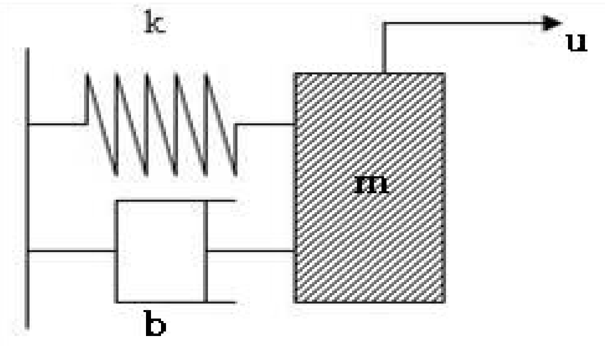
Mathematical Modeling of Dynamic Systems in State Space

A Simple Problem



Consider the system equation :

$$m\ddot{y} + b\dot{y} + ky = u$$

Laplace transform : $(ms^2 + bs + k)Y(s) = U(s)$

$$Transfer\ \ function: \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

Consider the system equation again : $\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u$

Choose variables : $x_1 = y, \quad x_2 = \dot{y}$

Then we get : $\dot{y} = \dot{x}_1 = x_2 \quad \ddot{y} = \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

Simplify the matrix : $\dot{x} = Ax + Bu$

State Space Modeling

State of a dynamic system : The smallest set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$. And the variables are called *state variables.*

State vector : *n* state variables which is need to completely describe the behavior of a given system can be considered the *n* components of a vector *x*. Such a vector is called a *state vector.*

State space : The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, …, x_n axis is called a *state space*.

State Space Modeling

If the system is linear time-invariant system, the system can be presented as *n* state variables, *r* input variables, and *m* output variables.

State equation :

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1r}u_r$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2r}u_r$$
$$\vdots$$
$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nr}u_r$$

Output equation :

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1r}u_r$$
$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2r}u_r$$
$$\vdots$$
$$y_m = c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n + d_{m1}u_1 + d_{m2}u_2 + \cdots + d_{mr}u_r$$

State Space Modeling

State space equation :

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix}$$

A : state matrix, B : input matrix, C : Output matrix, D : direct transmission matrix

Transformation of System Models

Step Response : sys=sys(A,B,C,D)

we use 'step(sys)' or 'step(A,B,C,D)' in MATLAB

Transfer Matrix : r inputs, $u_1, u_2, \cdots, u_r$ and m outputs, $y_1, y_2, \cdots, y_m$

We define those vectors.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Then, the transfer matrix G(s) expresses the relationship between Y(s) and U(s)

$$Y(s) = G(s)U(s)$$

Transformation of System Models

System equation : $\dot{x} = Ax + Bu$

$$y = Cx + Du$$

Laplace Transformation : $sX(s) - x(0) = AX(s) + BU(s)$

$$Y(s) = CX(s) + DU(s)$$

Assume, $x(0)=0$: $X(s) = (sI - A)^{-1} BU(s)$

$$Y(s) = \left[C(sI - A)^{-1} B + D \right] U(s) \rightarrow \therefore G(s) = C(sI - A)^{-1} B + D$$

ex)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

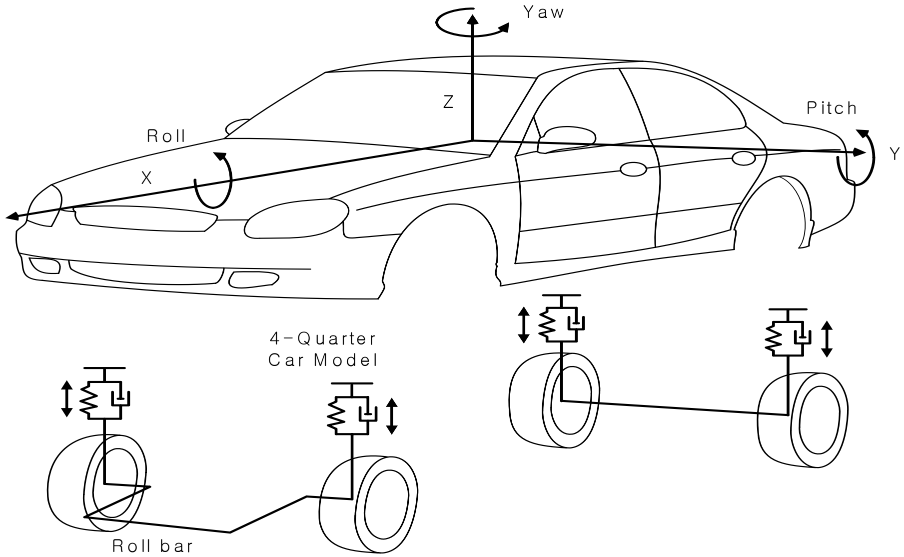
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -6.5 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s & -1 \\ 6.5 & s+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

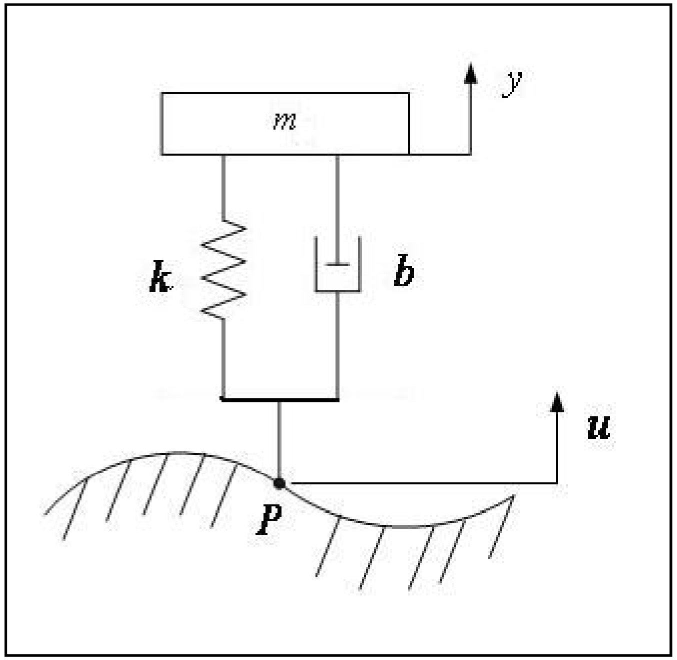
$$= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix}$$

