Transfer Function Approach to Modeling Dynamic Systems

The Concept of Transfer Function

Consider the linear time-invariant system defined by the following differential

equation :

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1}\dot{u} + b_m u \quad (n \geq m)$$

Where y is the output of the system, and x is the input. And the Laplace transform of

the equation is,

$$(a_0 S^n + a_1 S^{n-1} + \cdots + a_{n-1} S + a_n) Y(s)$$
$$= (b_0 S^m + b_1 S^{m-1} + \cdots + b_{m-1} S + b_m) U(s)$$

The Concept of Transfer Function

The ratio of the Lapalce transform of the output (response function) to the Laplace Transform of the input (driving function) under the assumption that all initial conditions are Zero.

$$Transfer\ Function = \frac{Y(s)}{U(s)} = G(s) = \frac{b_0 S^m + b_1 S^{m-1} + \cdots + b_{m-1} S + b_m}{a_0 S^n + a_1 S^{n-1} + \cdots a_{n-1} S + a_n}$$

$$= \frac{P(s)}{Q(s)} \qquad (n \geq m)$$

Comments on Transfer Function

1. A mathematical model.

2. Property of system itself.

   (Independent of the magnitude and nature of the input)

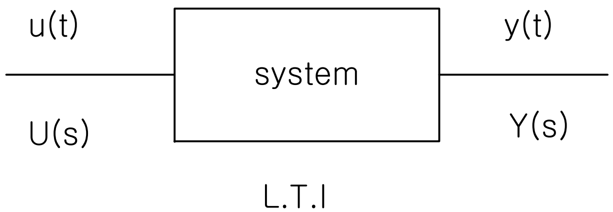
3. Includes the units

   (But, no information about the physical structure of the system)

$$\frac{Y(s)}{U(s)} = G(s) = \frac{[volt]}{[volt]} \qquad Y(s) = U(s)G(s)$$

Comments on Transfer Function

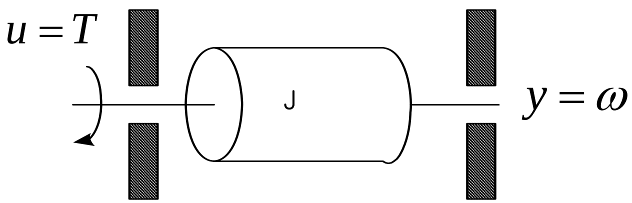
4. Analytic method and Experimental method



$$G(s)=\frac{Y(s)}{U(s)}$$

5. Different systems may have identical T.F

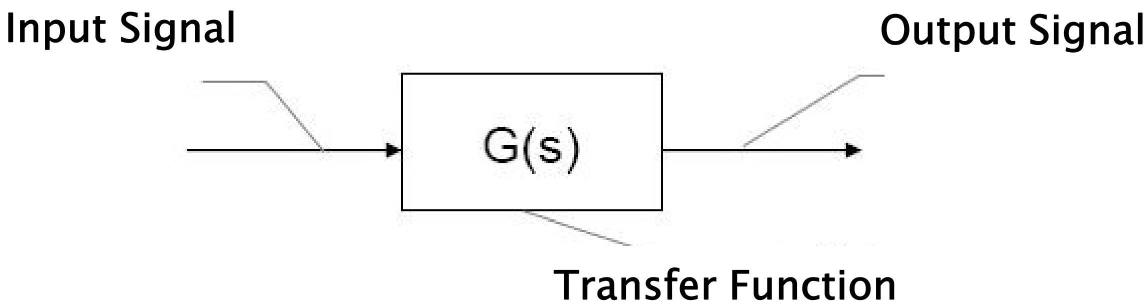
ex)

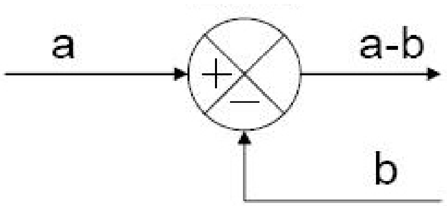


$$J\dot{\omega}+b\omega=T \qquad \frac{Y}{U}=\frac{\omega}{T}=\frac{1}{Js+b}$$

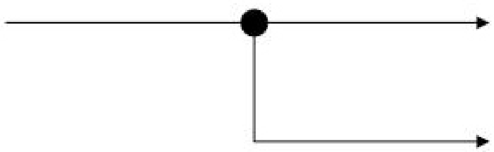
Electrical System $\qquad \dfrac{Y}{U}=\dfrac{R}{Ls+R}=\dfrac{1}{\frac{L}{R}s+1}$ (if b=1, J=L/R)

