E_{1t} = 0$	$E_{2t} = 0$
$\boldsymbol{a_{n2}} \times \boldsymbol{H_1} = \boldsymbol{J_S}$	$H_{2t} = 0$
$\boldsymbol{a_{n2}} \cdot \boldsymbol{D_1} = \rho_S$	$D_{2n} = 0$
$H_{1n} = 0$	$H_{2n} = 0$

Chap. 10 | Rectangular cavity resonators

- ***Why should we practice to drive the TM & TE waves in a resonator?***
  - In the textbook, only results shown with qualitative explanation → some important information missing!
  - Many useful EM concepts used in derivation!

TM wave

$$\begin{cases} E_x = -\dfrac{E_{z0}}{h^2}\left(\dfrac{m\pi}{a}\right)\left(\dfrac{p\pi}{d}\right)\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right)\sin\left(\dfrac{p\pi z}{d}\right) \\ E_y = -\dfrac{E_{z0}}{h^2}\left(\dfrac{n\pi}{b}\right)\left(\dfrac{p\pi}{d}\right)\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)\cos\left(\dfrac{p\pi z}{d}\right) \\ E_z = E_{z0}\sin\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right)\cos\left(\dfrac{p\pi z}{d}\right) \end{cases}$$

$$\begin{cases} H_x = \dfrac{j\omega\varepsilon E_{z0}}{h^2}\left(\dfrac{n\pi}{b}\right)\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi x}{b}\right)\cos\left(\dfrac{p\pi x}{d}\right) \\ H_y = -\dfrac{j\omega\varepsilon E_{z0}}{h^2}\left(\dfrac{m\pi}{a}\right)\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi x}{b}\right)\cos\left(\dfrac{p\pi x}{d}\right) \end{cases}$$

TE wave

$$\begin{cases} E_x = \dfrac{j\omega\mu H_{z0}}{h^2}\left(\dfrac{n\pi}{b}\right)\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right)\sin\left(\dfrac{p\pi z}{d}\right) \\ E_y = -\dfrac{j\omega\mu H_{z0}}{h^2}\left(\dfrac{m\pi}{a}\right)\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)\cos\left(\dfrac{p\pi z}{d}\right) \end{cases}$$

$$\begin{cases} H_x = -\dfrac{H_{z0}}{h^2}\left(\dfrac{m\pi}{a}\right)\left(\dfrac{p\pi}{d}\right)\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi x}{b}\right)\cos\left(\dfrac{p\pi x}{d}\right) \\ H_y = -\dfrac{H_{z0}}{h^2}\left(\dfrac{n\pi}{b}\right)\left(\dfrac{p\pi}{d}\right)\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi x}{b}\right)\cos\left(\dfrac{p\pi x}{d}\right) \\ H_z = H_{z0}\cos\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi x}{b}\right)\sin\left(\dfrac{p\pi x}{d}\right) \end{cases}$$

Chap. 10 | Derivation for E-fields for cavity resonators (1/3)

- ***Derivation for E-fields***

- Separation of variables

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_z+k^2E_z=0 \quad \cdots(1)$$

where $\boxed{E_z=X(x)Y(y)Z(z) \quad \cdots(2)}$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x+k^2E_x=0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_y+k^2E_y=0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_z+k^2E_z=0$$

We are solving this first, then apply the similar approaches to others

- By plugging (2) into (1), we get

$$\frac{\partial^2X(x)}{\partial x^2}Y(y)Z(z)+X(x)\frac{\partial^2Y(y)}{\partial y^2}Z(z)+X(x)Y(y)\frac{\partial^2Z(z)}{\partial z^2}+k^2X(x)Y(y)Z(z)=0$$

- By dividing above equation by *X*(*x*)*Y*(*y*)*Z*(*z*), we get

$$\frac{1}{X(x)}\frac{\partial^2X(x)}{\partial x^2}+\frac{1}{Y(y)}\frac{\partial^2Y(y)}{\partial y^2}+\frac{1}{Z(z)}\frac{\partial^2Z(z)}{\partial z^2}+k^2=0 \quad \cdots(3)$$

- Equation (3) to be satisfied for all *x*, *y*, *z*, we obtain the following three ODEs ⇨

$$\begin{cases}\dfrac{1}{X(x)}\dfrac{\partial^2X(x)}{\partial x^2}+k_x^2=0\\[2ex]\dfrac{1}{Y(y)}\dfrac{\partial^2Y(y)}{\partial y^2}+k_y^2=0 \quad \cdots(4)\\[2ex]\dfrac{1}{Z(z)}\dfrac{\partial^2Z(z)}{\partial z^2}+k_z^2=0\end{cases}$$

where $k_x^2+k_y^2+k_z^2=k^2 \quad \cdots(5)$

Chap. 10 | Derivation for E-fields for cavity resonators (2/3)

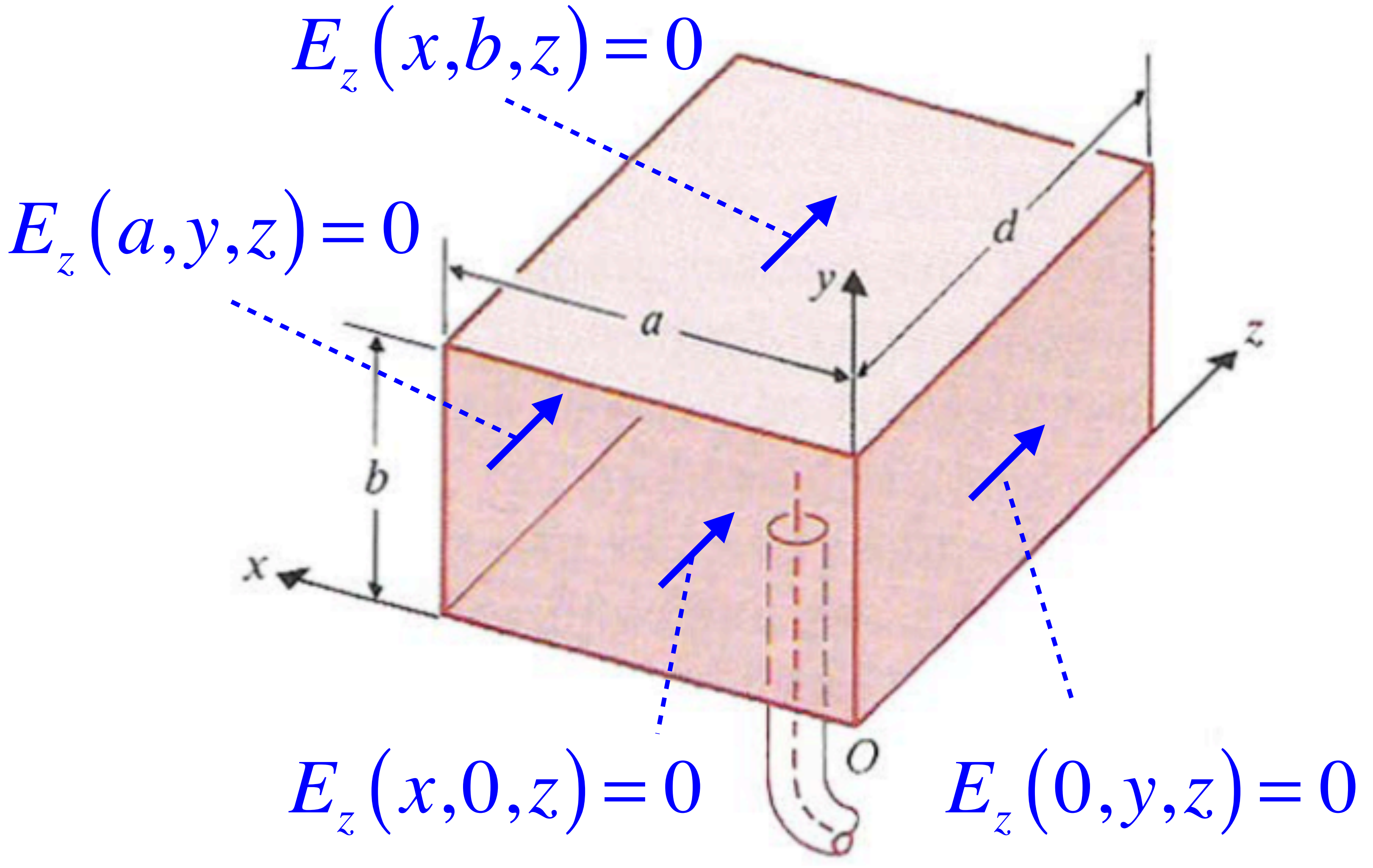
- ***Derivation for E-fields***

  - General solution to the equation for E_z: ***A combination of sinusoidal functions!***

$$E_z(x,y,z)=X(x)Y(y)Z(z)=\left(A_{z,xe}\cos k_x x+A_{z,xo}\sin k_x x\right)\left(A_{z,ye}\cos k_y y+A_{z,yo}\sin k_y y\right)\left(A_{z,ze}\cos k_z z+A_{z,zo}\sin k_z z\right)\quad\cdots(6)$$

  - Now, from B.C. where tangential E-fields should be zero,

$$E_z(x,y,z)=0\text{ at }\begin{cases}y=0\\y=b\end{cases},\text{ and at }\begin{cases}x=0\\x=a\end{cases}$$



  - ‣ $Y(y)=0$ at $y=0$. Then we get,

$$\boxed{Y(0)=A_{z,ye}=0}$$

  - ‣ $Y(y)=0$ at $y=b$. Then we get,

$$\boxed{Y(b)=A_{z,yo}\sin k_y b=0\quad\rightarrow\quad k_y b=n\pi\quad\rightarrow\quad k_y=\frac{n\pi}{b}}$$

  - Similarly applying the B.C. for $X(x)$, we should get

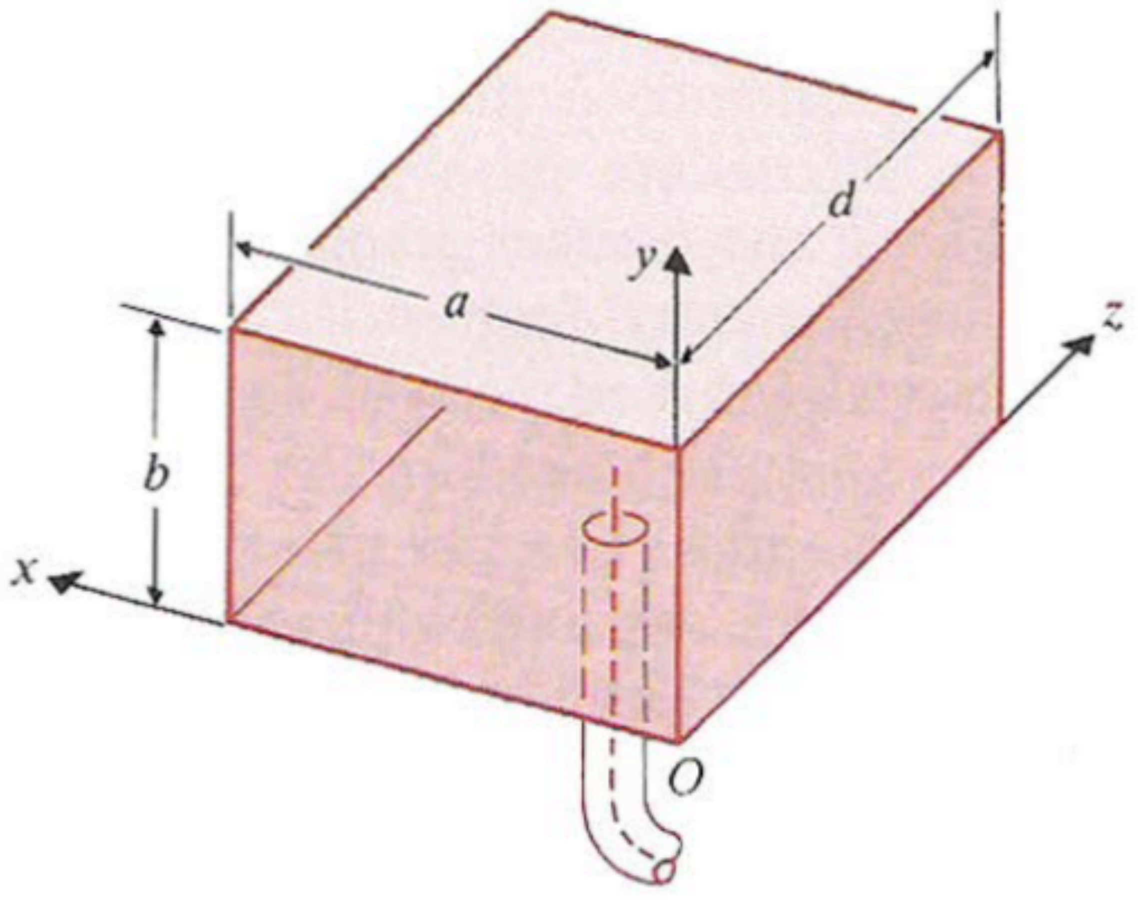
$$\boxed{X(0)=A_{z,xe}=0}\ \text{and}\ \boxed{k_x=\frac{m\pi}{a}}$$