Vehicle Suspension Problem

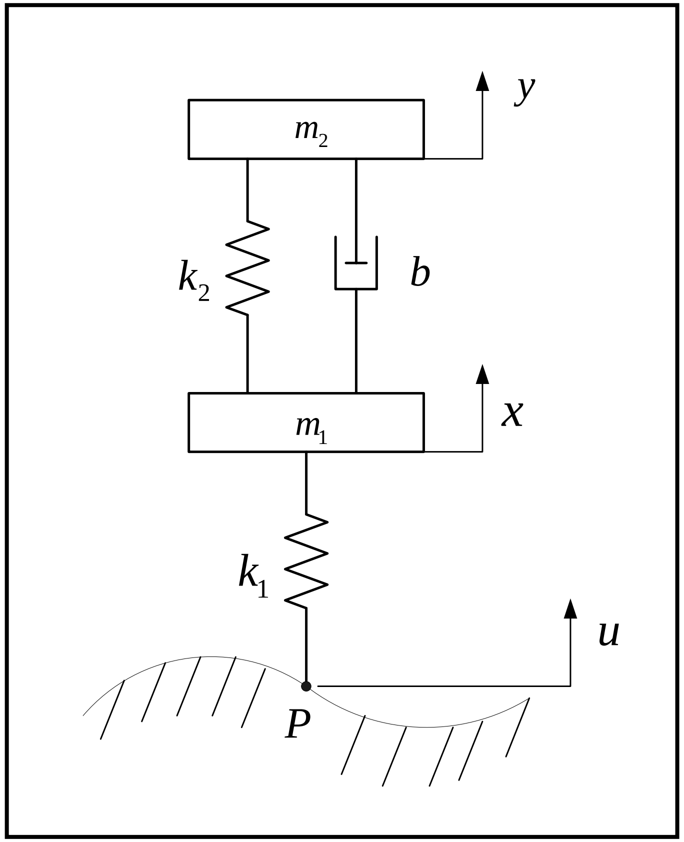


Vehicle Suspension Problem

ex1) Spring, damper, mass system

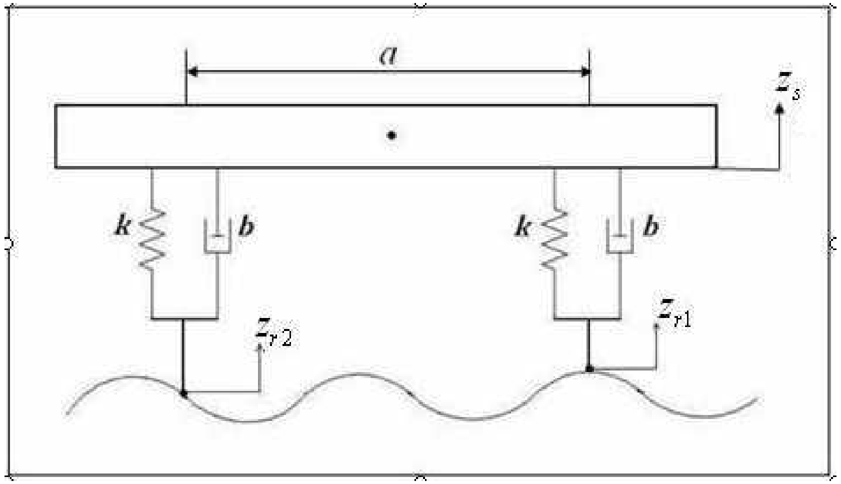


ex2) Body and tire model

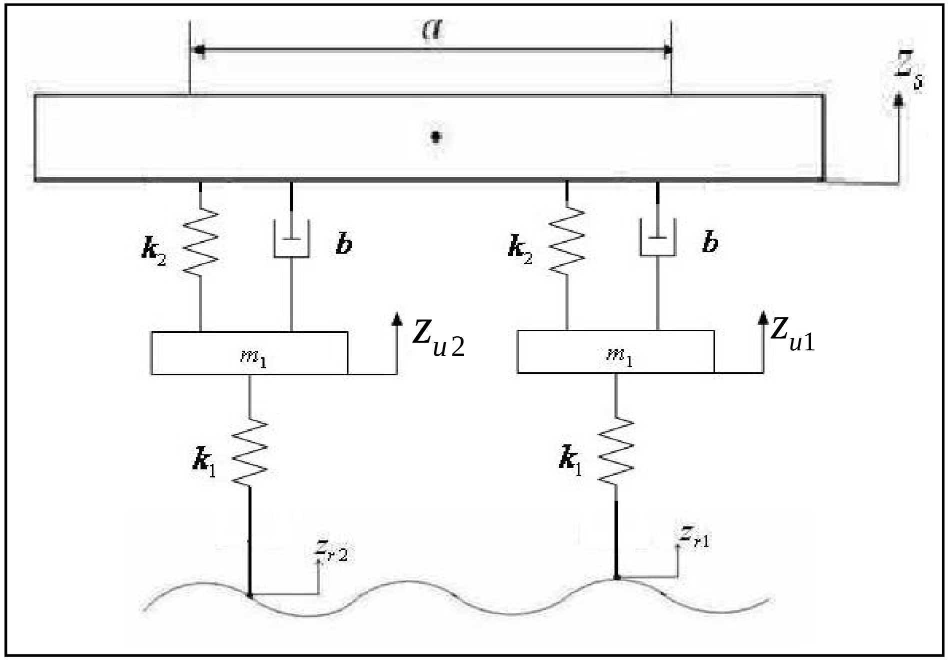


Vehicle Suspension Problem

ex3) Two inputs



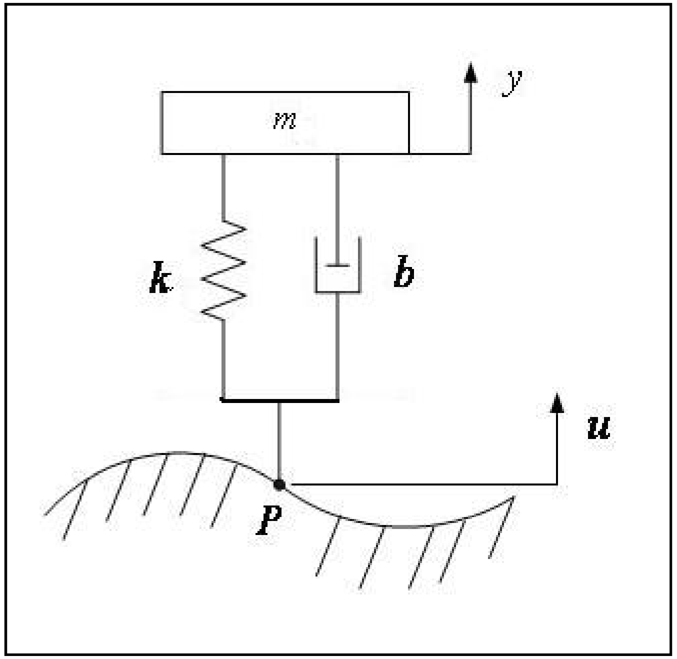
ex4) Two inputs, Body and Tire Model



Two inputs : z_{r1}, z_{r2} $\quad z_{r2}(t) = z_{r1}(t - \tau)$

Vehicle Suspension Problem

ex1) Spring, damper, mass system



- Design Considerations

1. Ride Quality

$\rightarrow$ *Sprung mass acceleration* : $\ddot{y}$

2. Rattle space

$\rightarrow$ *Suspension Deflection* : y

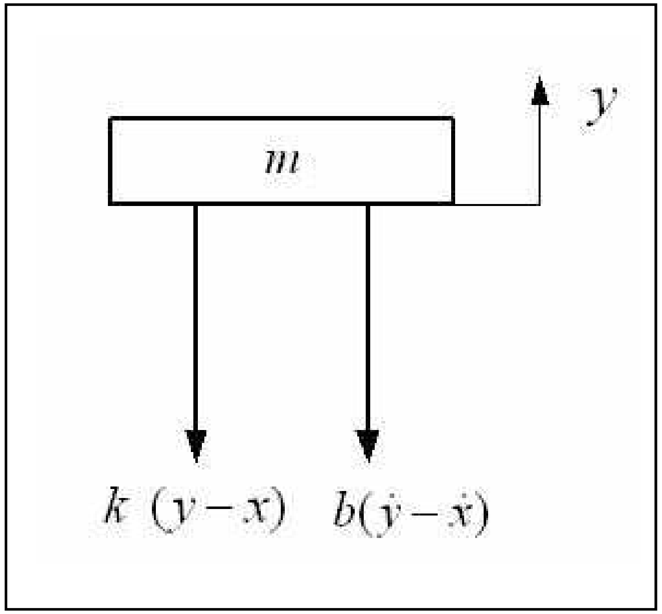
- Suspension Design Parameters

$\rightarrow$ *Spring Stiffness* : k

$\rightarrow$ *Damping Ratio* : b

Vehicle Suspension Problem

▪ Free Body Diagram



▪ Dynamic Equations

$$m\ddot{y}+b(\dot{y}-\dot{u})+k(y-u)=0$$

$$m\ddot{y}+b\dot{y}+ky=b\dot{u}+ku$$

▪ Laplace Transform

$$(ms^2+bs+k)Y(s)=(bs+k)X(s)$$

$$Transfer\ \ function:\ \frac{Y(s)}{U(s)}=\frac{bs+k}{ms^2+bs+k}$$

Vehicle Suspension Problem

