Chapter 2. Review of Hydrodynamics and Vector Analysis

2.1 Taylor series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots$$

$$f(x_0) = a_0$$

$$f'(x) = a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1} + \cdots$$

$$f'(x_0) = a_1$$

$$f''(x) = 2a_2 + \cdots + n(n-1)(x - x_0)^{n-2} + \cdots$$

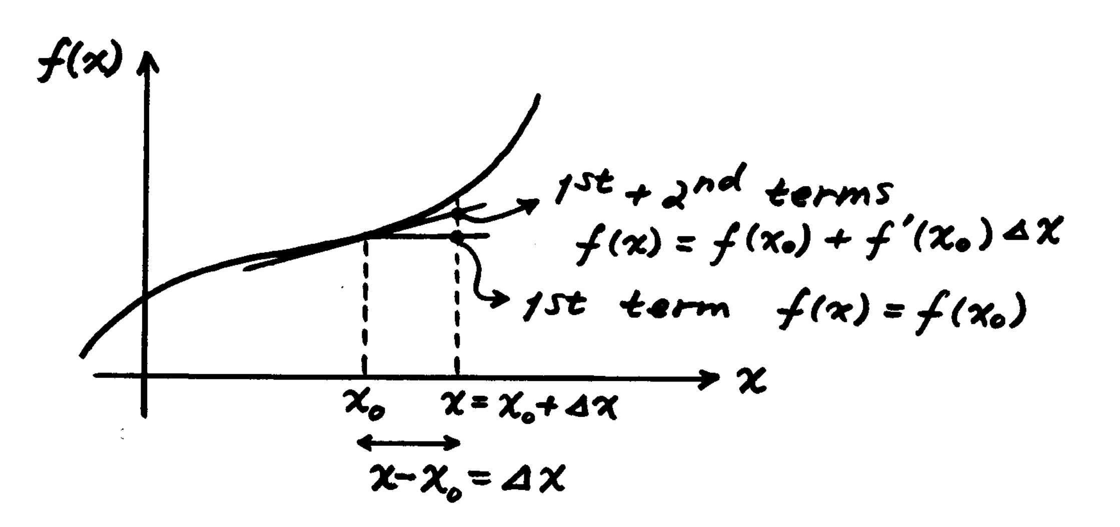
$$f''(x_0) = 2a_2 \;\Rightarrow\; a_2 = \frac{f''(x_0)}{2}$$

$$\vdots$$

$$\therefore f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} + \cdots$$

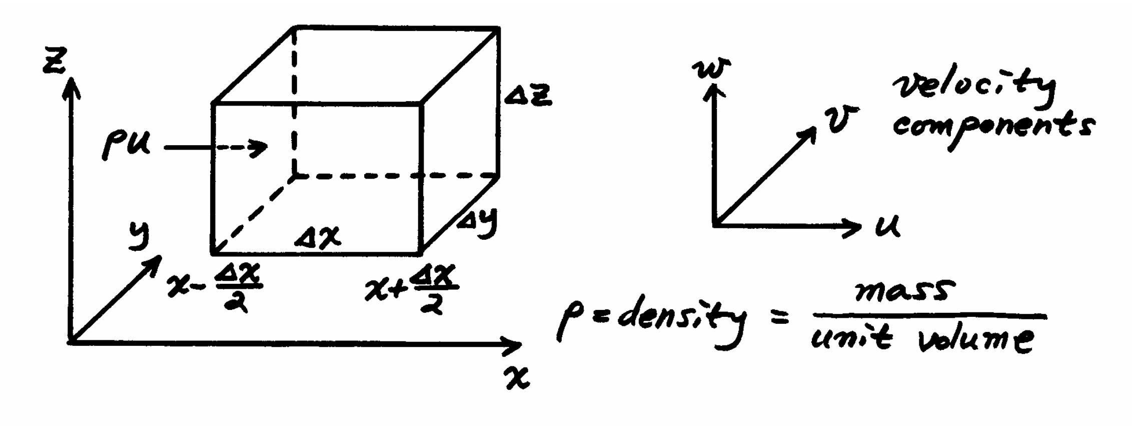
$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + f''(x_0)\frac{\Delta x^2}{2} + \cdots$$

If $f(x)$ is known at x_0, then it can be approximated at $x_0 + \Delta x$ by the Taylor series.



On the other hand, $f(x_0 - \Delta x) = f(x_0) - f'(x_0)\Delta x + f''(x_0)\frac{\Delta x^2}{2} - \cdots$

2.2 Conservation of mass



x -direction:

$$\rho\left(x-\frac{\Delta x}{2},y,z\right)u\left(x-\frac{\Delta x}{2},y,z\right)\Delta y\Delta z=\left[\rho(x,y,z)u(x,y,z)-\frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}+\cdots\right]\Delta y\Delta z$$

$$-\quad\rho\left(x+\frac{\Delta x}{2},y,z\right)u\left(x+\frac{\Delta x}{2},y,z\right)\Delta y\Delta z=\left[\rho(x,y,z)u(x,y,z)+\frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}+\cdots\right]\Delta y\Delta z$$

$$-\frac{\partial(\rho u)}{\partial x}\Delta x\Delta y\Delta z=[\text{mass flux}]_{\text{IN}_x}-[\text{mass flux}]_{\text{OUT}_x}$$