  - Thus, the solution form be simplified as

$$\boxed{E_z(x,y,z)=\sin k_x x\sin k_y y\left(E_{z,ze}\cos k_z z+E_{z,zo}\sin k_z z\right)\quad\cdots(7)}$$

where $E_{z,ze}=A_{z,ze}A_{z,xo}A_{z,yo}$ and $E_{z,zo}=A_{z,zo}A_{z,xo}A_{z,yo}$

Chap. 10 | Derivation for E-fields for cavity resonators (3/3)



- *Derivation for E-fields*
  - By going through the same procedures for Ex and Ey, we can obtain the following set of equations

$$\begin{cases} E_x(x,y,z) = \sin k_y y \sin k_z z \left(E_{x,xe} \cos k_x x + E_{x,xo} \sin k_x x\right) \\ E_y(x,y,z) = \sin k_x x \sin k_z z \left(E_{y,ye} \cos k_y y + E_{y,yo} \sin k_y z\right) \\ E_z(x,y,z) = \sin k_x x \sin k_y y \left(E_{z,ze} \cos k_z z + E_{z,zo} \sin k_z z\right) \end{cases} \quad \cdots(8) \quad \text{where } k_x = \frac{m\pi}{a},\ k_y = \frac{n\pi}{b},\ k_z = \frac{p\pi}{d}$$

(m, n, p: Integers)

- We can substitute equation (8) into (9) (Gauss Law)

$$\boxed{\nabla \cdot \boldsymbol{E} = 0 \quad \rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad \cdots(9)}$$

$$\frac{\partial E_x}{\partial x} = \sin k_y y \sin k_z z \left(-k_x E_{x,xe} \sin k_x x + k_x E_{x,xo} \cos k_x x\right)$$

$$\frac{\partial E_y}{\partial y} = \sin k_x x \sin k_z z \left(-k_y E_{y,ye} \sin k_y y + k_y E_{y,yo} \cos k_y y\right)$$

$$\frac{\partial E_z}{\partial z} = \sin k_x x \sin k_y y \left(-k_z E_{z,ze} \sin k_z z + k_z E_{z,zo} \cos k_z z\right)$$

- Equation (9) should hold at all points within the cavity and ***at the walls***
  - ***i.e.*** *At* (0, *y*, *z*), (*x*, 0, *z*), (*x*, *y*, 0)

$$\begin{array}{l} \left.\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right|_{(0,y,z)} = E_{x,xo} \sin k_y y \sin k_z z = 0 \quad \rightarrow \\ \left.\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right|_{(x,0,z)} = E_{y,yo} \sin k_x x \sin k_z z = 0 \quad \rightarrow \\ \left.\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right|_{(x,y,0)} = E_{z,zo} \sin k_x x \sin k_y y = 0 \quad \rightarrow \end{array} \boxed{\begin{array}{l} E_{x,xo} = 0 \\ E_{y,yo} = 0 \\ E_{z,zo} = 0 \end{array}}$$

- If we plug above condition back into equation (8), we get ***(next page)***

Chap. 10 | Derivation for H-fields for cavity resonators (1/2)

- ***Derivation for E-fields***

  - Equation (8) becomes

$$\begin{cases} E_x(x,y,z) = E_{x0}\cos(k_x x)\cdot\sin(k_y y)\cdot\sin(k_z z) \\ E_y(x,y,z) = E_{y0}\sin(k_x x)\cdot\cos(k_y y)\cdot\sin(k_z z) \\ E_z(x,y,z) = E_{z0}\sin(k_x x)\cdot\sin(k_y y)\cdot\cos(k_z z) \end{cases} \quad \cdots(10)$$

  - ***Gauss's law*** yields the following relation

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial x}+\frac{\partial E_z}{\partial x}=-\left(k_x E_{x0}+k_y E_{y0}+k_z E_{z0}\right)\sin k_x x\sin k_y y\sin k_z z=0$$

$$\rightarrow k_x E_{x0}+k_y E_{y0}+k_z E_{z0}=0 \quad \cdots(11)$$

→ Constants are simplified as $E_{xe,x}$ → E_{x0}, $E_{ye,y}$ → E_{y0}, $E_{ze,z}$ → E_{z0} for better readability.

- ***Derivation for H-fields***

  - From Faraday's law (Curl of ***E*** in Maxwell's equations),

$$\nabla\times\boldsymbol{E}=-j\omega\mu\boldsymbol{H}$$

$$\begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\left(\boldsymbol{a_x}H_x+\boldsymbol{a_y}H_y+\boldsymbol{a_z}H_z\right)$$

⇨

$$\begin{cases} H_x=\dfrac{j}{\omega\mu}\left(\dfrac{\partial E_z}{\partial y}-\dfrac{\partial E_y}{\partial z}\right) \\ H_y=\dfrac{j}{\omega\mu}\left(\dfrac{\partial E_x}{\partial z}-\dfrac{\partial E_z}{\partial x}\right) \\ H_z=\dfrac{j}{\omega\mu}\left(\dfrac{\partial E_y}{\partial x}-\dfrac{\partial E_x}{\partial y}\right) \end{cases} \quad \cdots(12)$$

Magnetic fields in terms of Electric fields

Chap. 10 | Derivation for H-fields for cavity resonators (2/2)

- ***Derivation for H-fields***
  - Now by substituting electric fields equations (10) into magnetic field equations (12),

$$H_x = \frac{j}{\omega\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = \frac{j}{\omega\mu}\left(k_y E_{z0}\sin k_x x\cos k_y y\cos k_z z - k_z E_{y0}\sin k_x x\cos k_y y\cos k_z z\right)$$

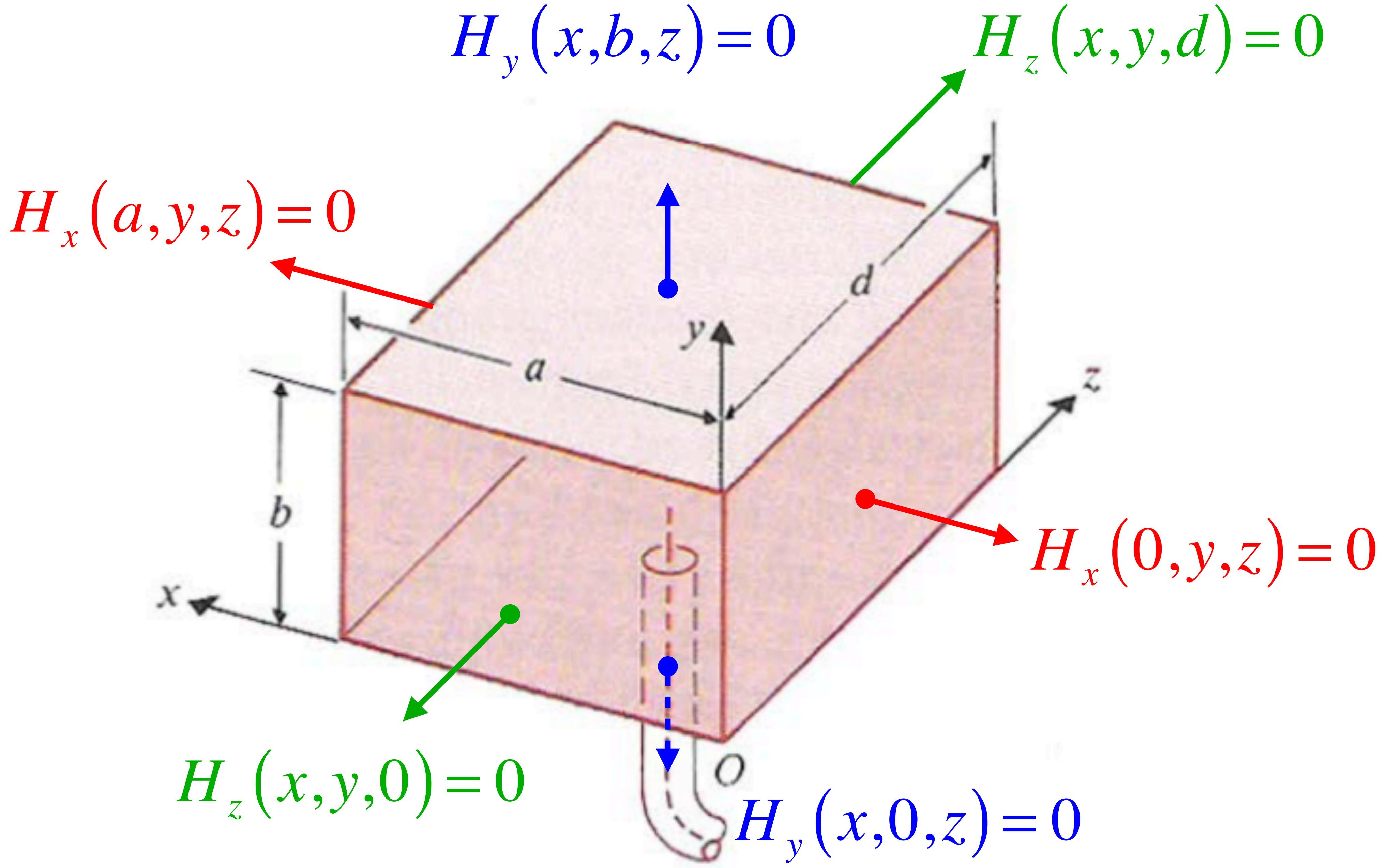
$$= \frac{j}{\omega\mu}\left(k_y E_{z0} - k_z E_{y0}\right)\sin k_x x\cos k_y y\cos k_z z$$

Medium 1 (dielectric)	Medium 2 (Conductor)
$E_{1t} = 0$	$E_{2t} = 0$
$\boldsymbol{a_{n2}} \times \boldsymbol{H_1} = \boldsymbol{J_S}$	$H_{2t} = 0$
$\boldsymbol{a_{n2}} \cdot \boldsymbol{D_1} = \rho_S$	$D_{2n} = 0$
$H_{1n} = 0$	$H_{2n} = 0$