Block Diagram

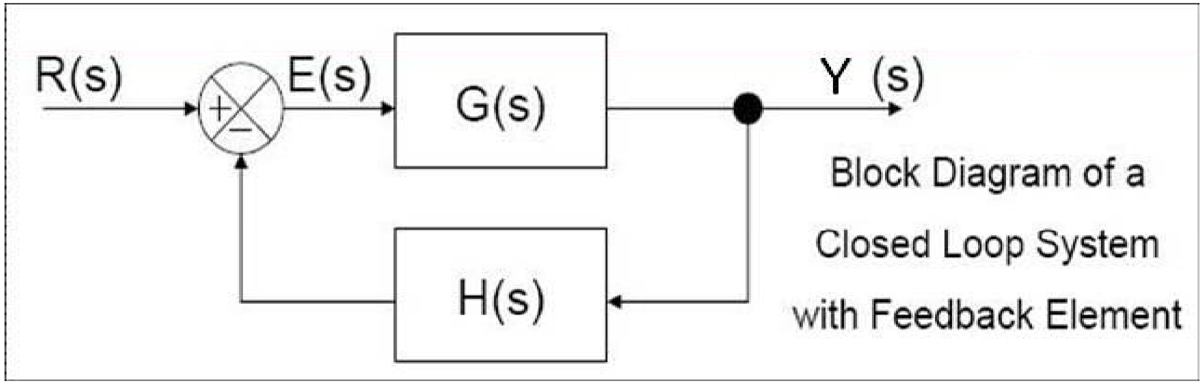




Summing Point


Pick Off Point

Closed Loop Transfer Function



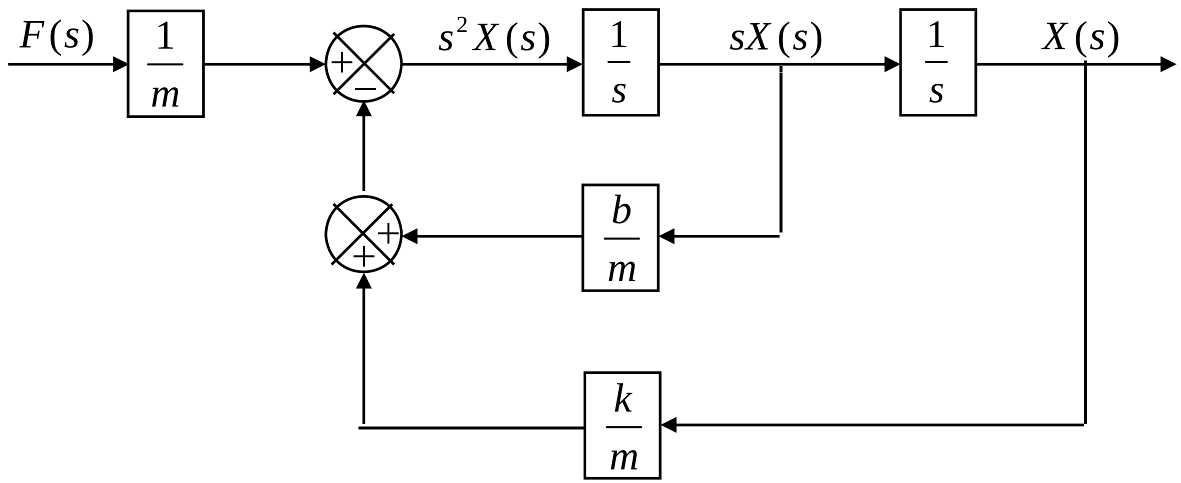
$$Y(s) = G(s)E(s) = G(s)[R(s) - H(s)Y(s)]$$

$$[1 + G(s)H(s)]Y(s) = G(s)R(s)$$

$$\therefore Transfer\ Function = \frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)H(s)}$$

Closed Loop Transfer Function

ex)



$$\frac{1}{m}F(s)-\frac{k}{m}X(s)-\frac{b}{m}sX(s)=s^2X(s)$$

$$F(s)-\left[kX(s)+bsX(s)\right]=ms^2X(s)$$

$$(ms^2+bs)X(s)=F(s)-kX(s),\qquad (ms^2+bs+k)X(s)=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{1}{ms^2+bs+k}\quad Transfer\ Function$$

Partial Fraction Expansion with MATLAB

$$\frac{B(s)}{A(s)} = \frac{num}{den} = \frac{b(1)S^h + b(2)S^{h-1} + \cdots + b(h+1)}{a(1)S^n + a(2)S^{n-1} + \cdots + a(n+1)}$$

$$num = [b(1) \quad b(2) \quad \cdots \quad b(h)], \quad den = [a(1) \quad a(2) \quad \cdots \quad a(h)]$$