= mass accumulated with time in cube by x -direction

$\therefore$ Net accumulation in 3 directions for unit time

$$=\left[-\frac{\partial(\rho u)}{\partial x}-\frac{\partial(\rho v)}{\partial y}-\frac{\partial(\rho w)}{\partial z}\right]\Delta x\Delta y\Delta z$$

On the other hand, mass increase in cube during Δt is

$$\rho(x,y,z,t+\Delta t)\Delta x\Delta y\Delta z-\rho(x,y,z,t)\Delta x\Delta y\Delta z$$

$$=\left[\rho(x,y,z,t)+\frac{\partial\rho}{\partial t}\Delta t\right]\Delta x\Delta y\Delta z-\rho(x,y,z,t)\Delta x\Delta y\Delta z=\frac{\partial\rho}{\partial t}\Delta t\Delta x\Delta y\Delta z$$

$$\therefore\frac{\partial\rho}{\partial t}\Delta t\Delta x\Delta y\Delta z=\left[-\frac{\partial(\rho u)}{\partial x}-\frac{\partial(\rho v)}{\partial y}-\frac{\partial(\rho w)}{\partial z}\right]\Delta x\Delta y\Delta z\Delta t$$

= accumulation in 3 directions during Δt

$$\boxed{\therefore\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0}$$ Exact conservation of mass equation

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}+v\frac{\partial\rho}{\partial y}+\rho\frac{\partial v}{\partial y}+w\frac{\partial\rho}{\partial z}+\rho\frac{\partial w}{\partial z}=0$$

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$ = total derivative or material derivative

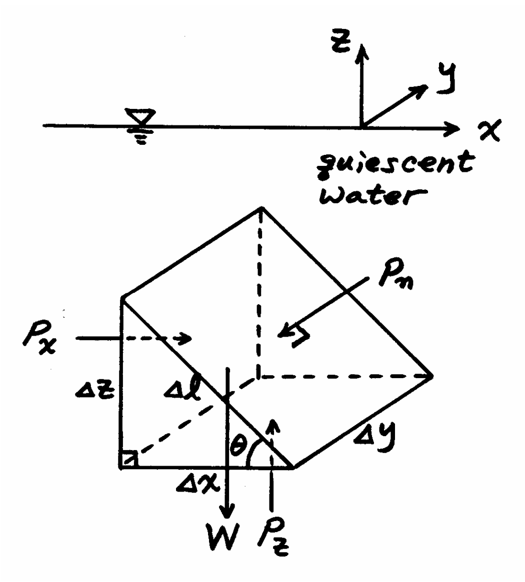
Bulk modulus, $E = \rho\frac{dp}{d\rho} \quad \Rightarrow \quad \frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{E}\frac{dp}{dt} \cong 0$

$E = 2.07 \times 10^9 \text{ N/m}^2$ for water (incompressible fluid)

For incompressible fluid,

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$ Continuity equation

2.3 Hydrostatic pressure



Newton's 2nd law:

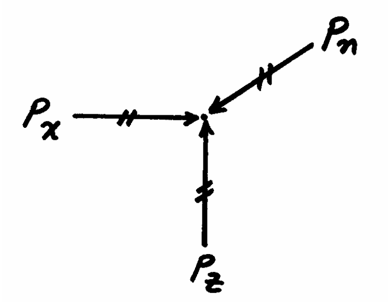
$$\vec{F} = m\vec{a} = 0 \quad (\because \text{no flow})$$

$$\sum F_x = p_x \Delta y \Delta z - p_n \sin\theta \Delta l \Delta y = p_x \Delta y \Delta z - p_n \Delta z \Delta y = 0 \quad \Rightarrow \quad p_x - p_n = 0$$

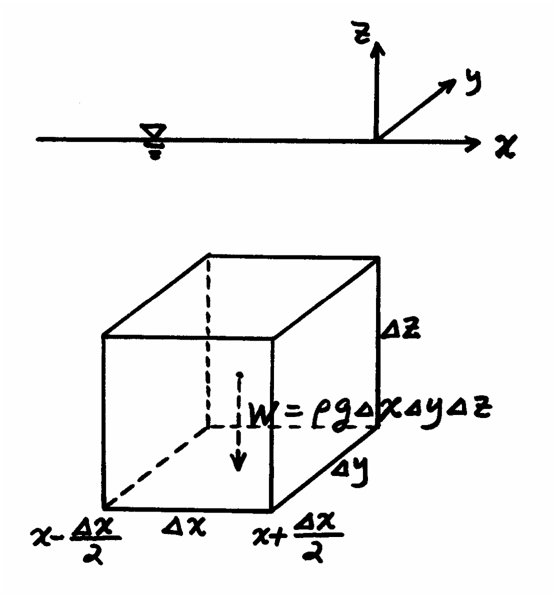
$$\sum F_z = p_z \Delta x \Delta y - p_n \cos\theta \Delta l \Delta y - \rho g \frac{1}{2} \Delta x \Delta z \Delta y = p_z \Delta x \Delta y - p_n \Delta x \Delta y - \rho g \frac{1}{2} \Delta x \Delta z \Delta y = 0$$

$$\Rightarrow \quad p_z - p_n - \frac{1}{2} \rho g \Delta z = 0$$

As the prism becomes smaller and smaller, or $\Delta x, \Delta z, W \to 0$, we have $p_x = p_z = p_n$