  - We can similarly obtain H_y and H_z as above

$$\boxed{\begin{cases} H_x = \dfrac{j}{\omega\mu}\left(k_y E_{z0} - k_z E_{y0}\right)\sin(k_x x)\cdot\cos(k_y y)\cdot\cos(k_z z) \\ H_y = \dfrac{j}{\omega\mu}\left(k_z E_{x0} - k_x E_{z0}\right)\cos(k_x x)\cdot\sin(k_y y)\cdot\cos(k_z z) \qquad \cdots(13) \\ H_z = \dfrac{j}{\omega\mu}\left(k_x E_{y0} - k_y E_{x0}\right)\cos(k_x x)\cdot\cos(k_y y)\cdot\sin(k_z z) \end{cases}}$$

  - Above magnetic fields satisfy the ***B.C. such that $H_n = 0$!***
  - Above magnetic fields satisfy $\nabla \cdot \boldsymbol{H} = 0$



Chap. 10 | TM modes for rectangular cavity resonators (1/2)

Complete expressions for E and H-fields in cavity

$$\begin{cases} E_x = E_{x0}\cos(k_x x)\cdot\sin(k_y y)\cdot\sin(k_z z) \\ E_y = E_{y0}\sin(k_x x)\cdot\cos(k_y y)\cdot\sin(k_z z) \\ E_z = E_{z0}\sin(k_x x)\cdot\sin(k_y y)\cdot\cos(k_z z) \end{cases}$$

$$\begin{cases} H_x = \dfrac{j}{\omega\mu}\left(k_y E_{z0} - k_z E_{y0}\right)\sin(k_x x)\cdot\cos(k_y y)\cdot\cos(k_z z) \\ H_y = \dfrac{j}{\omega\mu}\left(k_z E_{x0} - k_x E_{z0}\right)\cos(k_x x)\cdot\sin(k_y y)\cdot\cos(k_z z) \\ H_z = \dfrac{j}{\omega\mu}\left(k_x E_{y0} - k_y E_{x0}\right)\cos(k_x x)\cdot\cos(k_y y)\cdot\sin(k_z z) \end{cases}$$

where $k_x = \dfrac{m\pi}{a},\ k_y = \dfrac{n\pi}{b},\ k_z = \dfrac{p\pi}{d}$

$(m, n, p : \text{Interger values})$

and $k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0$ ***from Gauss's Law***

- *TM modes*
  - $H_z = 0$

$$\begin{cases} k_x E_{y0} - k_y E_{x0} = 0 & \text{(from } H_z = 0\text{)} \\ k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0 & \text{(from Gauss's Law)} \end{cases}$$

  - Two equations with three variables → ***E_{x0}, E_{y0} in terms of E_{z0}***

$$\begin{cases} E_x = -E_{z0}\dfrac{k_x k_z}{k_x^2 + k_y^2}\cos(k_x x)\cdot\sin(k_y y)\cdot\sin(k_z z) \\ E_y = -E_{z0}\dfrac{k_y k_z}{k_x^2 + k_y^2}\sin(k_x x)\cdot\cos(k_y y)\cdot\sin(k_z z) \\ E_z = E_{z0}\sin(k_x x)\cdot\sin(k_y y)\cdot\cos(k_z z) \end{cases}$$

$$\begin{cases} H_x = j\omega\varepsilon E_{z0}\dfrac{k_y}{k_x^2 + k_y^2}\sin(k_x x)\cdot\cos(k_y y)\cdot\cos(k_z z) \\ H_y = -j\omega\varepsilon E_{z0}\dfrac{k_x}{k_x^2 + k_y^2}\sin(k_x x)\cdot\cos(k_y y)\cdot\sin(k_z z) \end{cases}$$

Chap. 10 | TM modes for rectangular cavity resonators (2/2)

- *TM modes (in the textbook notation)*

$$\gamma^2 + k^2 = h^2$$

where $\gamma = jk_z$ & $k^2 = k_x^2 + k_y^2 + k_z^2 \rightarrow h^2 = k_x^2 + k_y^2$

and $k_x = \frac{m\pi}{a},\ k_y = \frac{n\pi}{b},\ k_z = \frac{p\pi}{d}$

$$\begin{cases} E_x = -\frac{E_{z0}}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right) \\ E_y = -\frac{E_{z0}}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) \\ E_z = E_{z0}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) \end{cases}$$

$$\begin{cases} H_x = \frac{j\omega\varepsilon E_{z0}}{h^2}\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi x}{b}\right)\cos\left(\frac{p\pi x}{d}\right) \\ H_y = -\frac{j\omega\varepsilon E_{z0}}{h^2}\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)\cos\left(\frac{p\pi x}{d}\right) \end{cases}$$

- No power flow in any directions

- All the E-fields are *in time phase*
- E-fields and H-fields are ***in time quadrature*** (π/2 phase difference)

$$\rightarrow \quad \therefore \boldsymbol{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E}\times\boldsymbol{H}^*\right) = 0$$

- Resonant frequency

- By definition,

$$k^2 = \omega^2\mu\varepsilon = k_x^2 + k_y^2 + k_z^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$$\rightarrow \omega_r = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

$$\rightarrow f_r = \frac{u}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

- Lowest resonant frequency for TM wave:

  m ≠ 0 & *n* ≠ 0 (Not to make $E_z = 0$) → ***TM$_{110}$ mode!***

Chap. 10 | TE & TM modes for rectangular cavity resonators

- *TE modes*

$$\begin{cases} E_x = \dfrac{j\omega\mu H_{z0}}{h^2}\left(\dfrac{n\pi}{b}\right)\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right)\sin\left(\dfrac{p\pi z}{d}\right) \\ E_y = -\dfrac{j\omega\mu H_{z0}}{h^2}\left(\dfrac{m\pi}{a}\right)\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)\cos\left(\dfrac{p\pi z}{d}\right) \end{cases}$$

$$\begin{cases} H_x = -\dfrac{H_{z0}}{h^2}\left(\dfrac{m\pi}{a}\right)\left(\dfrac{p\pi}{d}\right)\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi x}{b}\right)\cos\left(\dfrac{p\pi x}{d}\right) \\ H_y = -\dfrac{H_{z0}}{h^2}\left(\dfrac{n\pi}{b}\right)\left(\dfrac{p\pi}{d}\right)\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi x}{b}\right)\cos\left(\dfrac{p\pi x}{d}\right) \\ H_z = H_{z0}\cos\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi x}{b}\right)\sin\left(\dfrac{p\pi x}{d}\right) \end{cases}$$

- Resonant frequency

- Exactly same as TM modes:

$$k^2 = \omega^2\mu\varepsilon = k_x^2 + k_y^2 + k_z^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$$\boxed{\rightarrow f_r = \frac{u}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}}$$

- Different modes with same resonant frequency: ***Degenerate modes***
- Lowest resonant frequency for TE wave:

  $p \neq 0$ & ($n \neq 0 + m \neq 0$) (Not to make $H_z = 0$) → ***TE$_{011}$* or *TE$_{101}$ mode!***

- Dominant mode of rectangular cavity resonators?

- TE$_{011}$, TE$_{101}$, and TM$_{110}$ should be compared

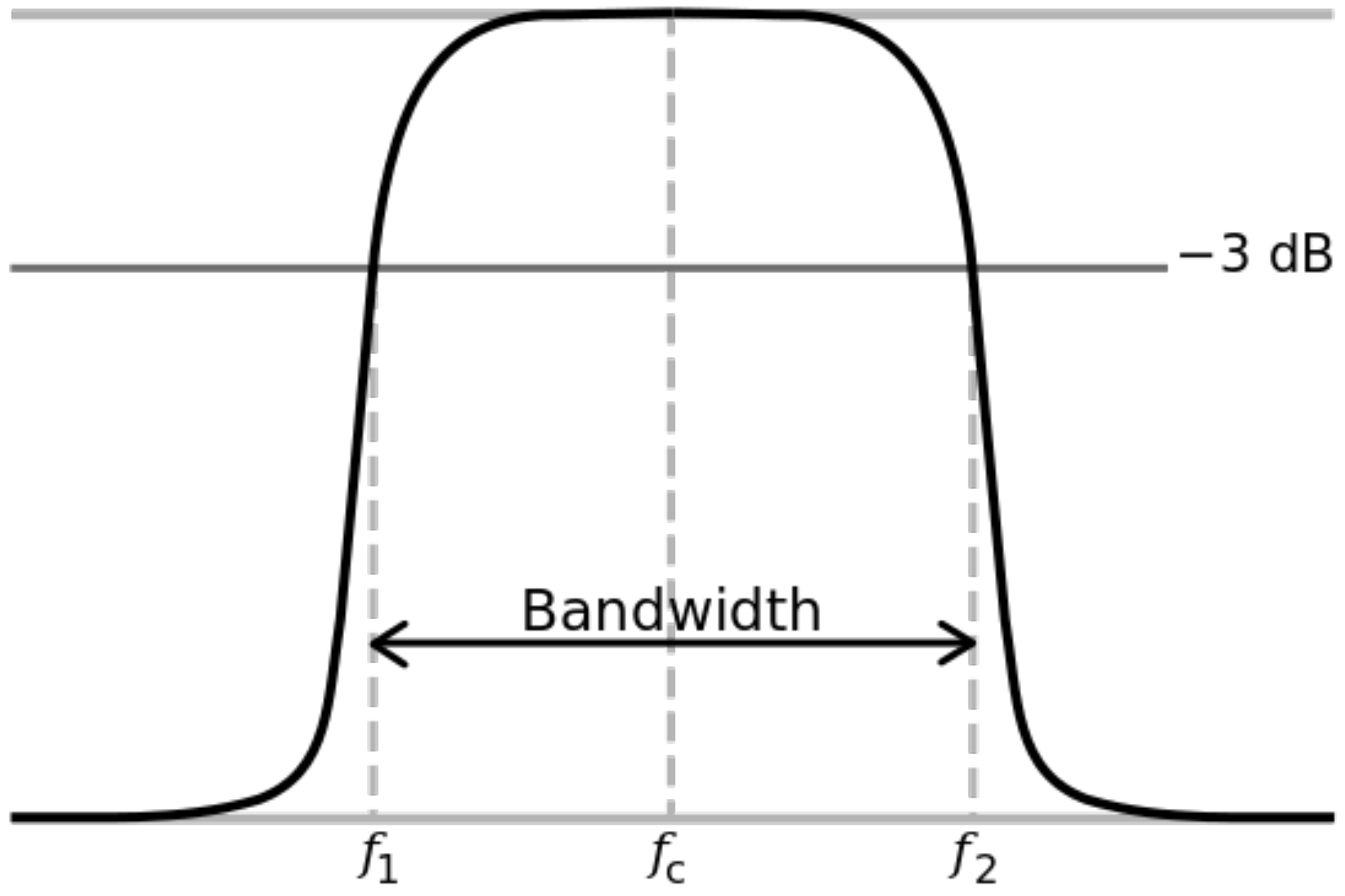
$$f_{TE011} = \frac{u}{2}\sqrt{\left(\frac{1}{b}\right)^2 + \left(\frac{1}{d}\right)^2},\ f_{TE101} = \frac{u}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2},\ f_{TM110} = \frac{u}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

- Depending on the dimensions of cavity!
  - If $a > b > d$ → TM110 dominant!
  - If $a > d > b$ → TE101 dominant!
  - If $a = b = d$ → all three are degenerately dominant

Chap. 10 | Quality factor, *Q*

- ***Comments on circular cavity resonator***
  - *E* and *H*-fields can be obtained by ***using cylindrical coordinates***
  - However, derivation is quite complicated and beyond our scope; Please refer to ***microwave engineering*** for details