[r, p, k] = residue (num, den)

$$\frac{B(s)}{A(s)} = k(s) + \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \cdots + \frac{r(n)}{s-p(n)}$$

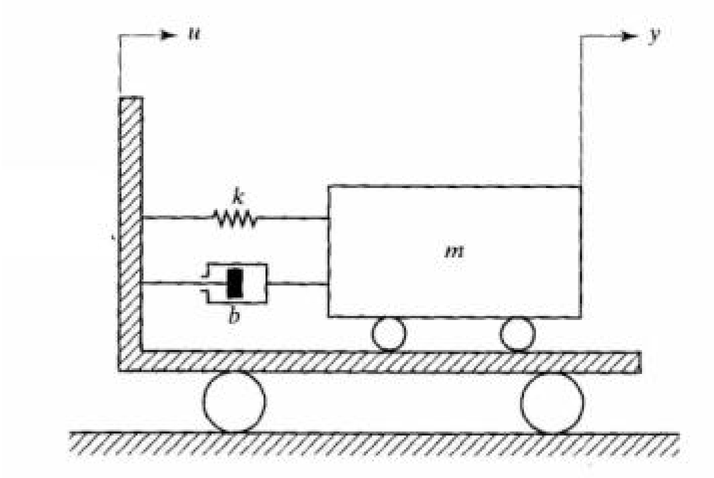
ex) $$\frac{B(s)}{A(s)} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6}$$

```
>>num=[1  8  16  9  6];
>>den=[1  6  11  6];
>>[r,p,k]=residue(num,den)
r=
    -6.0000
    -4.0000
     3.0000
p=
    -3.0000
    -2.0000
    -1.0000
k=
     1     2
```

$$= s + 2 + \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1}$$

Transient Response Analysis with MATLAB

ex)



$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$or \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

Assume the initial condition is 0,

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

$$Transfer\ Function = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

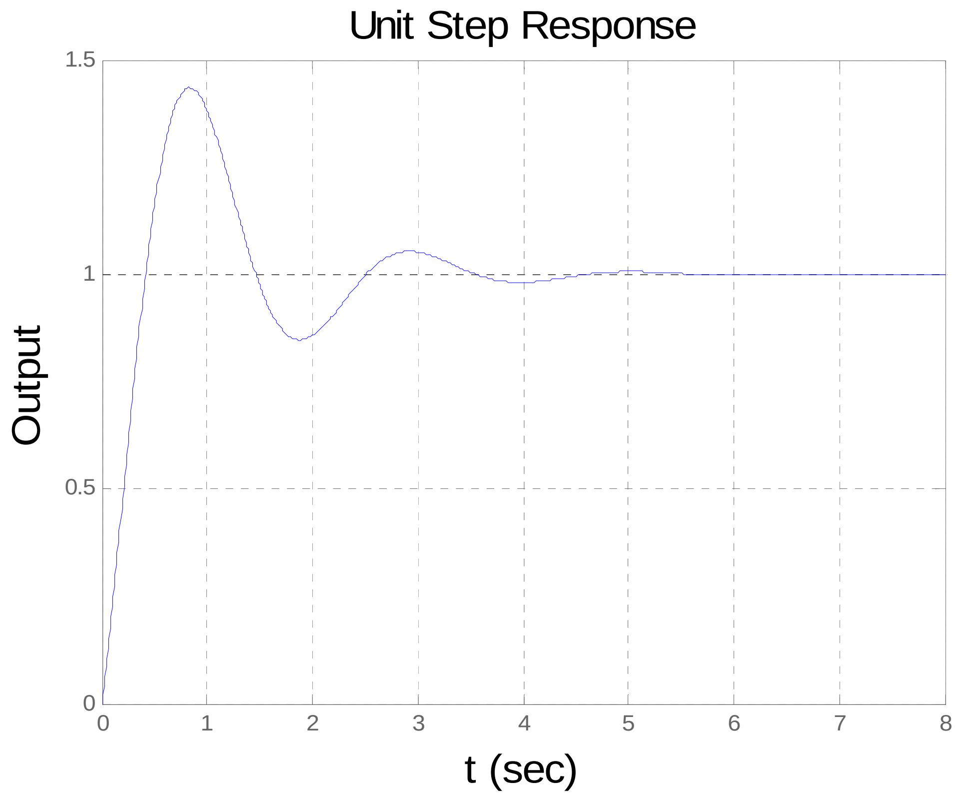
If, m=10kg, b=20N-s/m, k=100N/m

$$\frac{Y(s)}{U(s)} = \frac{20s+100}{10s^2+20s+100} = \frac{2s+10}{s^2+2s+10}, \qquad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2s+10}{s^2+2s+10}\frac{1}{s} = \frac{2s+10}{s^3+2s^2+10s}$$