- General Form of State Equation $\dot{x} = Ax + Bu$

- The State variables $(\mathrm{u} = z_r)$

$$x_1 = z_s - z_r \quad : \textit{Suspension Deflection}$$

$$x_2 = \dot{z}_s \quad : \textit{absolute velocity of body}$$

$$\dot{x}_1 = \dot{z}_s - \dot{z}_r = x_2 - \dot{z}_r,$$

$$\dot{x}_2 = \ddot{z}_s = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{z}_r \quad : \textit{acceleration of body}$$

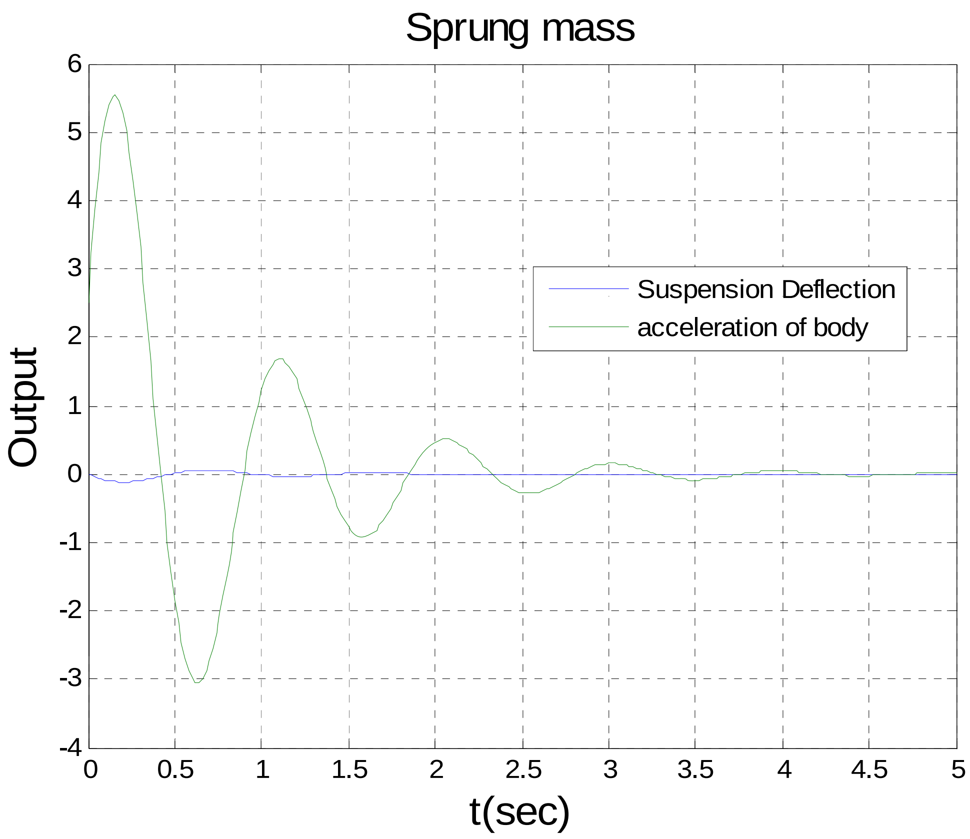
$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{b}{m} \end{bmatrix} \mathrm{u}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \ddot{z}_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{m} \end{bmatrix} \mathrm{u}$$

Vehicle Suspension Problem

Parameter : M=400kg,   b=1000Ns/m, k=18000N/m

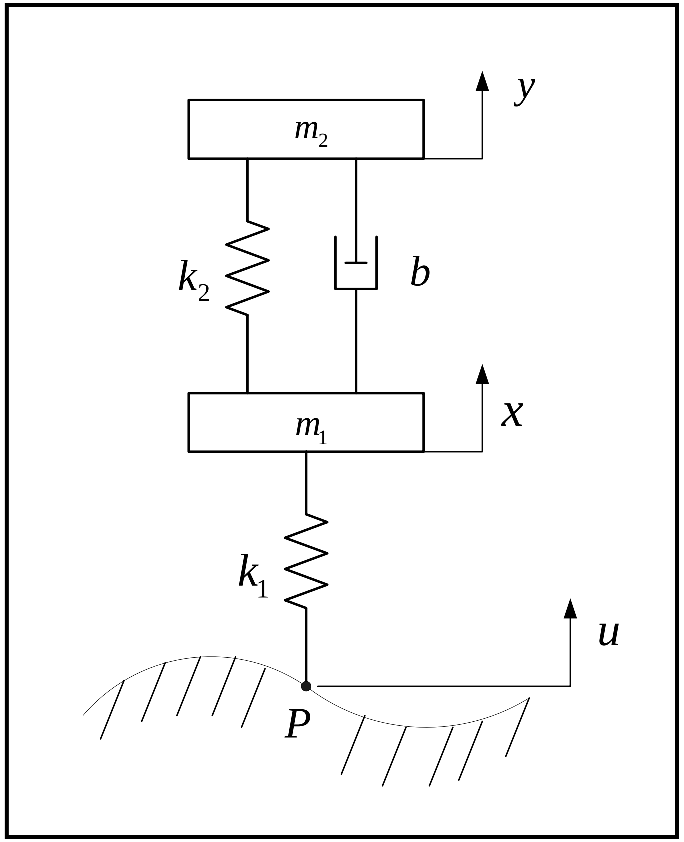
Input : Step input

```
>>t=0:0.02:25
>>A=[0 1;-45 -2.5];
>>B=[-1;2.5];
>>C=[1 0;-45 -2.5];
>>D=[0;2.5];
>>sys=ss(A,B,C,D);
>>[y,t]=step(sys,t);
>>plot(t,y)
>>grid
>>title('Step input VS Sine input','Fontsize',15)
>>xlabel('t(sec)','Fontsize',15)
>>ylabel('Output x','Fontsize',15)
>>legend('Step input','Sine input')
```