Pressure is a scalar. But the pressure force acting on a submerged body acts in the direction normal to the surface.



$$\sum F_x = 0 = p\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y \Delta z - p\left(x + \frac{\Delta x}{2}, y, z\right)\Delta y \Delta z$$

$$= \left[p(x, y, z) - \frac{\partial p}{\partial x}\frac{\Delta x}{2} + \cdots\right]\Delta y \Delta z - \left[p(x, y, z) + \frac{\partial p}{\partial x}\frac{\Delta x}{2} + \cdots\right]\Delta y \Delta z$$

$$= -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z$$

$$\therefore \frac{\partial p}{\partial x} = 0 \quad \Rightarrow \quad p = C_1(y, z) \qquad (1)$$

Similarly,

$$\sum F_y = 0 \quad \Rightarrow \quad \frac{\partial p}{\partial y} = 0 \quad \Rightarrow \quad p = C_2(x, z) \qquad (2)$$

From (1) and (2),

$$p = p(z) \qquad (3)$$

In z-direction,

$$\sum F_z = 0 = p\left(x, y, z - \frac{\Delta z}{2}\right)\Delta x \Delta y - p\left(x, y, z + \frac{\Delta z}{2}\right)\Delta x \Delta y - \rho g \Delta x \Delta y \Delta z$$

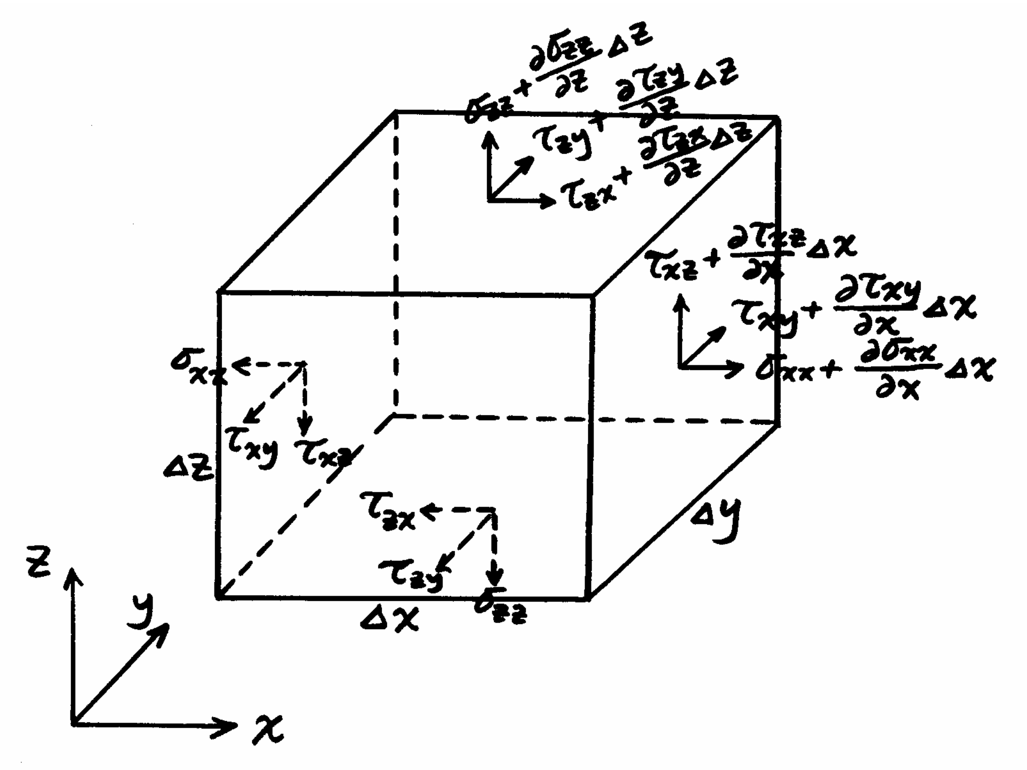
$$-\frac{\partial p}{\partial z}\Delta z \Delta x \Delta y - \rho g \Delta x \Delta y \Delta z = 0$$

$$\therefore \frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad p = -\rho g z + C_3(x, y) \qquad (4)$$

From (3) and (4), $C_3 = \text{constant}$. Because $p = 0$ at $z = 0$, we have $C_3 = 0$. Finally

$$p = -\rho g z$$

2.4 Equation of motion

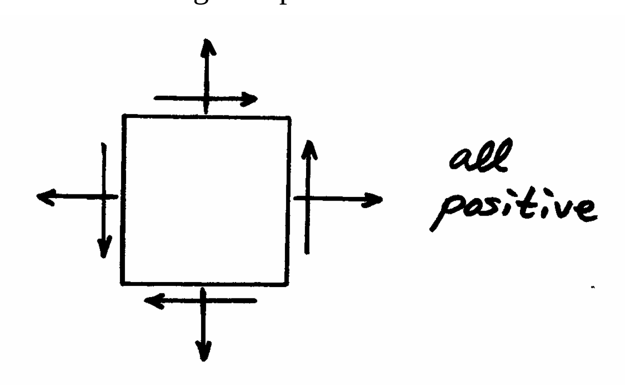


Surface forces:

$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ = normal stresses

$\tau_{xy}, \tau_{xz}, \tau_{zx}$, etc = shear stresses

The 1st subscript indicates the plane, while the 2nd subscript indicates the direction.
Sign convention: Stresses are positive if positive direction on positive plane or negative direction on negative plane.