- ***Quality factor, Q***
  - ***Energy loss in the cavity resonator***
    - Cavity stores EM energy in the form of ***standing waves*** for particular modes
    - Amplification of wave by resonance → ***infinity***, if ***energy loss*** = 0
    - Finite conductivity (σ) of the walls → ***energy loss per reflection (Eventually attenuated unless continuously supported)***



Resonant frequency

$$\boxed{\begin{aligned} Q &= 2\pi \frac{\text{Time-average stored energy at a resonant frequency}}{\text{Energy lost per cycle}} \\ &= 2\pi f_r \frac{\text{Time-average stored energy at a resonant frequency}}{\text{Energy loss}} = \omega \frac{W}{P_L} \end{aligned}}$$

$$W = W_e + W_m = \frac{1}{4}\mathrm{Re}\left(\varepsilon \boldsymbol{E}\cdot\boldsymbol{E}^* + \mu \boldsymbol{H}\cdot\boldsymbol{H}^*\right)$$

$$P_L = \oint_S \frac{1}{2}|\boldsymbol{J}_s| R_s ds = \oint_S \frac{1}{2}|\boldsymbol{H}_t| R_s ds$$

Where $\boldsymbol{J_s}$ = surface current density,

$\boldsymbol{H_t}$: tangential H-fields,

R_s: wall resistance

Practice it! (Example 10-17)

Electromagnetics

<Chap. 10> Transmission Lines

Section 9.1 ~ 9.2

(2nd of **week 9**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 9 | Contents for 2nd class of week 9

Sec 1. Introduction

Sec 2. Transverse Electromagnetic Wave along a Parallel-Plate Transmission Line

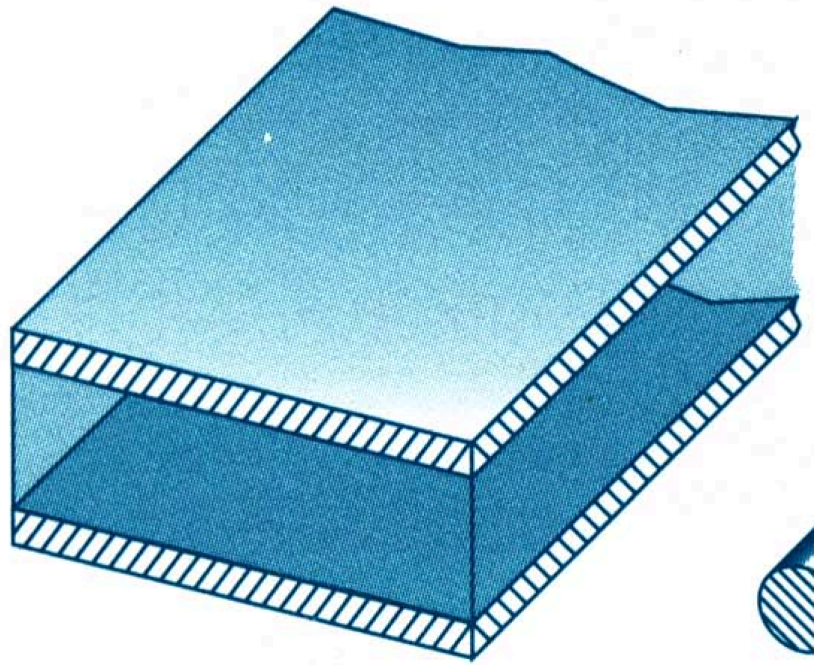
- Derivation of general expression for TEM waves
- Ideal transmission-line equations
- Equivalent circuit model

Chap. 9 | Intro to transmission lines

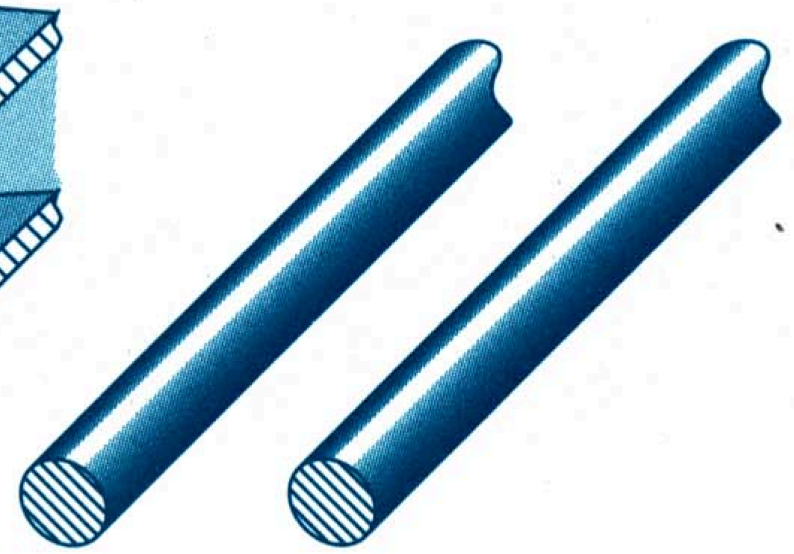
- *Transmission line (TR line)*
  - *A pair of electric conductors*
  - Used as ***cables*** for efficient transmission of ***AC signal*** at distance ***at radio frequency (RF > 30 kHz where wave characteristics matters)***
  - Why?
    - Signal ***radiates off*** the regular electric cables at *RF* → Loss! *(∵ Antenna [Chap. 12])*
    - Signal ***reflected at*** connectors or joints at *RF* → Loss! *(∵ Impedance miss-matching [Sec. 9-7])*
  - Signal guided within TR lines in form of ***"TEM" wave***
  - Easy to do "Impedance-matching" → Minimized reflection loss

<Power line: two-wire>

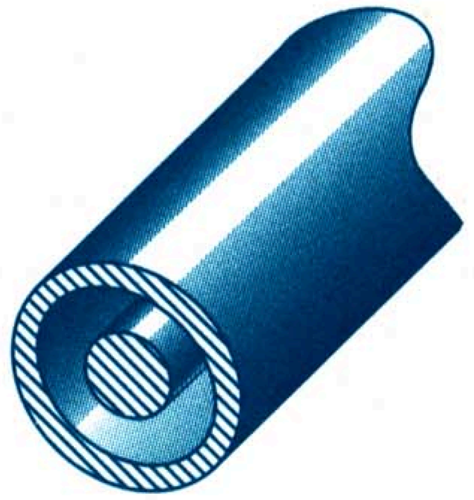
<TV cables: coaxial>



<Parallel-plate>



<Two-wire>



<Coaxial>

	Waveguide	**Transmission lines**
Structure	Hollow metallic structure through which EM propagates	A pair of conductors carrying AC electrical signal
Operating modes	TE and TM modes	TEM or quasi-TEM modes
Operating frequency	Microwave (0.3 ~ 300 GHz)	Radio frequency (30 kHz ~ 300 GHz)

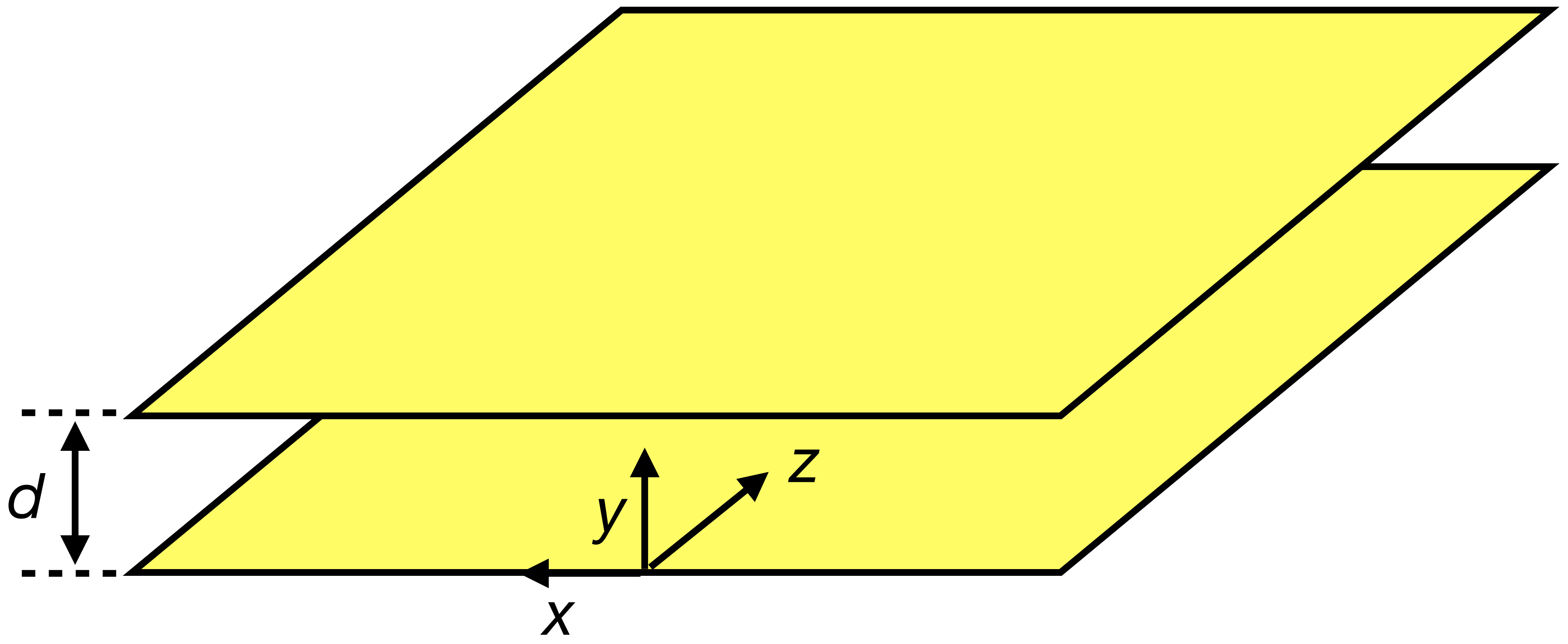
$$R_{Tr} \propto \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

→ ***Significant loss at microwave frequency for TR lines!***

Img src: alphaplatforms.com, Sewell Direct

Chap. 9 | Infinite parallel-plate TR line

- ***Infinite parallel-plate TR line***
  - Two ***perfectly conducting*** plates ($\sigma_c \to \infty$) separated by a ***dielectric*** medium (μ, ε)
  - All ***TEM, TM, TE*** waves ***propagating in z-direction***
  - ***Infinite in extent*** in *x*-direction
    - ***Fields do not vary in x-direction*** → $\dfrac{\partial \boldsymbol{E}}{\partial x}=0,\ \dfrac{\partial \boldsymbol{H}}{\partial x}=0\ \left(\boldsymbol{E}\neq 0,\ \boldsymbol{H}\neq 0\right)$



- ***Electric and magnetic fields for TM modes (H*** $H_z = 0$***)***
  - Wave equation for E_z

$$\nabla^2 E_z + k^2 E_z = 0, \quad \text{where} \ \ E_z(y,z) = E_z^0(y)e^{-\gamma z}$$

$$\boxed{\rightarrow \ \frac{d^2E_z^0}{dy^2} + h^2E_z^0 = 0 \ \left(\because h^2 = k^2+\gamma^2\right)}$$

  - Boundary condition ($E_t = 0$ at conducting interface)

$$E_z^0(y) = 0, \quad \text{where} \ \ y=0 \ \ \text{and} \ \ y=d$$

  - Solution

$$\boxed{E_z^0(y) = A_n\sin(hy) = A_n\sin\left(\frac{n\pi}{d}y\right), \quad (n=1,2,\ldots)}$$