Vehicle Suspension Problem

ex2) Body and tire model



- Design Considerations

  1. Ride Quality

  $\rightarrow$ *Sprung mass acceleration* : $\ddot{y}$

  2. Rattle space

  $\rightarrow$ *Suspension Deflection* : $y - x$

  3. Tire Force Vibration

  $\rightarrow$ *Tire Deflection* : $x - u$

- Suspension Design Parameters

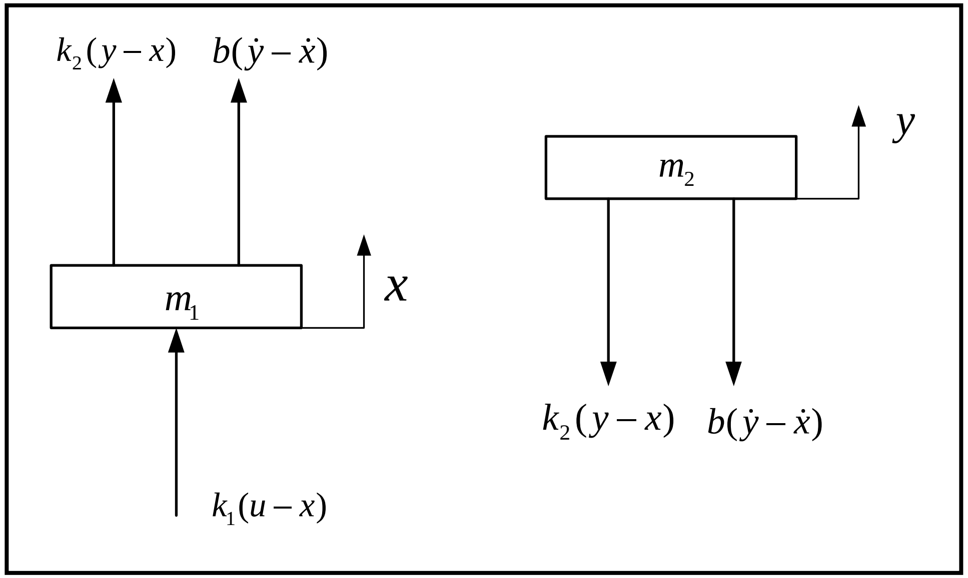
  $\rightarrow$ *Spring Stiffness* : k_2

  $\rightarrow$ *Damping Ratio* : b

  $\rightarrow$ *Tire Stiffness* : k_1

Vehicle Suspension Problem

- Free Body Diagram



- Dynamic Equations

$$m_1\ddot{x} = k_2(y-x) + b(\dot{y}-\dot{x}) + k_1(u-x)$$

$$m_2\ddot{y} = -k_2(y-x) - b(\dot{y}-\dot{x})$$

- Laplace Transform

$$[m_1 s^2 + bs + (k_1 + k_2)]X(s)$$

$$= (bs + k_2)Y(s) + k_1 U(s)$$

$$[m_2 s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$$

Vehicle Suspension Problem

- Displacement of Mass

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs+k_2)}{m_1m_2s^4+(m_1+m_2)bs^3+[(k_2m_1+(k_1+k_2)m_2]s^2+k_1bs+k_1k_2}$$

$$\frac{X(s)}{U(s)} = \frac{k_1(m_2s^2+bs+k_2)}{m_1m_2s^4+(m_1+m_2)bs^3+[(k_2m_1+(k_1+k_2)m_2]s^2+k_1bs+k_1k_2}$$

- Design Considerations

$$G_1(s) = \frac{s^2Y(s)}{U(s)} = \frac{s^2k_1(bs+k_2)}{m_1m_2s^4+(m_1+m_2)bs^3+[(k_2m_1+(k_1+k_2)m_2]s^2+k_1bs+k_1k_2}$$ $\rightarrow$ *Sprung mass acceleration* : $\ddot{y}$

$$G_2(s) = \frac{Y(s)-X(s)}{U(s)} = \frac{-k_1m_2s^2}{m_1m_2s^4+(m_1+m_2)bs^3+[(k_2m_1+(k_1+k_2)m_2]s^2+k_1bs+k_1k_2}$$ $\rightarrow$ *Suspension Deflection* : $y-x$

$$G_3(s) = \frac{X(s)-U(s)}{U(s)} = \frac{-m_1m_2s^4-(m_1+m_2)bs^3-k_2(m_1+m_2)s^2}{m_1m_2s^4+(m_1+m_2)bs^3+[(k_2m_1+(k_1+k_2)m_2]s^2+k_1bs+k_1k_2}$$ $\rightarrow$ *Tire Deflection* : $x-u$

Vehicle Suspension Problem

- General Form of State Equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Dynamic Equations

$$m_1\ddot{z}_u = k_2(z_s - z_u) + b(\dot{z}_s - \dot{z}_u) + k_1(u - z_u)$$

$$m_2\ddot{z}_s = -k_2(z_s - z_u) - b(\dot{z}_s - \dot{z}_u)$$

- The State variables $(x = z_u, \quad y = z_s)$

$x_1 = z_s - z_u$: *Suspension Deflection*

$x_2 = \dot{z}_s$: *absolute velocity of sprung mass*

$x_3 = z_u - u$: *Tire Deflection*

$x_4 = \dot{z}_u$: *absolute velocity of unsprung mass*

Vehicle Suspension Problem

- 1st order State equations

$$\dot{x}_1 = \dot{z}_s - \dot{z}_u = x_2 - x_4$$

$$\dot{x}_2 = -\frac{k_2}{m_2}(z_s - z_u) - \frac{b}{m_2}(\dot{z}_s - \dot{z}_u) = -\frac{k_2}{m_2}x_1 - \frac{b}{m_2}x_2 + \frac{b}{m_2}x_4$$

$$\dot{x}_3 = \dot{z}_u - \dot{u} = x_4 - \dot{u}$$

$$\dot{x}_4 = \frac{k_2}{m_1}(z_s - z_u) + \frac{b}{m_1}(\dot{z}_s - \dot{z}_u) - \frac{k_1}{m_1}(u - z_u) = \frac{k_2}{m_1}x_1 + \frac{b}{m_1}x_2 - \frac{k_1}{m_1}x_3 - \frac{b}{m_1}x_4$$