Body force:

$$\rho\Delta x\Delta y\Delta z(X+Y+Z)$$

where X, Y, and Z are body forces per unit mass in x, y, and z directions, respectively.

Net forces acting on the cube of mass $\rho\Delta x\Delta y\Delta z$ are

$$x:\ F_x=\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\rho X\right)\Delta x\Delta y\Delta z$$

$$y:\ F_y=\left(\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{zy}}{\partial z}+\rho Y\right)\Delta x\Delta y\Delta z$$

$$z:\ F_z=\left(\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\rho Z\right)\Delta x\Delta y\Delta z$$

On the other hand, Newton's 2nd law gives

$$F_x=(\rho\Delta x\Delta y\Delta z)\times a_x=\rho\Delta x\Delta y\Delta z\frac{Du}{Dt}=\rho\Delta x\Delta y\Delta z\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$

$$F_y=(\rho\Delta x\Delta y\Delta z)\times a_y=\rho\Delta x\Delta y\Delta z\frac{Dv}{Dt}=\rho\Delta x\Delta y\Delta z\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)$$

$$F_z=(\rho\Delta x\Delta y\Delta z)\times a_z=\rho\Delta x\Delta y\Delta z\frac{Dw}{Dt}=\rho\Delta x\Delta y\Delta z\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)$$

Note: Newton's 2nd law applies to a fluid particle consisting of the same molecules or a system, so the total acceleration (or local acceleration plus convective accelerations) should be used.

$$\rho\frac{Du}{Dt}=\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\rho X$$

$$\rho\frac{Dv}{Dt}=\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{zy}}{\partial z}+\rho Y$$

$$\rho \frac{Dw}{Dt} = \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho Z$$

Introduce pressure (scalar):

$$p = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \quad \text{positive as compression}$$

Define

$$\tau_{xx} = \sigma_{xx} + p$$

$$\tau_{yy} = \sigma_{yy} + p$$

$$\tau_{zz} = \sigma_{zz} + p$$

Then

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho X$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho Y$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho Z$$

For $u = v = w = 0$ and $\tau_{xx} = \tau_{xy} = \cdots = 0$,

$$-\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y} - \frac{\partial p}{\partial z} + \rho(X + Y + Z) = 0$$

If z is positive vertically upwards, $\partial p / \partial x = \partial p / \partial y = 0$. If gravity is the only body force, $X = Y = 0$ and $Z = -g$. Then

$$\frac{\partial p}{\partial z} = -\rho g$$

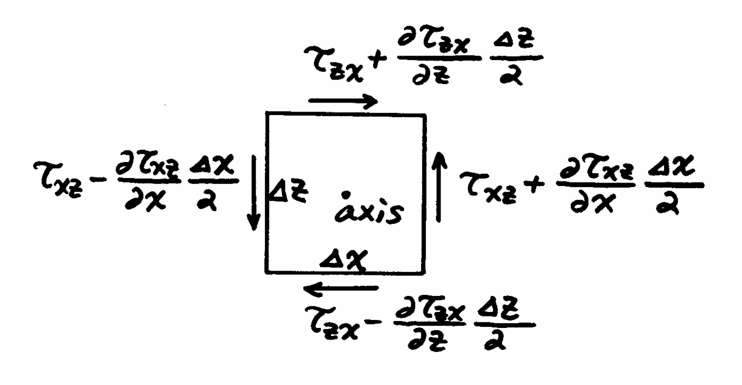
$$p = -\rho g z \quad \leftarrow \text{hydrostatic pressure}$$

Assume all the shear stresses are zero (inviscid fluid), that is valid for most water wave problems. Then

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Euler equation (Equation of motion for inviscid fluid)

Consider angular momentum:



$$\sum M = I\dot{\omega}$$

where M = moment, I = moment of inertia, and $\dot{\omega}$ = angular acceleration.

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y \frac{\Delta z}{2} - \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}\frac{\Delta x}{2}\right)\Delta z \Delta y \frac{\Delta x}{2}$$
$$+\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y \frac{\Delta z}{2} - \left(\tau_{xz} - \frac{\partial \tau_{xz}}{\partial x}\frac{\Delta x}{2}\right)\Delta z \Delta y \frac{\Delta x}{2} = \frac{1}{12}\rho \Delta x \Delta y \Delta z \left(\Delta x^2 + \Delta z^2\right)\dot{\omega}$$

$$\tau_{zx} - \tau_{xz} = \frac{1}{12}\rho\left(\Delta x^2 + \Delta z^2\right)\dot{\omega} \sim O\left(\Delta x^2\right)\dot{\omega}$$