  - Transverse field components

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma\cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu\cancel{\dfrac{\partial H_z^0}{\partial y}}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial E_z^0}{\partial y} - j\omega\mu\cancel{\dfrac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma\cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon\dfrac{\partial E_z^0}{\partial y}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma\cancel{\dfrac{\partial H_z^0}{\partial y}} + j\omega\varepsilon\cancel{\dfrac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

➡

$$\boxed{\begin{cases} E_x^0(y) = 0 \\ E_y^0(y) = -\dfrac{\gamma}{h^2}A_n\cos\left(\dfrac{n\pi y}{d}\right) \\ H_x^0(y) = \dfrac{j\omega\varepsilon}{h}A_n\cos\left(\dfrac{n\pi y}{d}\right) \\ H_y^0(y) = 0 \end{cases}}$$

Chap. 9 | TEM mode in Parallel-plate TR line

- *Special case of TM modes = TEM mode*

Longitudinal:
$$\begin{cases} E_z(y,z) = A_n \sin\left(\dfrac{n\pi}{d}y\right)e^{-\gamma z} \\ H_z(y,z) = 0 \end{cases}$$

Transverse:
$$\begin{cases} E_x(y,z) = 0 \\ E_y(y,z) = -\dfrac{\gamma}{h^2}A_n \cos\left(\dfrac{n\pi y}{d}\right)e^{-\gamma z} \\ H_x(y,z) = \dfrac{j\omega\varepsilon}{h}A_n \cos\left(\dfrac{n\pi y}{d}\right)e^{-\gamma z} \\ H_y(y,z) = 0 \end{cases}$$

Propagation constant:
$$\gamma = \sqrt{h^2 - k^2} = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \omega^2\mu\varepsilon}$$

Cutoff frequency ($\gamma = 0$)
$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2d\sqrt{\mu\varepsilon}} \ \text{(Hz)}$$

What if
n = 0?
(h → 0)

Longitudinal:
$$\begin{cases} E_z(y,z) = 0 \\ H_z(y,z) = 0 \end{cases}$$

Transverse:
$$\begin{cases} E_x(y,z) = 0 \\ E_y(y,z) = E_0 e^{-\gamma z} \\ H_x(y,z) = -\dfrac{E_0}{\eta}e^{-\gamma z} \quad \text{where} \ \ \eta = -\dfrac{E_y}{H_x} = \sqrt{\dfrac{\mu}{\varepsilon}} \\ H_y(y,z) = 0 \end{cases}$$

Propagation constant:
$$\gamma = \sqrt{-k^2} = j\omega\sqrt{\mu\varepsilon} \triangleq j\beta$$

Cutoff frequency
$$f_c = 0$$

- TM_0 = TEM!
- TEM is ***a dominant mode of the parallel-plate***! (∵ lowest f_c)

Chap. 9 | Derivation of TEM mode in TR line (1/2)

- *Derivation of the TEM mode*

$$\nabla\times\boldsymbol{E} = -j\omega\mu\boldsymbol{H}$$

$$\begin{cases} \cancel{\dfrac{\partial E_z}{\partial y}} - \dfrac{\partial E_y}{\partial z} = -j\omega\mu H_x & \cdots(\mathrm{a})\\ -\cancel{\dfrac{\partial E_z}{\partial x}} + \dfrac{\partial E_x}{\partial z} = -j\omega\mu H_y & \cdots(\mathrm{b})\\ \cancel{\dfrac{\partial E_y}{\partial x}} - \dfrac{\partial E_x}{\partial y} = \cancel{-j\omega\mu H_z} & \cdots(\mathrm{c})\end{cases}$$

$$\nabla\times\boldsymbol{H} = j\omega\varepsilon\boldsymbol{E}$$

$$\begin{cases} \cancel{\dfrac{\partial H_z}{\partial y}} + \dfrac{\partial H_y}{\partial z} = j\omega\varepsilon E_x & \cdots(\mathrm{e})\\ -\cancel{\dfrac{\partial H_z}{\partial x}} + \dfrac{\partial H_x}{\partial z} = j\omega\varepsilon E_y & \cdots(\mathrm{f})\\ \cancel{\dfrac{\partial H_y}{\partial x}} - \dfrac{\partial H_x}{\partial y} = \cancel{j\omega\varepsilon E_z} & \cdots(\mathrm{g})\end{cases}$$

where $\begin{cases} E_z(y,z)=0\\ H_z(y,z)=0\end{cases}$ and $\dfrac{\partial\boldsymbol{E}}{\partial x}=0,\ \dfrac{\partial\boldsymbol{H}}{\partial x}=0$

(by definition) *(Assumption)*

$\begin{cases} \boldsymbol{E}_t=0\\ \boldsymbol{H}_n=0\end{cases}$ *at the conducting boundary (y = 0 and y = b)*

- From equations **(c)**, we know that

$E_x(y,z)=C\cdot E_x^0(z)$. From B.C., $E_x(d \text{ or } 0,z)=0 \ \rightarrow\ C=0$

$$\boxed{\therefore E_x(y,z)=0}$$

- By substituting $E_x = 0$ into equation **(b)**, we get

$$\boxed{\therefore H_y(y,z)=0}$$

- Equation **(e)** also vanishes accordingly, since $E_x = H_y = 0$.

- From equation **(g)**, we know that

$$\boxed{H_x(y,z)=H_0H_x^0(z) \quad \cdots(1)}$$

- By differentiating equation **(f)** with *z*,

equation **(a)**

$$\frac{\partial^2 H_x}{\partial z^2}=\left[j\omega\varepsilon\frac{\partial E_y}{\partial z}=j\omega\varepsilon(j\omega\mu H_x)=-\omega^2\mu\varepsilon H_x\right]$$

$$\boxed{\rightarrow \frac{d^2H_x}{dz^2}+\omega^2\mu\varepsilon H_x=0 \quad \cdots(2)}$$

(Why ODE, not PDE?)

Chap. 9 | Derivation of TEM mode in TR line (2/2)

- ***Derivation of the TEM mode***

  - By solving the ODE,

$$\frac{d^2 H_x}{dz^2} + \omega^2 \mu\varepsilon H_x = 0 \quad \rightarrow \quad H_x(y,z) = H_0 e^{-j(\omega\sqrt{\mu\varepsilon})z} + \cancel{H_1} e^{j(\omega\sqrt{\mu\varepsilon})z}$$

(∵there is no reflection!)

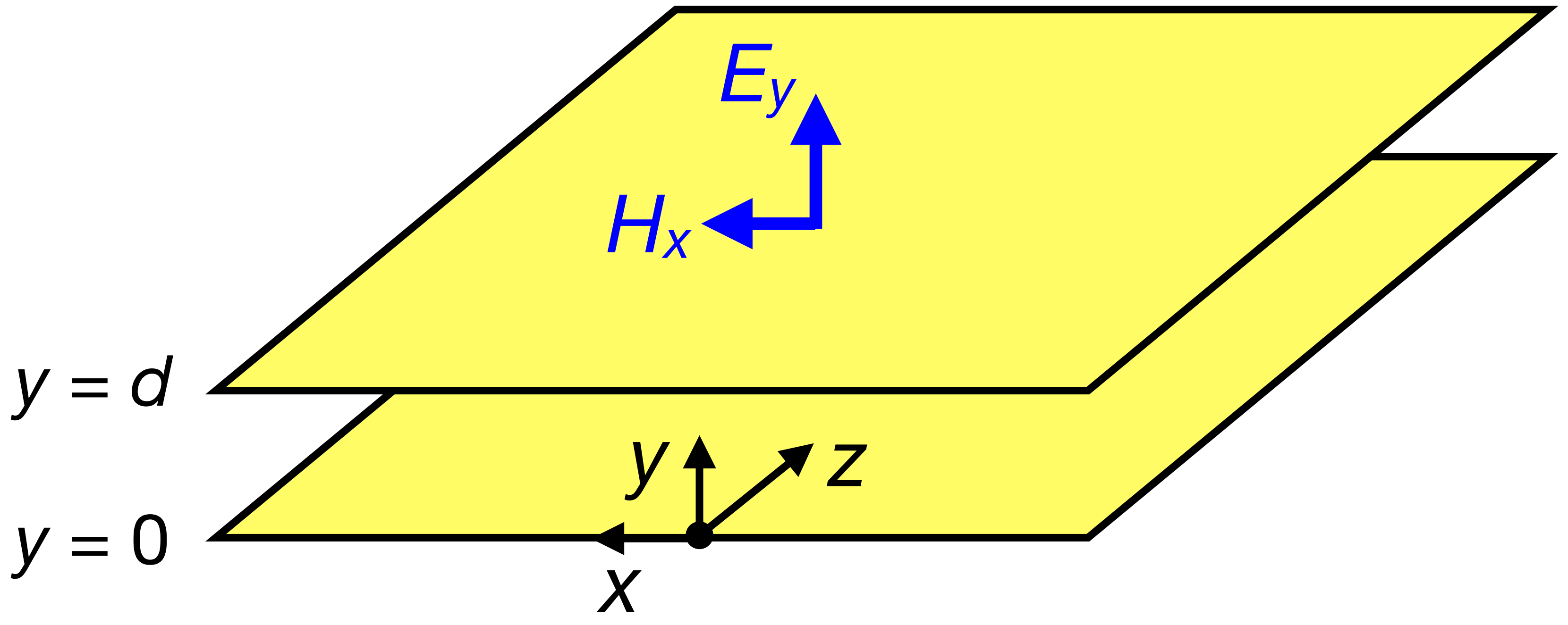
$$\therefore H_x(y,z) = H_0 H_x^0(z) = H_0 e^{-j\beta z}$$

where $\beta = \omega\sqrt{\mu\varepsilon}$

  - By substituting H_x back into equation **(f)**, we get

$$\frac{\partial H_x}{\partial z} = j\omega\varepsilon E_y \quad \rightarrow \quad E_y = \frac{1}{j\omega\varepsilon}\frac{\partial H_x}{\partial z} = \frac{1}{j\omega\varepsilon}\cdot\left(-j\beta H_0 e^{-\beta z}\right)$$

$$\therefore E_y(y,z) = -\sqrt{\frac{\mu}{\varepsilon}} H_0 e^{-j\beta z} = E_0 e^{-j\beta z}$$



- In summary, TEM wave characterized as

$$\begin{cases} E_z(y,z) = 0 \\ H_z(y,z) = 0 \end{cases}$$

$$\begin{cases} E_y(y,z) = E_0 e^{-j\beta z} \\ H_x(y,z) = -\dfrac{E_0}{\eta} e^{-j\beta z} \end{cases}$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ and $-\dfrac{E_y}{H_x} = \sqrt{\dfrac{\mu}{\varepsilon}} \triangleq \eta$