- Matrix Form of State equations (system matrix & output matrix)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_2}{m_2} & -\frac{b}{m_2} & 0 & \frac{b}{m_2} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_1} & \frac{b}{m_1} & -\frac{k_1}{m_1} & -\frac{b}{m_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \dot{u}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{k_2}{m_2} & -\frac{b}{m_2} & 0 & \frac{b}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$y_1 = \ddot{x}_2 = -\frac{k_2}{m_2}x_1 - \frac{b}{m_2}x_2 + \frac{b}{m_2}x_4 \quad : Sprung\ mass\ acceleration$$

$$y_2 = z_s - z_u = x_1 \quad : Suspension\ Deflection, \quad y_3 = z_u - u = x_3 \quad : Tire\ Deflection$$

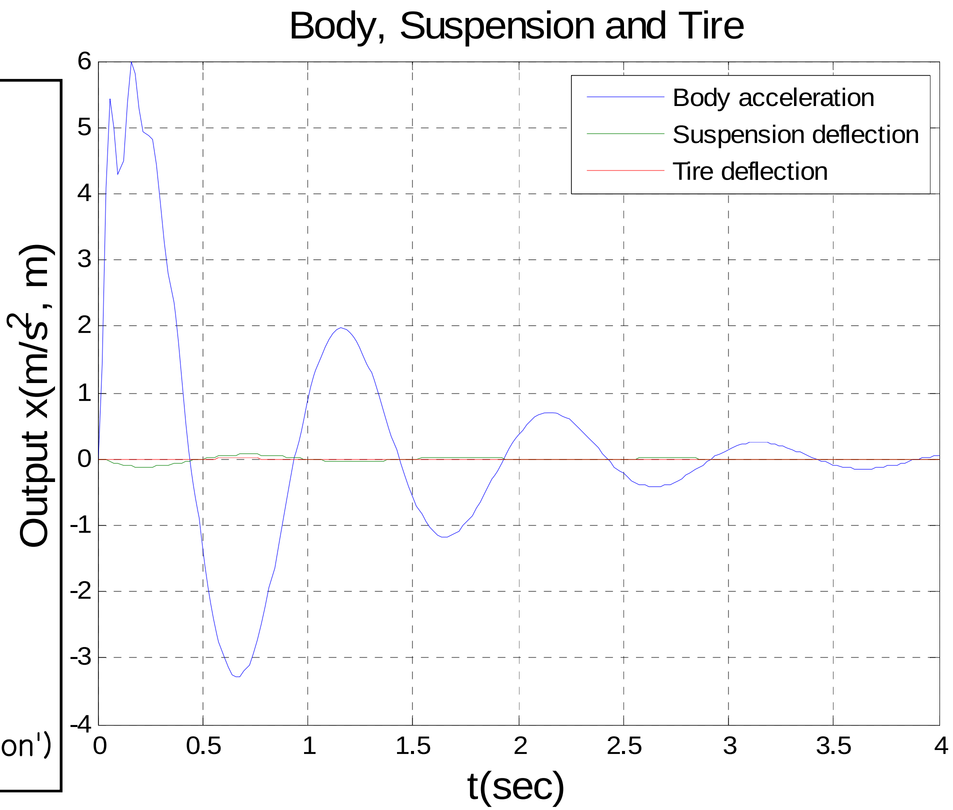
Vehicle Suspension Problem

Parameter : M1(tire)=55kg, M2(body)=400kg, b=1000Ns/m

k1(tire)=180000N/m, k2(suspension)=18000N/m

Input : Step input

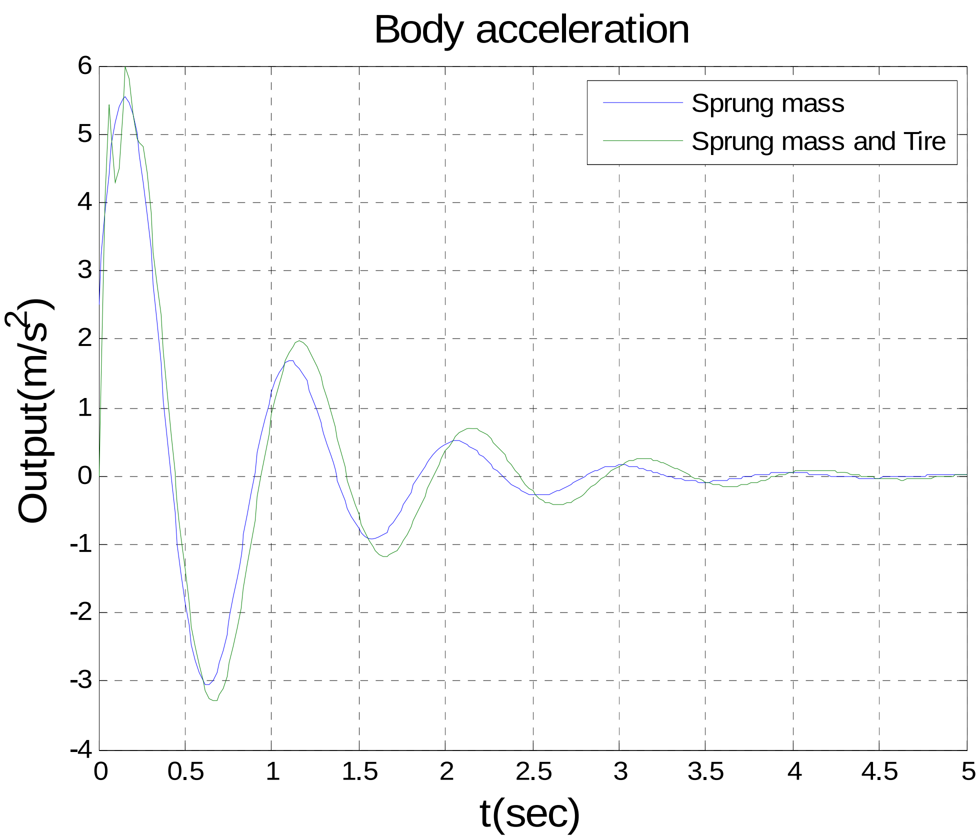
```
>>t=0:0.02:4;
>>A=[0 1 0 -1;-45 -2.5 0 2.5;0 0 0 1
      327.3 18.18 -3273 -18.18];
>>B=[0;0;-1;0];
>>C=[-45 -2.5 0 2.5;1 0 0 0;0 0 1 0];
>>D=0;
>>sys=ss(A,B,C,D);
>>[y,t]=step(sys,t);
>>plot(t,y)
>>grid
>>title('Body, Suspension and Tire','Fontsize',15)
>>xlabel('t(sec)','Fontsize',15)
>>ylabel('Output x(m/s^2, m)','Fontsize',15)
>>legend('Body acceleration','Suspension deflection','Tire deflection')
```