As the cube becomes smaller and smaller, $\dot{\omega}$ increases, that implies fast spinning of water particle. Therefore, $\tau_{zx} - \tau_{xz}$ must be zero. In general,

$$\tau_{ij} = \tau_{ji} \quad (i, j = x, y, z)$$

For laminar flow, shear stress is proportional to rate of strain, so that

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}$$

where μ = coefficient of viscosity. Substituting into x-direction momentum equation,

$$\begin{aligned}\rho\frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} + \mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z} + \mu\frac{\partial w}{\partial x}\right) + \rho X \\ &= -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \rho X\end{aligned}$$

Using the continuity equation, we obtain the Navier-Stokes equation, equation of motion for incompressible, Newtonian fluid:

$$\boxed{\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + X \\ \frac{Dv}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + Y \\ \frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + Z\end{aligned}}$$

2.5 Review of vector analysis

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

where $\vec{i}, \vec{j}, \vec{k}$ = unit vectors in x, y, z directions. The length or magnitude of the vector $\vec{a}$ is given by

$$|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

2.5.1 Dot product

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta = \vec{b} \cdot \vec{a} \quad \text{(commutative)}$$

where θ = angle between the two vectors. For unit vectors, we have

$$\vec{i} \cdot \vec{i} = 1 \times 1 \times \cos 0° = 1 = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k}$$

$$\vec{i} \cdot \vec{j} = 1 \times 1 \times \cos 90° = 0 = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k}$$

Using the above relation,

$$\vec{a} \cdot \vec{b} = (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \cdot (b_x \vec{i} + b_y \vec{j} + b_z \vec{k}) = a_x b_x + a_y b_y + a_z b_z$$

Note: 1) Dot product is a scalar.

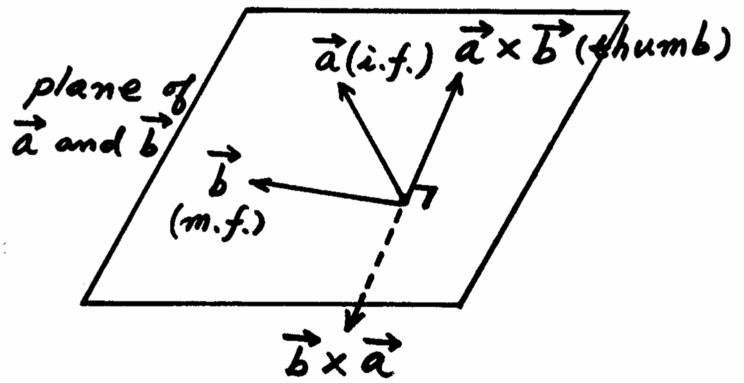
2) If $|a| \neq 0$, $|b| \neq 0$, but $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$ $(\because \cos\theta = 0)$

3) Projection of $\vec{a}$ onto $\vec{b}$ = $\vec{a} \cdot \vec{b} / |b|$

2.5.2 Cross product

$$\vec{a}\times\vec{b}=|a||b|\sin\theta\neq\vec{b}\times\vec{a}\quad\text{(not commutative)}$$

Cross product of vectors is a vector (magnitude + direction). The magnitude is given by the above expression, and the direction is given by the right-hand rule.



For unit vectors,

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0\quad(\because\sin\theta=0)$$

$$\vec{i}\times\vec{j}=\vec{k},\quad \vec{j}\times\vec{i}=-\vec{k},\quad \vec{j}\times\vec{k}=\vec{i},\quad \vec{k}\times\vec{j}=-\vec{i},\quad \vec{k}\times\vec{i}=\vec{j},\quad \vec{i}\times\vec{k}=-\vec{j}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=(a_yb_z-a_zb_y)\vec{i}+(a_zb_x-a_xb_z)\vec{j}+(a_xb_y-a_yb_x)\vec{k}$$

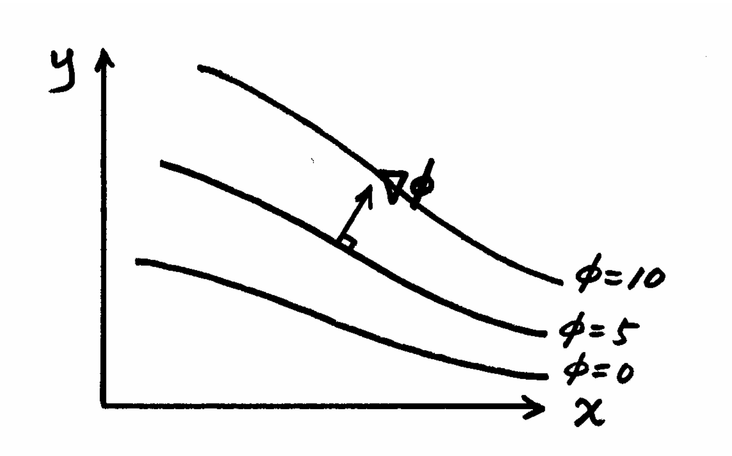
Note: If $|a|\neq 0$, $|b|\neq 0$, but $\vec{a}\times\vec{b}=0$, then $\vec{a}\,//\,\vec{b}$ $(\because\sin\theta=0)$