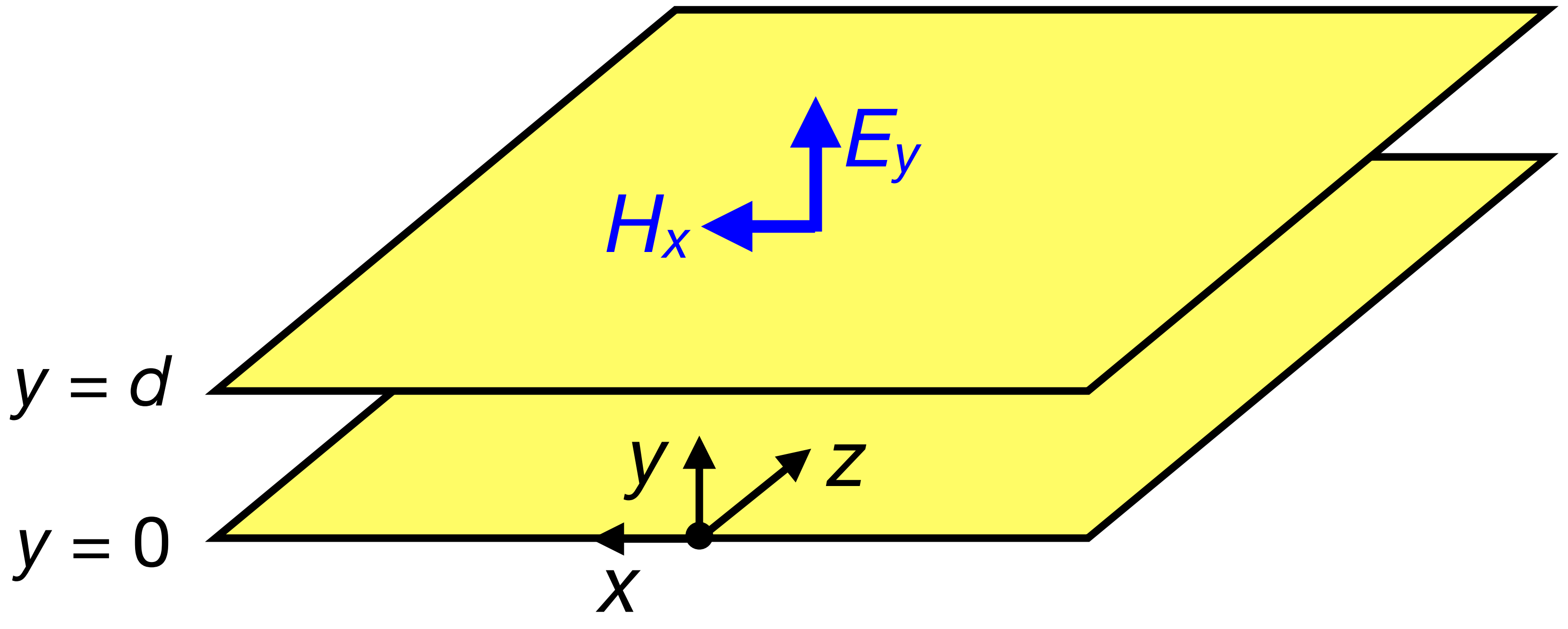
Characteristics of "TEM waves" guided within TR lines

=

Those of "Uniform plane wave" propagating in an unbounded dielectric

Chap. 9 | TR line equations (1/3)

- *Surface currents and charges at the plates*
  - Recall the B.C. for dielectric / conductor interface



Medium 1 (dielectric)	*Medium 2* (Conductor)	
$E_{1t} = 0$	$E_{2t} = 0$	*tangential*
$\boldsymbol{a_n} \times \boldsymbol{H}_1 = \boldsymbol{J_S}$	$H_{2t} = 0$	
$\boldsymbol{a_n} \cdot \boldsymbol{D}_1 = \rho_S$	$D_{2n} = 0$	*Normal*
$H_{1n} = 0$	$H_{2n} = 0$	

where $\begin{cases} \boldsymbol{H}_1 = \boldsymbol{a_x} H_x \\ \boldsymbol{D}_1 = \boldsymbol{a_y} \varepsilon E_y \end{cases}$, $\boldsymbol{a}_n$ → *surface normal* from conductor to dielectric

At the ***upper*** plate (*y* = *d*)

- $\boldsymbol{a}_n = -\boldsymbol{a}_y$

$$-\boldsymbol{a_y} \cdot \left(\boldsymbol{a_y} \varepsilon E_y\right) = \rho_{su} \quad \rightarrow \quad \rho_{su} = -\varepsilon E_y = -\varepsilon E_0 e^{-j\beta z}$$

$$-\boldsymbol{a_y} \times \left(\boldsymbol{a_x} H_x\right) = \boldsymbol{J}_{su} \quad \rightarrow \quad \boldsymbol{J}_{su} = \boldsymbol{a_z} H_x = -\boldsymbol{a_z} \frac{E_0}{\eta} e^{-j\beta z}$$

At the ***lower*** plate (*y* = *0*)

- $\boldsymbol{a}_n = \boldsymbol{a}_y$

$$\boldsymbol{a_y} \cdot \left(\boldsymbol{a_y} \varepsilon E_y\right) = \rho_{sl} \quad \rightarrow \quad \rho_{sl} = \varepsilon E_y = \varepsilon E_0 e^{-j\beta z}$$

$$\boldsymbol{a_y} \times \left(\boldsymbol{a_x} H_x\right) = \boldsymbol{J}_{sl} \quad \rightarrow \quad \boldsymbol{J}_{sl} = -\boldsymbol{a_z} H_x = \boldsymbol{a_z} \frac{E_0}{\eta} e^{-j\beta z}$$

Surface current (J_s) & Surface charges (ρ_s)
varies *sinusoidally* as E_y and H_x!

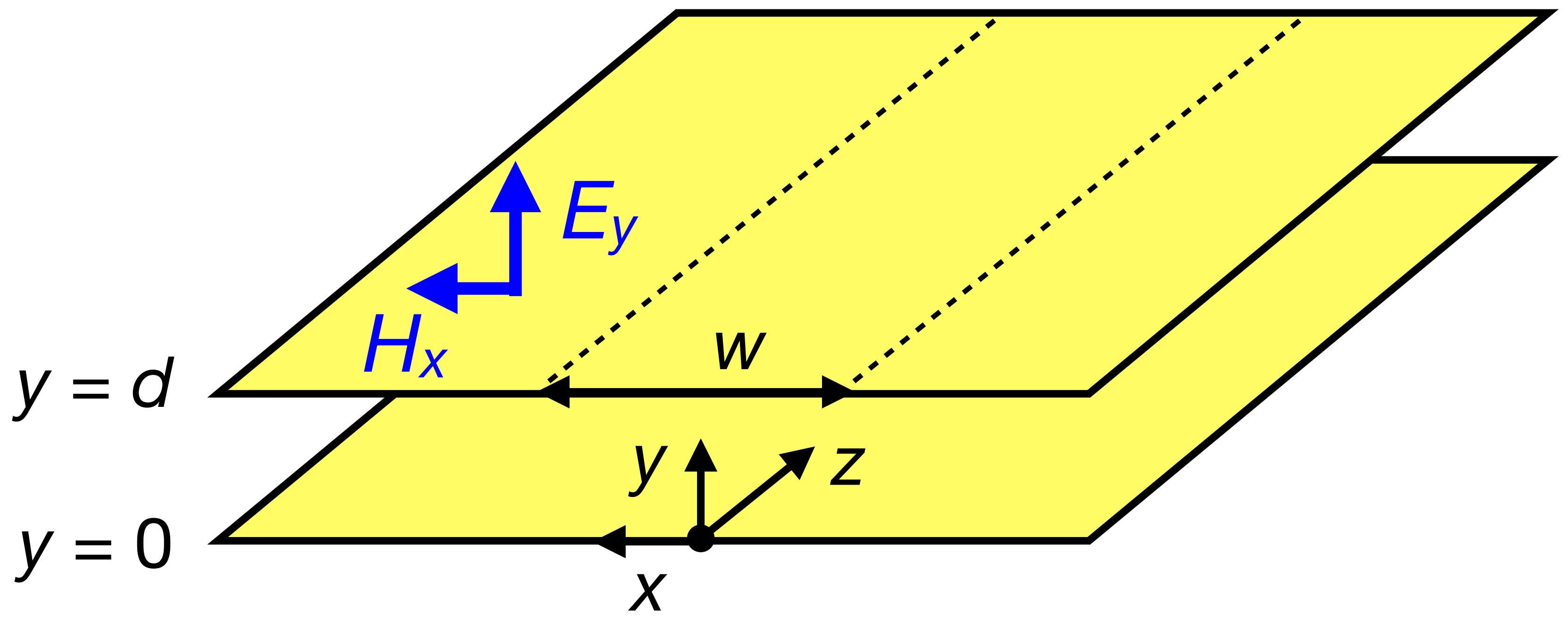
Chap. 9 | TR line equations (2/3)

- *TR line equations*

  - From curl equations **(b)** and **(e)**, we have two ODEs as

$$\begin{cases} \dfrac{dE_y}{dz} = j\omega\mu H_x & \cdots(1) \\ \dfrac{dH_x}{dz} = j\omega\varepsilon E_y & \cdots(2) \end{cases}$$

Why ODE?
(→E_y and H_x only functions of z)



$$\begin{cases} E_y(y,z) = E_0 e^{-j\beta z} \\ H_x(y,z) = -\dfrac{E_0}{\eta} e^{-j\beta z} \end{cases} \quad \text{where } \beta = \omega\sqrt{\mu\varepsilon}$$

$$\text{At } y = d, \quad \begin{cases} \rho_{su} = -\varepsilon E_y \\ \boldsymbol{J}_{su} = \boldsymbol{a_z} H_x \end{cases}$$

$$\text{At } y = 0, \quad \begin{cases} \rho_{sl} = \varepsilon E_y \\ \boldsymbol{J}_{sl} = -\boldsymbol{a_z} H_x \end{cases}$$

  - If we integrate equation **(1)** from *y* = 0 to *y* = *d*

$$\left[\frac{d}{dz}\underline{\int_0^d E_y(y,z)dy} = -\frac{dV(z)}{dz}\right] = \left[j\omega\mu\int_0^d H_x\,dy = j\omega\mu H_x d = j\omega\mu J_{su}(z)d\right]$$

$$\rightarrow -\frac{dV(z)}{dz} = j\omega\underline{\left(\mu\frac{d}{w}\right)}\underline{\left[J_{su}(z)w\right]}$$ (J_{su}: surface current flowing in z-direction)

where $V(z) \triangleq -\int_0^d E_y\,dy$: Potential difference (voltage) between two plates

$L \triangleq \mu\dfrac{d}{w}$ (H/m) : ***Inductance*** per unit length of parallel-plate transmission line