Transient Response Analysis with MATLAB

```
t=0:0.01:8;
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
step(sys,t)
grid
title('Unit Step Response','Fontsize',15')
xlabel('t(sec)','Fontsize',15')
ylabel('Output','Fontsize',15')
```



The law of conservation of momentum

By Newton's 2nd law,

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv) \quad \rightarrow \quad F \cdot dt = d(mv)$$

$$\int_{t_1}^{t_2} F dt = \int_{v_1}^{v_2} d(mv) = mv_2 - mv_1$$

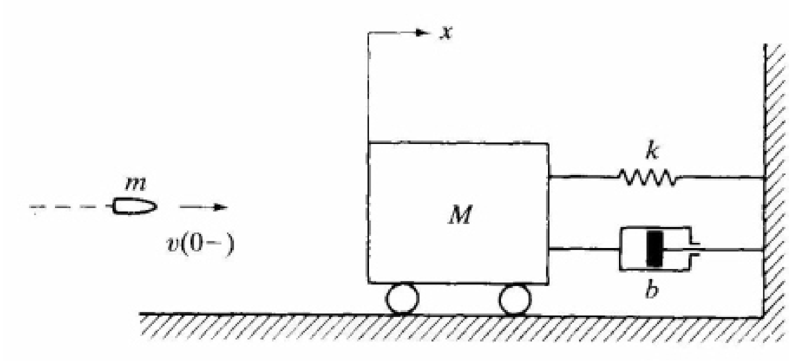
And if There is no input force, then we get

$$d(mv) = 0, \quad mv = const.$$ The law of conservation of momentum

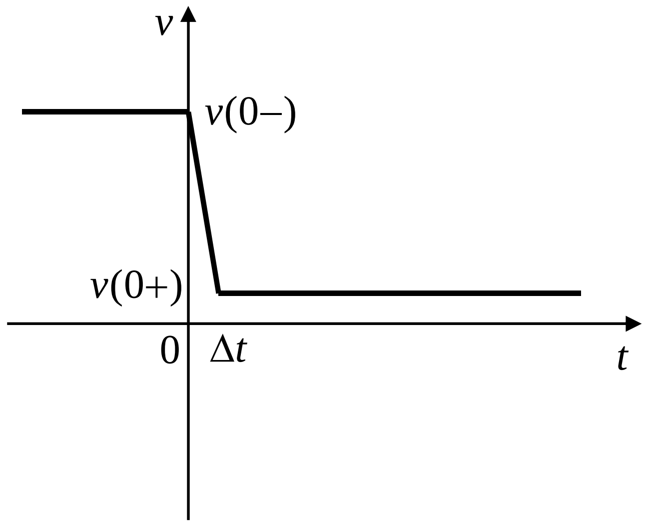
And also,

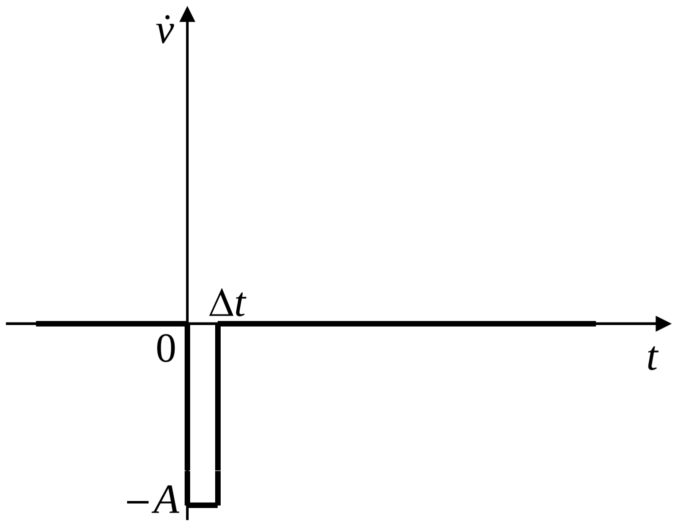
$$j\omega = const.$$ The law of conservation of angular momentum

Impulse Input



A bullet stuck in mass M





Sudden change in the velocity and acceleration of bullet

Impulse Input

System equation : $(M+m)\ddot{x}+b\dot{x}+kx=F(t)$

Impulse force : $F(t)=-m\dot{v}=A\Delta t\ \delta(t),\quad A\Delta t$: Magnitude of impulse force

$$\int_{0-}^{0+} A\Delta t\ \delta(t)\ dt=-m\int_{0-}^{0+}\dot{v}\ dt,\quad A\Delta t=mv(0-)-mv(0+)$$

Noting that, Magnitude of impulse force is equal to change of momentum!!