2.5.3 Vector differential operator

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla\phi = \text{gradient of } \phi \text{ (scalar)} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

Gradient indicates the spatial rate of change of a scalar. Gradient is a vector whose direction indicates the maximum rate of change.



$$\nabla\cdot\vec{u} = \text{divergence of } \vec{u}$$
$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(u_x\vec{i} + u_y\vec{j} + u_z\vec{k}\right)$$
$$= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

If $\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$,

$$\nabla\cdot\vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$\boxed{\nabla\cdot\vec{u} = 0}$ continuity equation in vector notation

Laplacian operator:

$$\nabla^2\phi = \nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

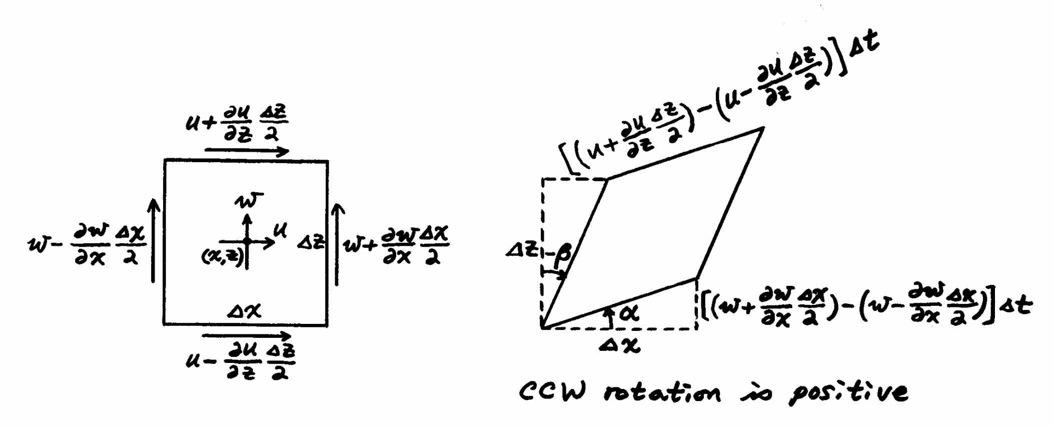
$$\begin{aligned}
\nabla\times\vec{u} &= \text{curl of } \vec{u} \\
&= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\times\left(u_x\vec{i} + u_y\vec{j} + u_z\vec{k}\right) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} \\
&= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\vec{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\vec{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\vec{k}
\end{aligned}$$

Useful vector identities:

Divergence of curl is zero, $\nabla\cdot\left(\nabla\times\vec{u}\right) = 0$

Curl of gradient is zero, $\nabla\times\nabla\phi = 0$

2.6 Rotation of fluid particle



At time t $\rightarrow$ At time $t+\Delta t$

$$\alpha \cong \tan\alpha = \frac{\left[\left(w+\frac{\partial w}{\partial x}\frac{\Delta x}{2}\right)-\left(w-\frac{\partial w}{\partial x}\frac{\Delta x}{2}\right)\right]\Delta t}{\Delta x} = \frac{\partial w}{\partial x}\Delta t$$

$$\therefore\ \frac{\partial \alpha}{\partial t} = \frac{\partial w}{\partial x} \quad \leftarrow \text{angular velocity (or rate of rotation)}$$

$$\frac{\partial \beta}{\partial t} = -\frac{\partial u}{\partial z}$$

The mean angular velocity, or vorticity is given by

$$\omega = \frac{1}{2}\left(\frac{\partial \alpha}{\partial t}+\frac{\partial \beta}{\partial t}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)$$

For irrotational flow, $\omega = 0$. Note that in an irrotational flow, the overall shape of the fluid particle can be distorted, but the mean angular velocity (or vorticity) must be zero.

2.7 Velocity potential

Definition (3-D):

If $\omega = \nabla \times \vec{u} = (\partial w/\partial y - \partial v/\partial z)\vec{i} + (\partial u/\partial z - \partial w/\partial x)\vec{j} + (\partial v/\partial x - \partial u/\partial y)\vec{k} = 0$, then the flow is irrotational.

Theorem:

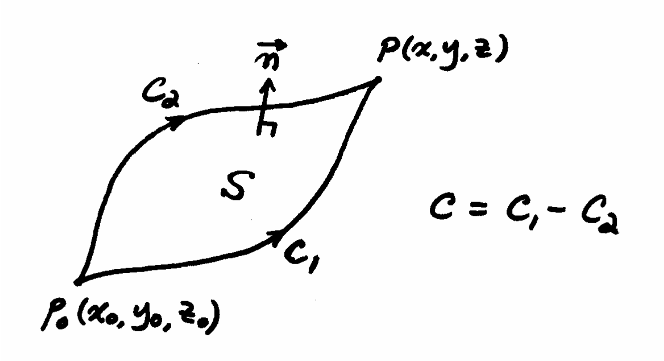
$\omega = \nabla \times \vec{u} = 0$ if and only if there exists a scalar field ϕ such that $\vec{u} = \nabla\phi$.

Proof (Greenberg, 1978. Foundations of Applied Mechanics, Prentice-Hall, 170-171):

1) Assume that there is a scalar ϕ such that $\vec{u} = \nabla\phi$.