$I(z) \triangleq J_{su}(z)w$: Total current flowing in z-direction in the upper plate

$$\boxed{\therefore -\frac{dV(z)}{dz} = j\omega L I(z) \quad \cdots(3)}$$

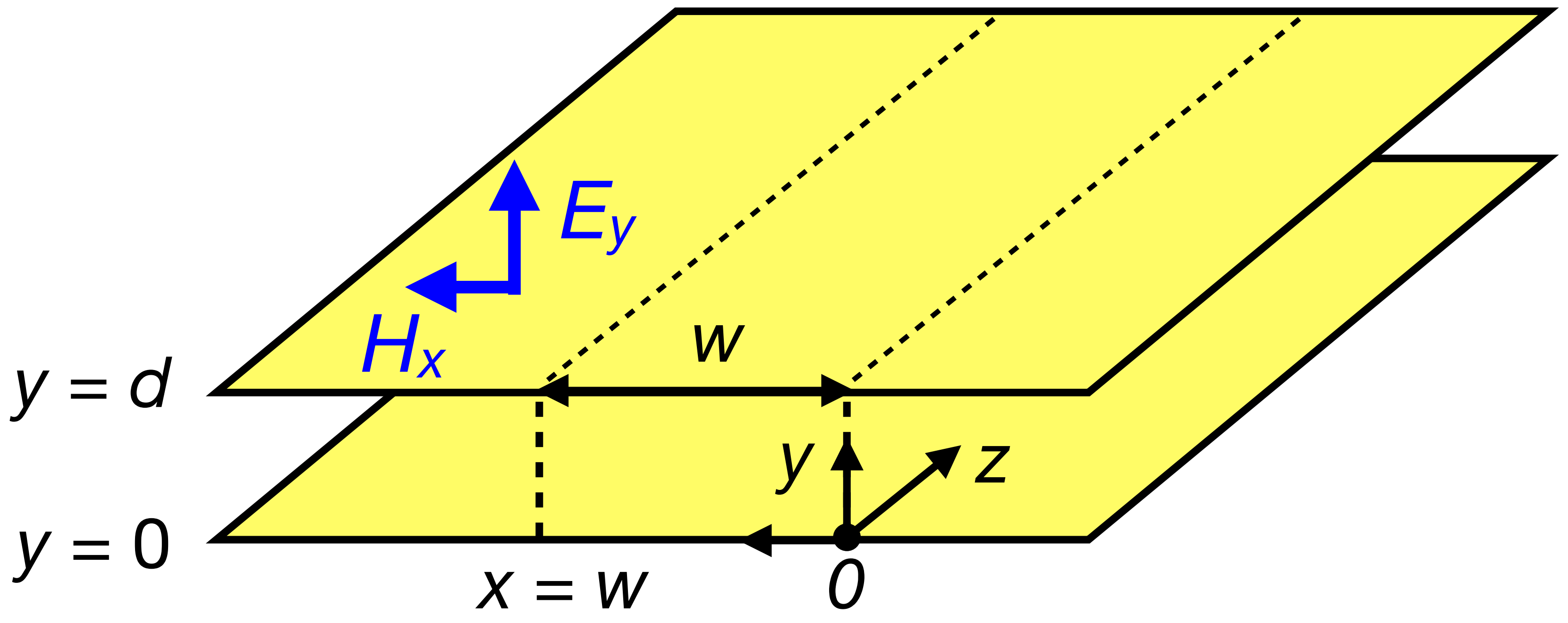
Chap. 9 | TR line equations (3/3)

- *TR line equations*

  - Similarly, by integrating equation **(2)** from <u>$x = 0$ to $x = w$</u>

$$\left[\frac{d}{dz}\int_0^w \underline{H_x(y,z)dx} = \frac{dI(z)}{dz}\right] = \left[j\omega\varepsilon\int_0^w E_y\,dy = j\omega\varepsilon E_y w\right]$$

$$\rightarrow \frac{dI(z)}{dz} = -j\omega\underline{\left(\varepsilon\frac{w}{d}\right)}\underline{\left[-E_y d\right]}$$



$$\boxed{\rightarrow \frac{dI(z)}{dz} = -j\omega C V(z) \quad \cdots(4) \quad \text{where} \quad C \triangleq \varepsilon\frac{w}{d} \ (\text{F/m})}$$: ***Capacitance*** per unit length of parallel-plate transmission line

∴ ***A pair of time-harmonic transmission line equations***

$$\begin{cases} -\dfrac{dV(z)}{dz} = j\omega L I(z) \\[2ex] -\dfrac{dI(z)}{dz} = j\omega C V(z) \end{cases} \Rightarrow \begin{cases} \dfrac{d^2V(z)}{dz^2} = -\omega^2 LCV(z) \\[2ex] \dfrac{d^2I(z)}{dz^2} = -\omega^2 LCI(z) \end{cases} \Rightarrow \boxed{\begin{cases} V(z) = V_0 e^{-j\beta z} \\ I(z) = I_0 e^{-j\beta z} \end{cases} \ \text{where} \ \beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} \quad \because \begin{cases} L \triangleq \mu\dfrac{d}{w} \ (\text{H/m}) \\[2ex] C \triangleq \varepsilon\dfrac{w}{d} \ (\text{F/m}) \end{cases}}$$

- ***Characteristic Impedance*** (By plugging *V*(*z*) and *I*(*z*) to the paired equations)

$$-\frac{d}{dz}\left(V_0 e^{-j\beta z}\right) = j\omega L I_0 e^{-j\beta z}$$

$$\rightarrow j\beta V_0 e^{-j\beta z} = j\omega L I_0 e^{-j\beta z}$$

$$\rightarrow \boxed{Z_0 = \frac{V(z)}{I(z)} = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}} = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}} = \frac{d}{w}\eta \ \ (\Omega)}$$

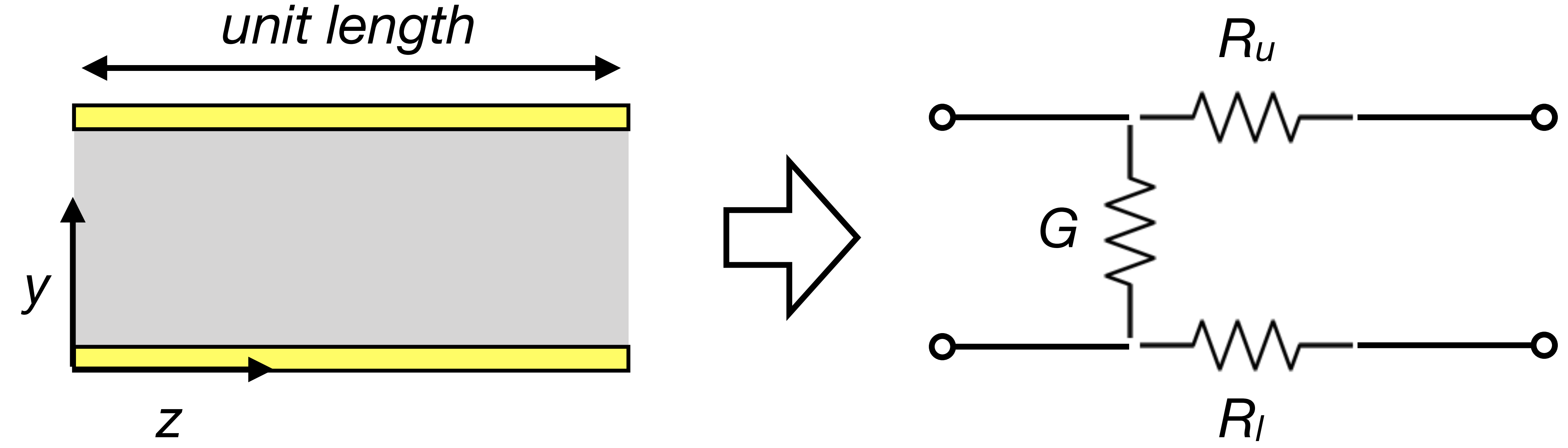
- ***Velocity of propagation***

$$\boxed{u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} \quad (m/s)}$$

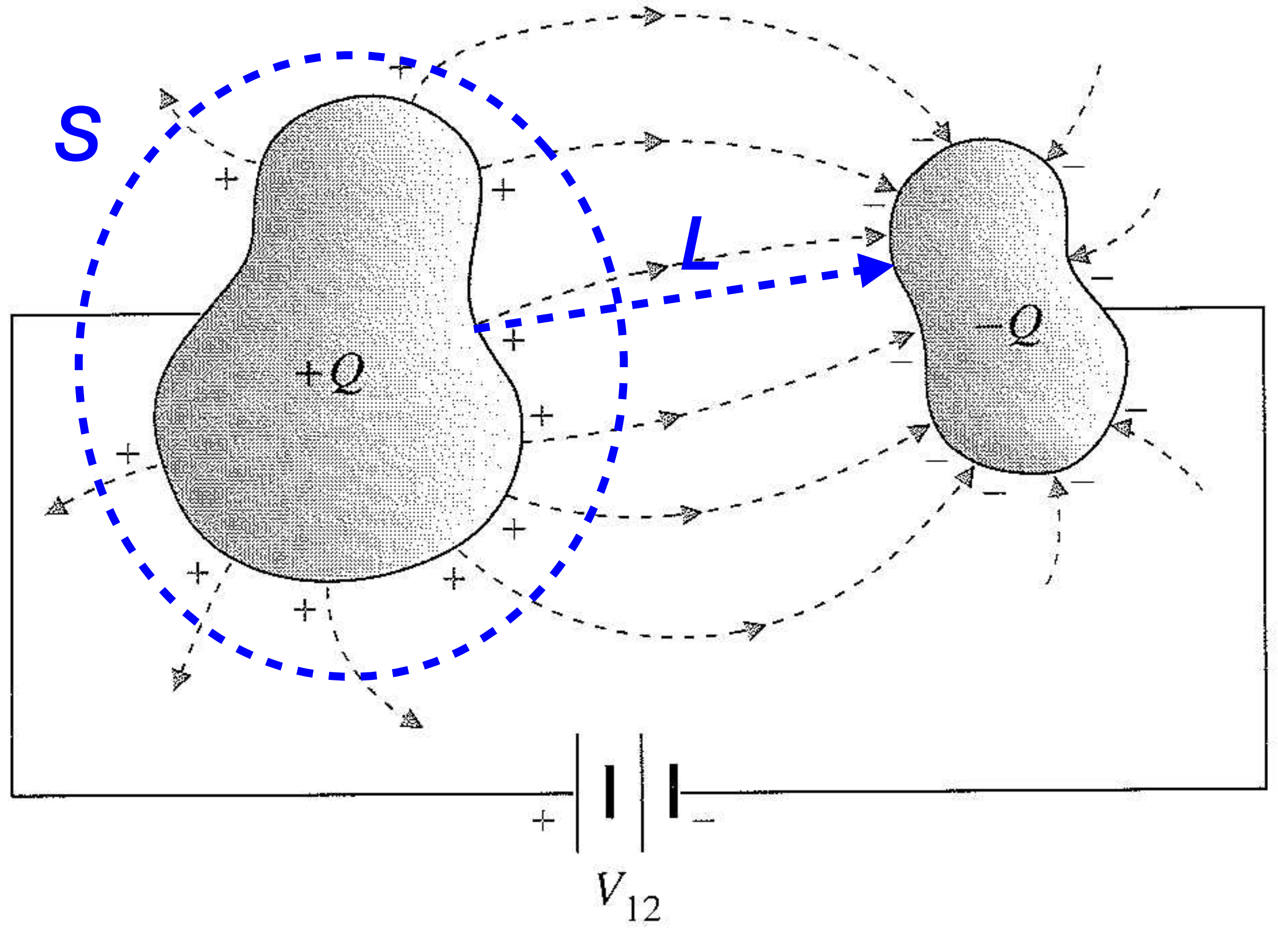
Chap. 9 | Lossy TR lines: Equivalent circuit model (1/3)

- ***Attenuation in the parallel-plate transmission lines caused by…***
  - (1) Lossy dielectric ($\sigma \neq 0$)
  - (2) Imperfectly conducting walls ($\sigma_c \neq \infty$)

$$\alpha = \alpha_d + \alpha_c = \frac{\sigma}{2}\eta + \frac{1}{d}\sqrt{\frac{\pi f \varepsilon}{\sigma_c}}$$ (Will be derived in next class)



<Circuit representation>



$$C = \frac{Q}{V} = \frac{\oint_S \boldsymbol{D}\cdot d\boldsymbol{s}}{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}} = \frac{\oint_S \varepsilon\boldsymbol{E}\cdot d\boldsymbol{s}}{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}$$

$$R = \frac{V}{I} = \frac{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}{\oint_S \boldsymbol{J}\cdot d\boldsymbol{s}} = \frac{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}{\oint_S \sigma\boldsymbol{E}\cdot d\boldsymbol{s}}$$

$$\rightarrow RC = \frac{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}{\sigma\oint_S \boldsymbol{E}\cdot d\boldsymbol{s}} \cdot \frac{\varepsilon\oint_S \boldsymbol{E}\cdot d\boldsymbol{s}}{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}} = \frac{\varepsilon}{\sigma} = \frac{C}{G}$$