$v(0+)=\dot{x}(0+)=$ Initial velocity of M+m

Thus the system equation becomes

$$(M+m)\ddot{x}+b\dot{x}+kx=F(t)=\left[mv(0-)-m\dot{x}(0+)\right]\delta(t)$$

Impulse Input

By taking the Laplace Transform,

$$(M+m)\left[s^2X(s)-sx(0-)-\dot{x}(0-)\right]+b\left[sX(s)-x(0-)\right]+kX(s)=mv(0-)-m\dot{x}(0+)$$

Noting that $\dot{x}(0-)=x(0-)=0$ we obtain $X(s)=\dfrac{mv(0-)-m\dot{x}(0+)}{(M+m)s^2+bs+k}$

$$\dot{x}(0+)=\lim_{t\to 0+}\dot{x}(t)=\lim_{s\to\infty}s\left[sX(s)\right]=\lim_{s\to\infty}\frac{s^2\left[mv(0-)-m\dot{x}(0+)\right]}{(M+m)s^2+bs+k}=\frac{mv(0-)-m\dot{x}(0+)}{M+m}$$

$$\rightarrow \quad \dot{x}(0+)=\frac{m}{M+2m}v(0-)$$

$$\therefore X(s)=\frac{(M+m)\dot{x}(0+)}{(M+m)s^2+bs+k}=\frac{1}{(M+m)s^2+bs+k}\frac{(M+m)mv(0-)}{M+2m}$$

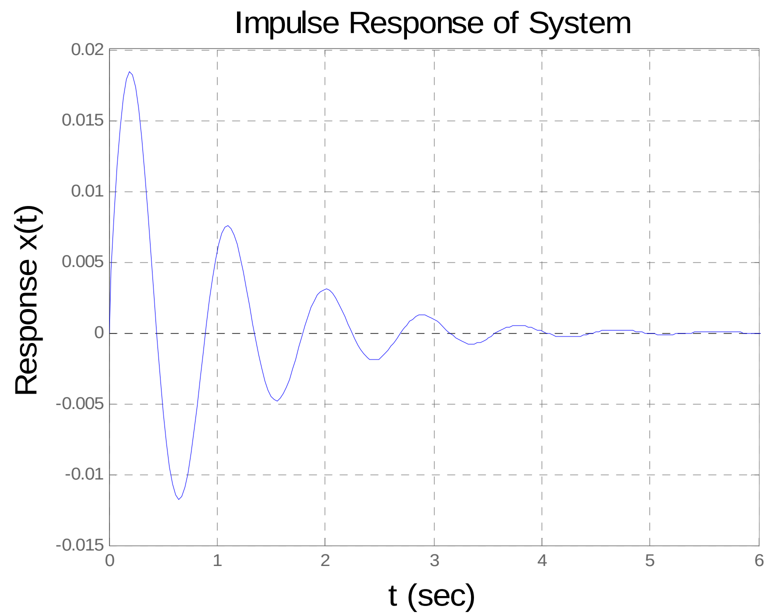
Impulse Input

ex)

$$M = 50kg, \quad m = 0.01kg, \quad b = 100Ns/m, \quad k = 2500N/m, \quad v(0-) = 800m/s$$

$$X(s) = \frac{1}{50.01s^2 + 100s + 2500} \frac{50.01 \times 0.01 \times 800}{50.02} = \frac{7.9984}{50.01s^2 + 100s + 2500}$$

```
num=[7.9984];
den=[50.01 100 2500];
sys=tf(num,den);
impulse(sys)
grid
title('Impulse Response of System','Fontsize',15')
xlabel('t(sec)','Fontsize',15')
ylabel('Response x(t)','Fontsize',15')
```



Ramp Response

$$\frac{Y(s)}{U(s)} = \frac{2s+10}{s^2+2s+10}$$

M=10kg, b=20Ns/m, k=100N/m

$$u(t): Unit\ ramp\ input, \quad u = \alpha t, \quad \alpha = 1$$

```
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
t=0:0.01:4;
u=t;
lsim(sys,u,t)
grid
title('Unit-Ramp Response','Fontsize',15)
xlabel('t')
ylabel('Output y(t) and Input u(t)=t','Fontsize',15)
text(0.8,0.4,'y','Fontsize',12)
text(0.4,0.8,'u','Fontsize',12)
legend('y')
```