2) Then $\omega = \nabla \times \vec{u} = \nabla \times (\nabla\phi) = 0$ ($\because$ curl of gradient = 0)

3) We must show that $\nabla \times \vec{u} = 0$ implies the existence of a scalar ϕ such that $\vec{u} = \nabla\phi$.



By Stokes theorem,

$$\int_C \vec{u} \cdot d\vec{r} = \int_S \vec{n} \cdot \nabla \times \vec{u} d\sigma = 0 \quad (\because \ \nabla \times \vec{u} = 0)$$

However,

$$\int_C \vec{u}\cdot d\vec{r} = 0 = \int_{C_1} \vec{u}\cdot d\vec{r} + \int_{-C_2} \vec{u}\cdot d\vec{r} = \int_{C_1} \vec{u}\cdot d\vec{r} - \int_{C_2} \vec{u}\cdot d\vec{r}$$

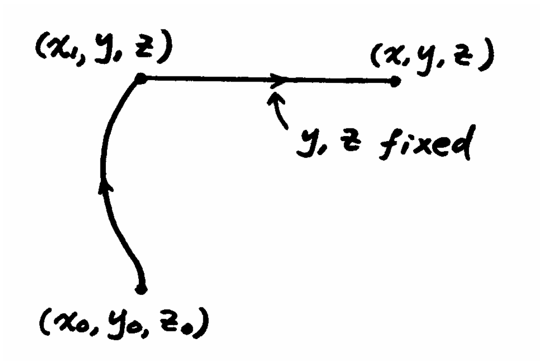
Therefore,

$$\int_{C_1} \vec{u}\cdot d\vec{r} = \int_{C_2} \vec{u}\cdot d\vec{r} \quad \leftarrow \text{independent of path of integration}$$

Define

$$\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{u}\cdot d\vec{r} \equiv \phi(x,y,z)$$

Then the velocity potential $\phi(x,y,z)$ is uniquely defined by the point $P(x,y,z)$, not by the path of integration.



Using

$$\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

we have

$$\phi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{u}\cdot d\vec{r} = \int_{(x_0,y_0,z_0)}^{(x_1,y,z)} (udx + vdy + wdz) + \int_{(x_1,y,z)}^{(x,y,z)} (udx + vdy + wdz)$$

Using the fundamental theorem of integral calculus,

$$\frac{d}{dx}\int_a^x f(\xi)d\xi = f(x)$$

we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\int_{(x_1,y,z)}^{(x,y,z)}(udx + vdy + wdz) = u$$

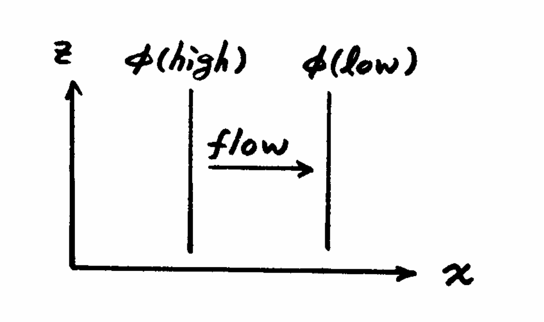
In the same way,

$$\frac{\partial\phi}{\partial y} = v, \quad \frac{\partial\phi}{\partial z} = w$$

Therefore,

$$\vec{u} = \nabla\phi$$

This is the end of the proof.



Considering the above diagram, physically the flow must be in positive direction. But according to the definition of the velocity potential, we have

$$u = \frac{\partial\phi}{\partial x} < 0$$

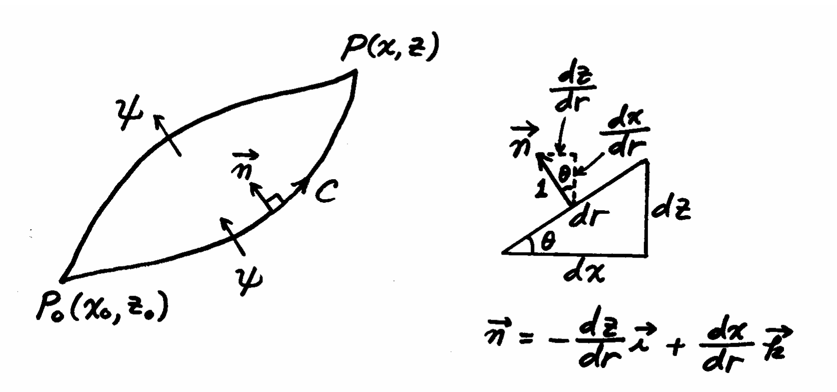
or the flow is in negative direction. Therefore, we redefine the velocity potential as

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \quad \rightarrow \quad \vec{u} = -\nabla \phi$$

so that the flow occurs in the direction from high potential to low potential.