Vehicle Suspension Problem

```
t=0:0.02:5;
A=[0 1;-45 -2.5];
B=[-1;2.5];
C=[-45 -2.5];
D=[2.5];
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
A2=[0 1 0 -1;-45 -2.5 0 2.5;0 0 0 1
    327.3 18.18 -3273 -18.18];
B2=[0;0;-1;0];
C2=[-45 -2.5 0 2.5];
D2=0;
sys2=ss(A2,B2,C2,D2);
[y2,t]=step(sys2,t);
plot(t,y,t,y2)
grid
title('Body acceleration','Fontsize',15)
xlabel('t(sec)','Fontsize',15)
ylabel('Output(m/s^2)','Fontsize',15)
legend('Sprung mass','Sprung mass and Tire')
```

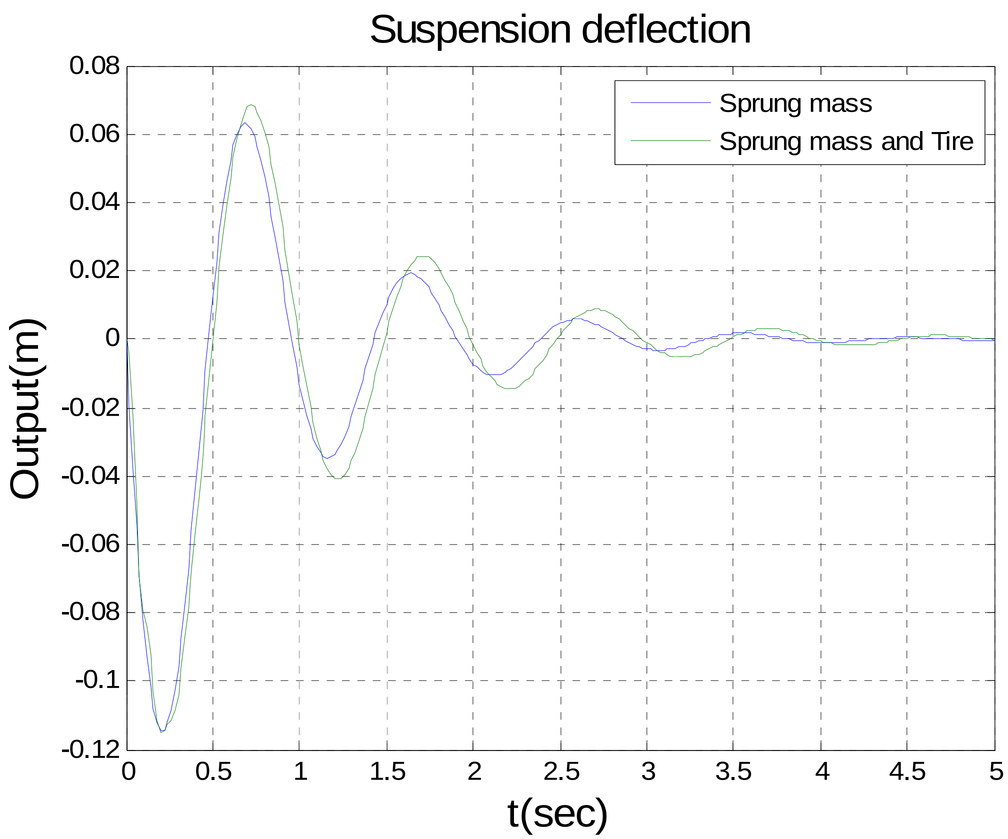
Comparison 1. Acceleration of sprung mass (body)



Vehicle Suspension Problem

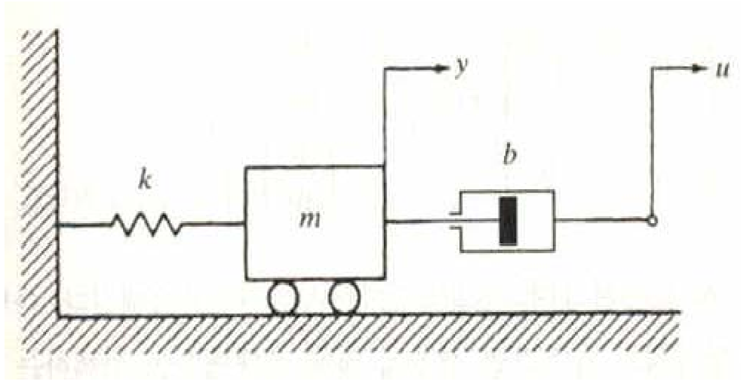
```
t=0:0.02:5;
A=[0 1;-45 -2.5];
B=[-1;2.5];
C=[1 0];
D=0;
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
A2=[0 1 0 -1;-45 -2.5 0 2.5;0 0 0 1
    327.3 18.18 -3273 -18.18];
B2=[0;0;-1;0];
C2=[1 0 0 0];
D2=0;
sys2=ss(A2,B2,C2,D2);
[y2,t]=step(sys2,t);
plot(t,y,t,y2)
grid
title('Suspension deflection','Fontsize',15)
xlabel('t(sec)','Fontsize',15)
ylabel('Output(m)','Fontsize',15)
legend('Sprung mass','Sprung mass and Tire')
```

Comparison 2. Suspension deflection



State-Space System Modeling with Input Derivatives

ex1) Consider a mechanical system,



$$m\ddot{y} = -ky - b(\dot{y} - \dot{u}), \quad \ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}$$

Choose state variables, $x_1 = y, \; x_2 = \dot{y}$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{u}$$

The right side includes $\dot{u}$ term. To explain the reason we should not include

differentiation of u, assume $u = \delta(t) \; (unit\ impulse\ \ function)$

$$x_2 = -\frac{k}{m}\int y\, dt - \frac{b}{m}y + \frac{k}{m}\delta(t)$$

x_2 includes $(k/m)\,\delta(t)$ term. It means $x_2(0) = \infty$ and cannot be accepted as a state variable.

State-Space System Modeling with Input Derivatives

To eliminate $\dot{u}$ term, $\ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u} \rightarrow \ddot{y} - \frac{b}{m}\dot{u} = -\frac{k}{m}y - \frac{b}{m}\dot{y}$

$$\frac{d}{dt}\left(\dot{y} - \frac{b}{m}u\right) = -\frac{k}{m}y - \frac{b}{m}\left(\dot{y} - \frac{b}{m}u\right) - \left(\frac{b}{m}\right)^2 u$$

So we choose state variables as, $x_1 = y,\ x_2 = \dot{y} - \frac{b}{m}u$

$$\dot{x}_2 = \ddot{y} - \frac{b}{m}\dot{u} = \left(-\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}\right) - \frac{b}{m}\dot{u} = -\frac{k}{m}x_1 - \frac{b}{m}\left(x_2 + \frac{b}{m}u\right)$$

$$= -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \left(\frac{b}{m}\right)^2 u$$

→ $\dot{u}$ term has been eliminated.