- ***Conductance (G) between two conductors per unit length***

$$G = C\frac{\sigma}{\varepsilon}$$ *(from right)*

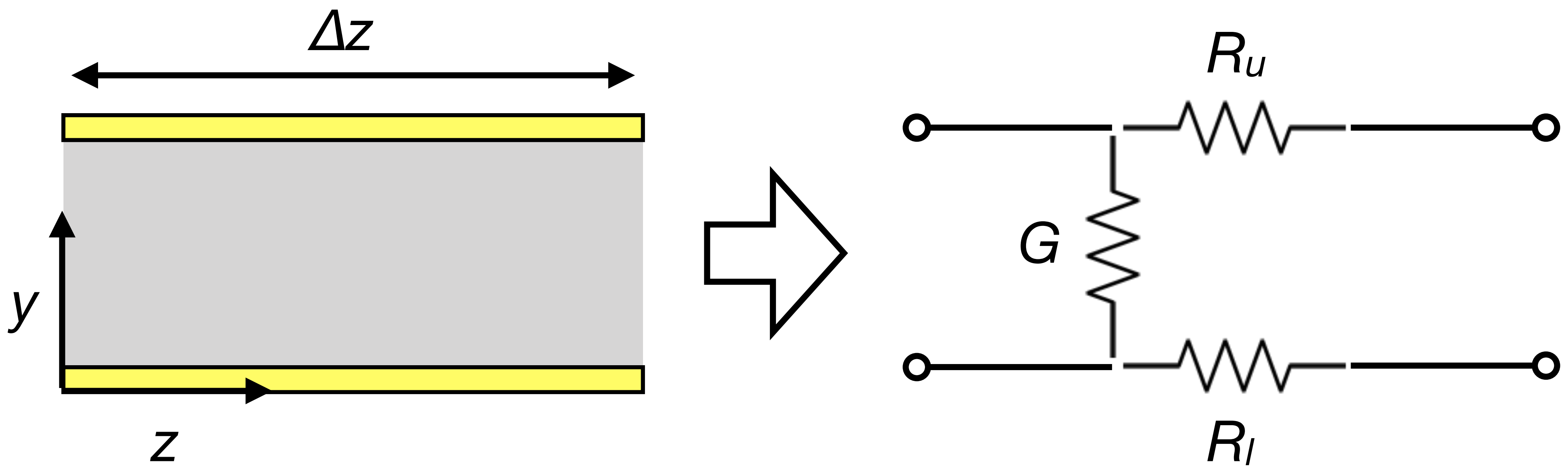
$$= \varepsilon\frac{w}{d}\cdot\frac{\sigma}{\varepsilon} = \sigma\frac{w}{d} \quad \text{(S/m)}$$

where σ is the conductivity of the dielectric

Chap. 9 | Lossy TR lines: Equivalent circuit model (2/3)

- ***Resistance** (R) along the conductors per unit length*
  - In actual cases, conductivity of the plate is ***finite*** ($\sigma_c \neq \infty$)
  - ∴*small, yet non-vanishing axial field (E_z) "induced!"* $\left(\because \mathbf{J}_s = \sigma_c \mathbf{E}_z\right)$

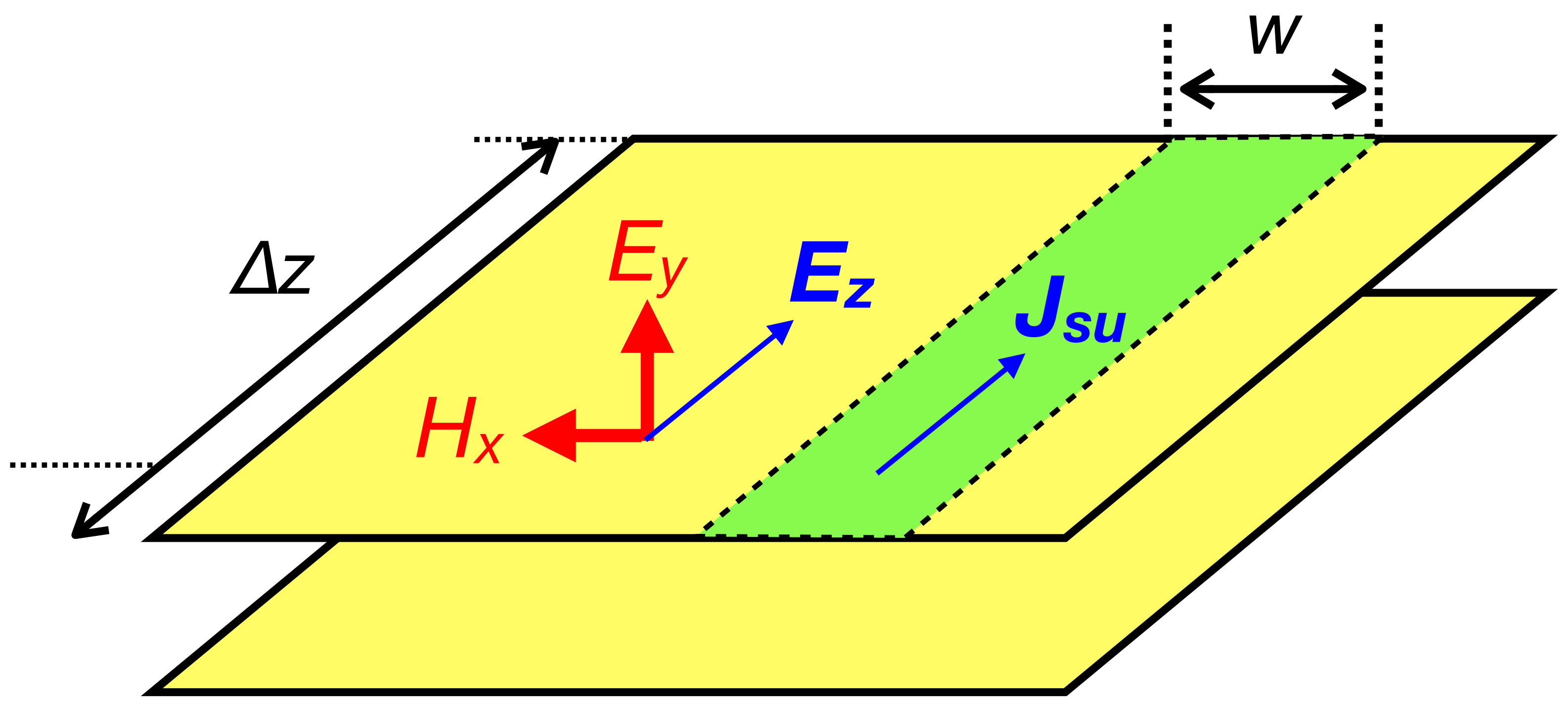
    → *"Quasi-TEM mode" in lossy transmission line!*



<Circuit representation>

- Average power dissipated per unit area due to E_z

$$\mathbf{P}_{av} = \mathbf{a}_y p_{\sigma_c} = \frac{1}{2}\mathrm{Re}\left(\mathbf{a}_z E_z \times \mathbf{a}_x H_x^*\right) \quad \cdots(1)$$



- Surface impedance (Z_s) by E_z

$$Z_s \triangleq \frac{E_t}{J_s} \rightarrow Z_s = \frac{E_z}{J_{su}} = \frac{E_z}{H_x} = \eta_c \quad \cdots(2)$$ *(Intrinsic impedance of the plate)*

$$\eta_c = R_s + jX_s = (1+j)\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega) \quad \cdots(3)$$ *(Refer to lecture note 3-2)*

$$\mathbf{J}_{su} = -\mathbf{a}_z \frac{E_0}{\eta} e^{-j\beta z} = \mathbf{a}_z H_x = \mathbf{a}_z \sigma_c E_z$$

- How significant is E_z?

$$\frac{|E_z|}{|E_y|} = \frac{|\eta_c H_x|}{|\eta H_x|} = \sqrt{\frac{\varepsilon}{\mu}}|\eta_c| = \sqrt{\frac{\varepsilon}{\mu}}\sqrt{2}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \sqrt{\frac{2\pi f \varepsilon}{\sigma_c}}$$

e.g.) For copper [σ_c = 5.8 x 10^7 (S/m)] and $\varepsilon = \varepsilon_0$ for dielectric at f = 3 (GHz),

$$|E_z| \simeq 5.3 \times 10^{-5} |E_y| \ll |E_y|$$

→ *E_z is thus, a slight perturbation and TEM approximation holds!*

Chap. 9 | Lossy TR lines: Equivalent circuit model (3/3)

- Ohmic power dissipation
  - From Equation **(1)**, we get

$$E_z = J_{su} Z_s \quad \text{and} \quad H_x = J_{su} \quad \cdots(4)$$

  - By plugging **(4)** into **(1)**,

$$p_{\sigma_c} = \frac{1}{2}\mathrm{Re}\left(|J_s|^2 Z_s\right) = \frac{1}{2}|J_s|^2 R_s \quad (\mathrm{W/m^2})$$

- Ohmic power dissipated in ***a unit length of the plate***

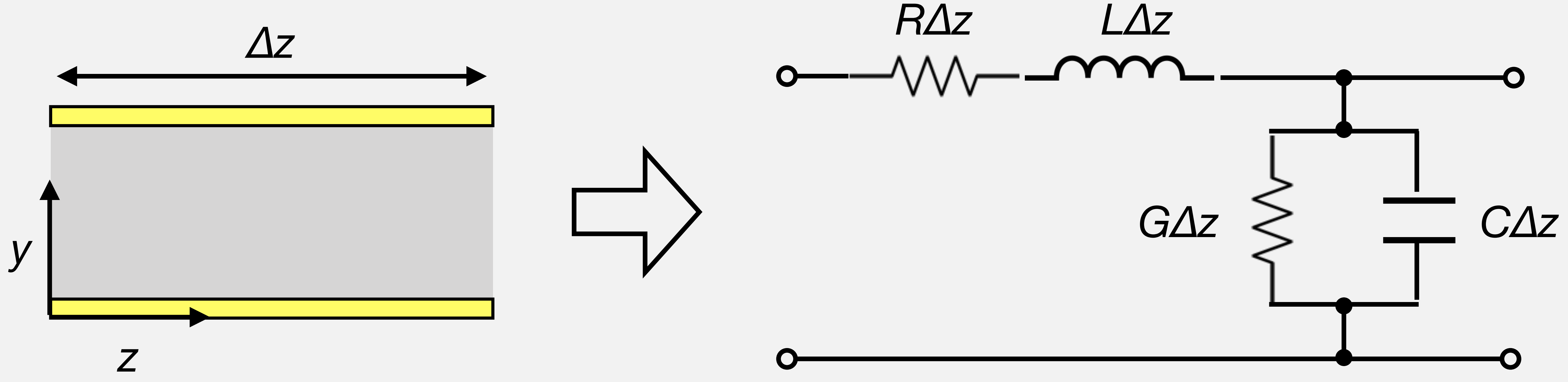
$$P_{\sigma_c} = p_{\sigma_c} w = \frac{1}{2}|J_s w|^2\left(\frac{R_s}{w}\right) = \frac{1}{2}I^2 R_u$$

where *I* is total current flowing through R_s/w

$$\boxed{\therefore R = R_u + R_l = 2\left(\frac{R_s}{w}\right) = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega/m)}$$

- ***Distributed parameters of parallel-plate transmission line (width = w, separation = d)***

Parameter	**Formula**	**Unit**
R	$\frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$	(Ω/m)
L	$\mu\frac{d}{w}$	(H/m)
G	$\sigma\frac{w}{d}$	(S/m)
C	$\varepsilon\frac{w}{d}$	(F/m)



<Equivalent circuit of a two conductor transmission line>
(More detail in next class!)