2.8 Stream function

Consider 2D $x - z$ plane



$$\Psi = \int_{(x_0,z_0)}^{(x,z)} \vec{u} \cdot \vec{n} dr \quad \leftarrow \text{flow rate across the line connecting } P_0 \text{ and } P$$

$$= \int_{(x_0,z_0)}^{(x,z)} \left(u\vec{i} + w\vec{k}\right) \cdot \left(-dz\vec{i} + dx\vec{k}\right)$$

$$= \int_{(x_0,z_0)}^{(x,z)} (-udz + wdx)$$

For conservation of mass, Ψ is independent of path of integration. For this, the integrand must be an exact differential, $d\Psi$. This requires that

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial z} dz = -udz + wdx$$

or

$$w = \frac{\partial \Psi}{\partial x}, \quad u = -\frac{\partial \Psi}{\partial z}$$

Using

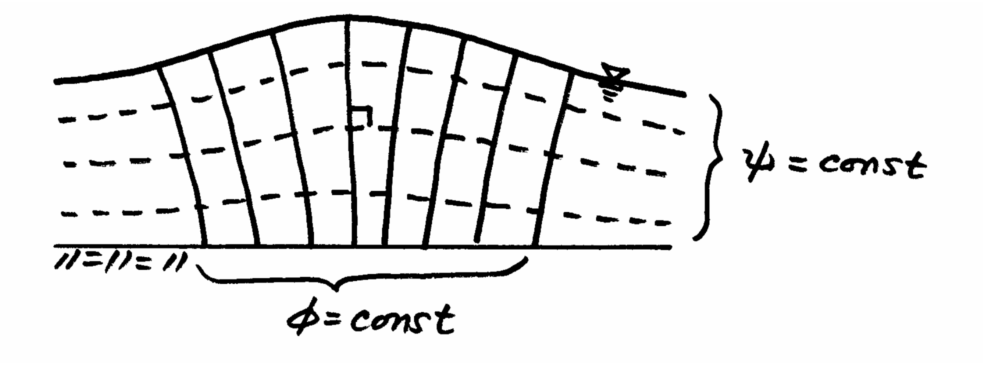
$$\frac{\partial w}{\partial z} = \frac{\partial^2 \Psi}{\partial x \partial z} \text{ and } \frac{\partial u}{\partial x} = -\frac{\partial^2 \Psi}{\partial z \partial x}$$

we have

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

which is 2-D continuity equation. Therefore, a stream function exists for 2-D incompressible flow.

$$\left.\begin{array}{l} \phi : \text{line integral along } C \\ \Psi : \text{line integral across } C \end{array}\right\} \rightarrow \phi \text{ and } \Psi \text{ are orthogonal}$$



Finally, ϕ and Ψ satisfies the Cauchy-Riemann conditions:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \Psi}{\partial z}$$
$$\frac{\partial \phi}{\partial z} = -\frac{\partial \Psi}{\partial x}$$

2.9 Bernoulli equation

Euler equation in x-direction is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Using 2-D irrotational flow condition, $\partial u / \partial z = \partial w / \partial x$,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{w^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similarly, in z - direction, we have

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial z}\left(\frac{w^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Introducing ϕ and $u = -\partial\phi / \partial x$, $w = -\partial\phi / \partial z$,

$$x: \quad \frac{\partial}{\partial x}\left[-\frac{\partial \phi}{\partial t} + \frac{(\partial\phi/\partial x)^2 + (\partial\phi/\partial z)^2}{2} + \frac{p}{\rho}\right] = 0$$

$$z: \quad \frac{\partial}{\partial z}\left[-\frac{\partial \phi}{\partial t} + \frac{(\partial\phi/\partial x)^2 + (\partial\phi/\partial z)^2}{2} + \frac{p}{\rho} + gz\right] = 0$$

Integration gives

$$x: \quad -\frac{\partial \phi}{\partial t} + \frac{(\partial\phi/\partial x)^2 + (\partial\phi/\partial z)^2}{2} + \frac{p}{\rho} = C_1(z,t)$$

$$z: \quad -\frac{\partial \phi}{\partial t} + \frac{(\partial\phi/\partial x)^2 + (\partial\phi/\partial z)^2}{2} + \frac{p}{\rho} = -gz + C_2(x,t)$$

The above two equations give

$$C_1(z,t) = -gz + C_2(x,t) \quad \Rightarrow \quad C_2(x,t) = C(t)$$

Therefore,

$$-\frac{\partial \phi}{\partial t} + \frac{(\partial \phi / \partial x)^2 + (\partial \phi / \partial z)^2}{2} + \frac{p}{\rho} + gz = C(t)$$

which is the Bernoulli equation for unsteady flow, giving relationship between pressure field and flow kinematics (u, w).

Using

$$C(t) = \frac{\partial f(t)}{\partial t}$$

the Bernoulli equation becomes

$$-\frac{\partial}{\partial t}\left[\phi + f(t)\right] + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + \frac{p}{\rho} + gz = 0$$

Defining $\phi' = \phi + f(t)$, we have

$$-\frac{\partial \phi'}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi'}{\partial x}\right)^2 + \left(\frac{\partial \phi'}{\partial z}\right)^2\right] + \frac{p}{\rho} + gz = 0$$

The Bernoulli term $C(t)$ is included in ϕ'. However, the flow kinematics are the same for ϕ' and ϕ:

$$\frac{\partial \phi'}{\partial x} = \frac{\partial \phi}{\partial x} = u, \quad \frac{\partial \phi'}{\partial z} = \frac{\partial \phi}{\partial z} = w$$