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ -\left(\frac{b}{m}\right)^2 \end{bmatrix} u$$

State-Space System Modeling with Input Derivatives

ex2) Consider a system defined by $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

Choose state variables,

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

Then we obtain,

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \dddot{y} = -6x_1 - 11x_2 - 6x_3 + 6u$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State-Space Representation of Dynamic Systems

System differential equations

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$Transfer\ Function = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots a_{n-1}s + a_n}$$

Method 2. Consider the second-order system, $\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} \quad \to \quad \frac{Z(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_2}, \quad \frac{Y(s)}{Z(s)} = b_0 s^2 + b_1 s + b_2$$

$$\ddot{z} + a_1\dot{z} + a_2 z = u, \quad b_0\ddot{z} + b_1\dot{z} + b_2 z = y$$

$$let, \quad x_1 = z, \quad x_2 = \dot{z} \quad \to \quad \dot{x}_2 = -a_2 x_1 - a_1 x_2 + u$$

$$b_0\ddot{z} + b_1\dot{z} + b_2 z = b_0(-a_2 x_1 - a_1 x_2 + u) + b_1 x_2 + b_2 x_1 = y$$

$$\therefore \quad \dot{x}_1 = x_2, \quad \dot{x}_2 - a_2 x_1 - a_1 x_2 + u$$

State-Space Representation of Dynamic Systems

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} b_2 - a_2 b_0 & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

N-th order differential equation, $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y$

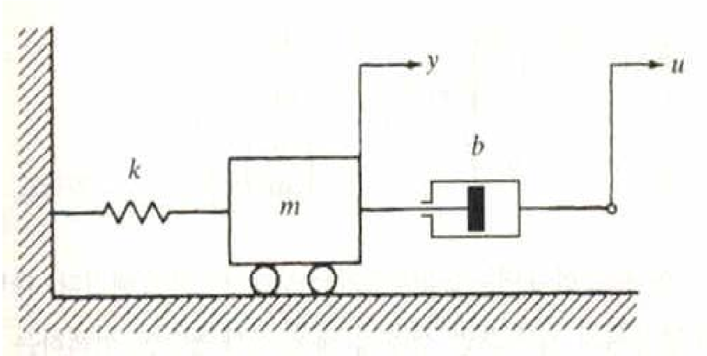
$$= b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

State-Space Representation of Dynamic Systems

ex1) Consider this mechanical system again,

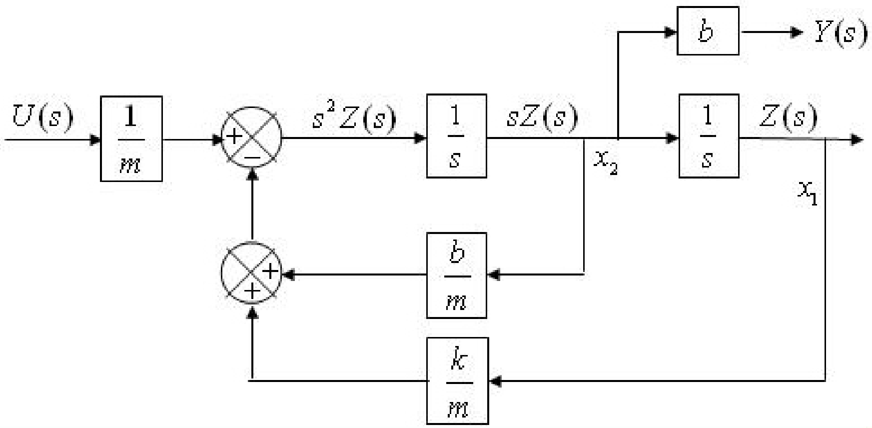


$$m\ddot{y} = -ky - b(\dot{y} - \dot{u}), \quad m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

$$\frac{Y(s)}{U(s)} = \frac{bs}{ms^2 + bs + k}, \quad Z(s) = \frac{Y(s)}{bs} = \frac{U(s)}{ms^2 + bs + k}$$

$$(ms^2 + bs + k)Z(s) = U(s), \quad bsZ(s) = Y(s)$$

$$s^2 Z(s) = \frac{1}{m}U(s) - \frac{b}{m}sZ(s) - \frac{k}{m}Z(s)$$



State variables,

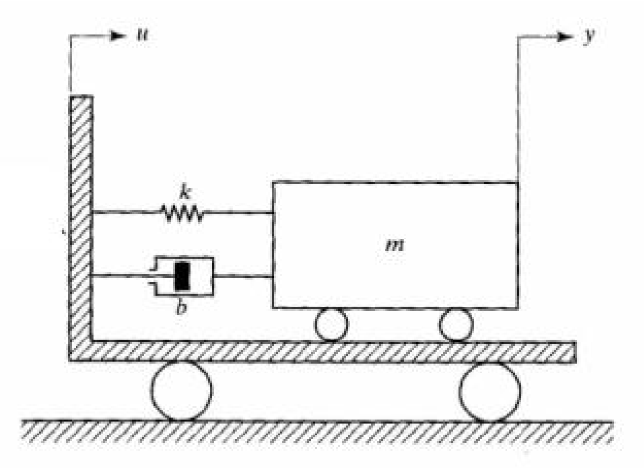
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$y = b\,x_2$$

State-Space Representation of Dynamic Systems

ex) Consider a spring-mass-damper system (refer to Chapter 4).



$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$\text{or} \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

$$Transfer\ Function = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

$$b_0 = 0, \quad b_2 - a_2 b_0 = \frac{k}{m} - \frac{k}{m} + 0 = \frac{k}{m}$$

$$b_1 - a_1 b_0 = \frac{b}{m} - \frac{b}{m} + 0 = \frac{b}{m}$$

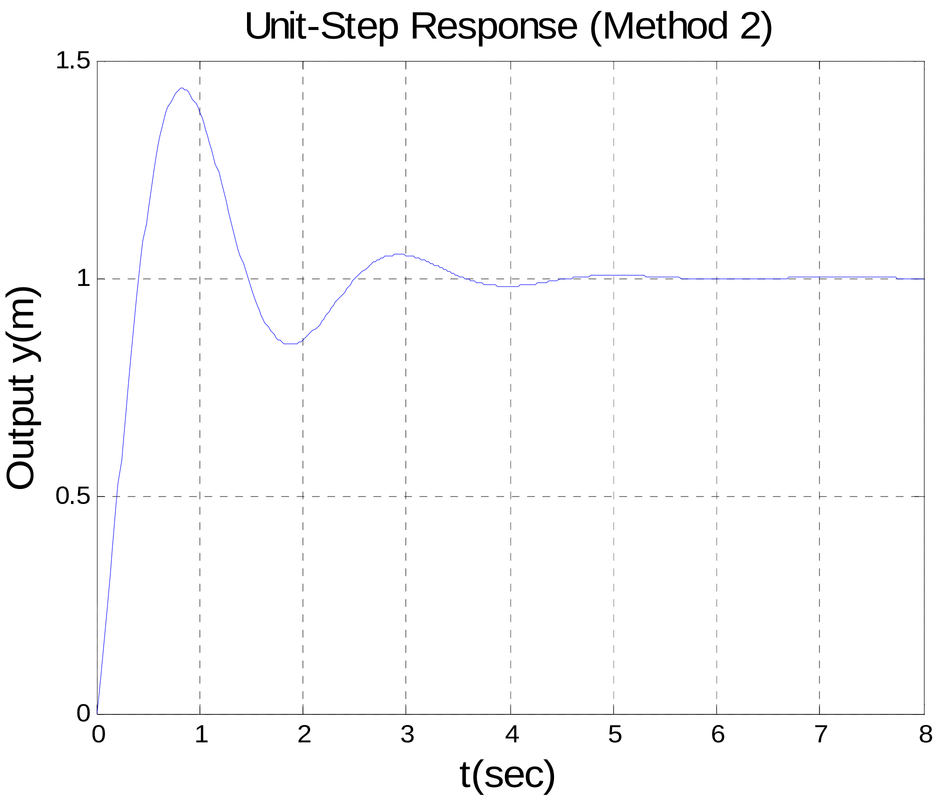
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{u}, \quad y = \begin{bmatrix} \frac{k}{m} & \frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-Space Representation of Dynamic Systems

If, m=10kg, b=20N-s/m, k=100N/m

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{u}, \quad y = \begin{bmatrix} 10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

```
t=0:0.02:8;
A=[0 1;-10 -2];
B=[0;1];
C=[10 2];
D=[0];
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
plot(t,y)
grid
title('Unit-Step Response (Method 2)','Fontsize',15)
xlabel('t(sec)','Fontsize',15)
ylabel('Output y(m)','Fontsize',15)
```



Transformation of Mathematical Models with MATLAB

$$\frac{Y(s)}{U(s)} = \frac{numerator\ \ polynomial\ in\ s}{denominator\ \ polynomial\ in\ s} = \frac{num}{den}$$

MATLAB command, [A, B, C, D] = tf2ss(num,den) gives a state space representation.

ex) Consider, $\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$

One of many possible state-space representations is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & 160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

```
>> num=[0 0 1 0];
>> den=[1 14 56 160];
>> [A,B,C,D]=tf2ss(num,den)
A =
  -14   -56  -160
    1     0     0
    0     1     0
B =
    1
    0
    0
C =
    0     1     0
D =
    0
```

Transformation of a State-Space Models into Another One

- Nonuniqueness of a set of state variables : Let $x_1, x_2, \ldots, x_n$ a set of state variables

  Another set of state variables any set of functions,

$$\begin{aligned}\hat{x}_1 &= X_1(x_1, x_2, \ldots, x_n)\\ \hat{x}_2 &= X_2(x_1, x_2, \ldots, x_n)\\ &\vdots\\ \hat{x}_n &= X_n(x_1, x_2, \ldots, x_n)\end{aligned}$$

  For every set of values $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ $\longrightarrow$ Unique set of values $x_1, x_2, \ldots, x_n$

  If x is a state vector, and P is a nonsingular matrix, $\hat{\mathrm{x}} = \mathrm{P}^{-1}\mathrm{x}$ is also a state vector

- A state-space model $\begin{aligned}\dot{\mathrm{x}} &= \mathrm{Ax} + \mathrm{Bu}\\ \mathrm{y} &= \mathrm{Cx} + \mathrm{Du}\end{aligned}$ is transformed into another state-space model

  by transforming the state vector x into state vector $\hat{\mathrm{x}}$ by means of the following

  transformation, $\mathrm{x} = \mathrm{P}\,\hat{\mathrm{x}}$

Transformation of a State-Space Models into Another One

- $\dot{x} = Ax + Bu$, $y = Cx + Du$ can be written as,

$$P\dot{\hat{x}} = AP\hat{x} + Bu \qquad \dot{\hat{x}} = P^{-1}AP\hat{x} + P^{-1}Bu$$
$$y = CP\hat{x} + Du \quad \text{or} \quad y = CP\hat{x} + Du$$

– Since infinitely many n x n matrices can be a transformation matrix P, there are infinitely many state-space models for a given system.

- Eigenvalues of an n x n matrix A are the roots of the characteristic equation.

$$|\lambda I - A| = 0$$

The eigenvalues are also called the characteristic roots.

ex) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $\quad |\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix} = \lambda^3 + 6\lambda^2 + 11\lambda + 6$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

Diagonalization of State Matrix A

Consider an n x n state matrix A :

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

If matrix A has distinct eigenvalues and the state vector x is transformed into another state vector z by use of a transformation matrix P ,

$$x = Pz, \quad where \; P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ {\lambda_1}^2 & {\lambda_2}^2 & \cdots & {\lambda_n}^2 \\ \vdots & \vdots & & \vdots \\ {\lambda_2}^{n-1} & {\lambda_2}^{n-1} & \cdots & {\lambda_n}^{n-1} \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$P^{-1}AP$ is a canonical matrix and each column of P is an eigenvector of matrix A

Jordan Canonical Form

If matrix A involves multiple eigenvalues, diagonalization is not possible but matrix A can be transformed into a Jordan Canonical Form.

Consider the 3 x 3 matrix A : $\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$

Assume that matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ *where* $\lambda_1 = \lambda_2 \neq \lambda_3$

$(\lambda \mathrm{I} - \mathrm{A}) v_1 = 0 \qquad \mathrm{A} : 3 \times 3\, matrix, \qquad \lambda_1, \lambda_2, \lambda_3 \qquad v_1, v_2, v_3$

Case 1. $rank\left(\lambda_1 \mathrm{I} - \mathrm{A}\right) = 1$ can determine two eigenvectors $v_1,\ v_2$.

$$Av_1 = \lambda_1 v_1,\ \ Av_2 = \lambda_1 v_2,\ \ Av_3 = \lambda_3 v_3$$

$$\mathrm{A}\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad \mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Jordan Canonical Form

Case 2. $rank\left(\lambda_1 \mathrm{I} - \mathrm{A}\right) = 2$ can determine one eigenvector v_1 .

$$\mathrm{A}\, v_1 = \lambda_1 v_1, \qquad \mathrm{A}\, v_3 = \lambda_3 v_3$$

Find v_2 such that $\left|\mathrm{A} - \lambda_1 \mathrm{I}\right| v_2 = v_1 \quad \mathrm{A}\, v_2 = v_1 + \lambda_1 v_2$

$$\mathrm{A}\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\mathrm{x} = \mathrm{S}\,\mathrm{z}$ *where* $\mathrm{S} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1{}^2 & 2\lambda_1 & \lambda_3{}^2 \end{bmatrix}$ will yield $\mathrm{S}^{-1}\mathrm{AS} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathrm{J}$

This is in the Jordan Canonical Form.