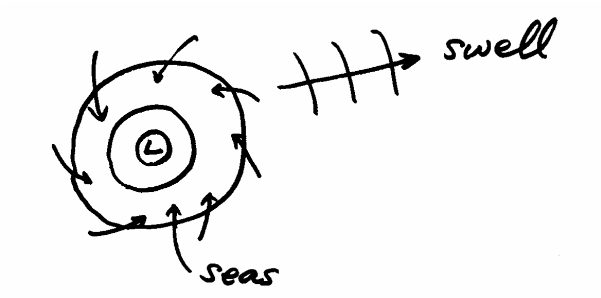
Chapter 2. Two-Dimensional Wave Equations and Wave Characteristics

2.1 Surface Gravity Waves

– Classification of waves (See handout, Shore Protection Manual, 1984, Fig. 2-1):

- capillary waves: $T \le 0.1\,\mathrm{s}$; surface tension is important
- gravity waves (or wind waves): $1 \le T \le 30\ \mathrm{s}$; mostly generated by wind, and gravity is primary restoring force; the most important in coastal engineering
  - seas (풍파): waves generated by local wind
  - swell (너울): waves propagating out of storms

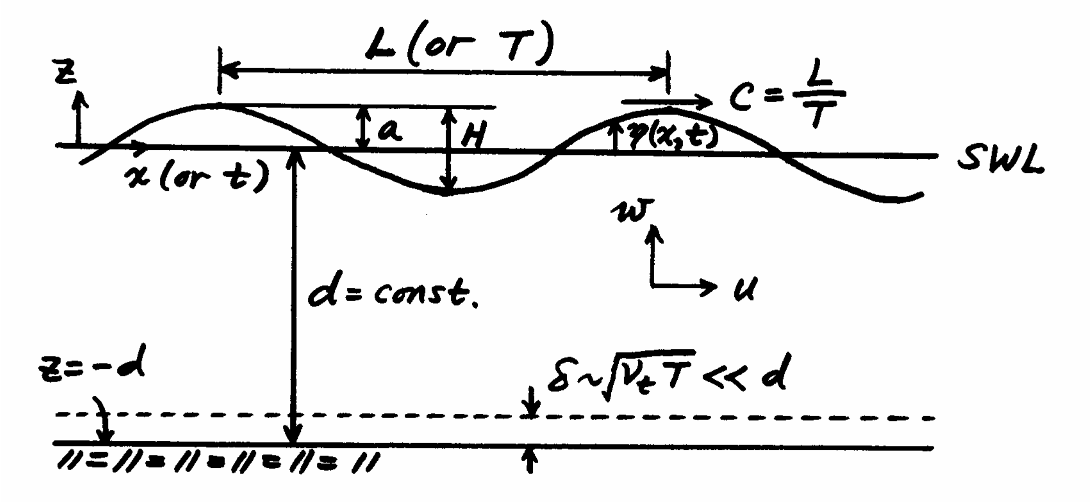


- infragravity waves: $30\ \mathrm{s} \le T \le 5\ \mathrm{min}$; important for surf beat and harbor oscillation
- long-period waves: $T \ge 5\ \mathrm{min}$; storm surge, tsunamis, tides

– Regular versus irregular waves:

- regular (or sinusoidal, monochromatic) waves: uniform period and amplitude
- irregular (or random) waves: superposition of a number of sinusoidal waves with different periods, amplitudes, directions, and phases (See handout, Goda, 2000, Fig. 2.8):

2.2 Small-Amplitude (or Linear) Wave Theory



d = water depth; L = wavelength; T = wave period; C = wave celerity (speed);
H = wave height; $a = H/2$ = wave amplitude; η = surface displacement;
δ = boundary layer thickness; ν_t = eddy viscosity

Assuming inviscid and incompressible fluid and irrotational flow motion, velocity potential $\phi(x,z,t)$ exists, which satisfies the Laplace equation (cf. Elementary Fluid Mechanics, Street et al.):

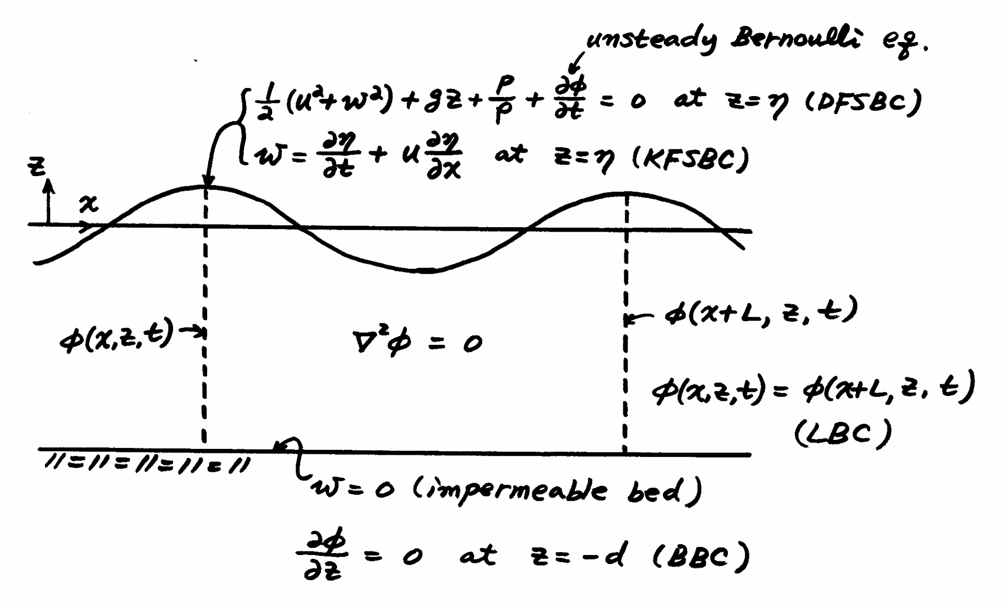
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\uparrow$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \text{(continuity equation)}$$

$$u = \frac{\partial\phi}{\partial x}, \quad w = \frac{\partial\phi}{\partial z} \quad \leftarrow \text{definition of velocity potential}$$

Boundary value problem



Assume the free surface varies sinusoidally in both space and time so that

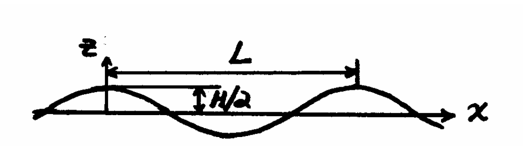
$$\eta(x,t)=\frac{H}{2}\cos(kx-\sigma t)$$

where

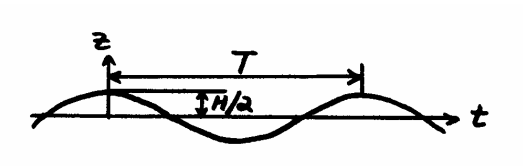
$$k=\frac{2\pi}{L} \quad \text{(wave number)}$$

$$\sigma=\frac{2\pi}{T} \quad \text{(wave angular frequency)} \qquad ※ \; f=\frac{1}{T} \quad \text{(wave frequency)}$$

At $t=0$, $\eta(x,0)=\frac{H}{2}\cos kx$

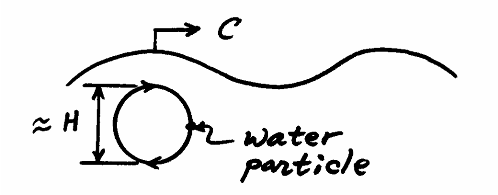


At $x=0$, $\eta(0,t)=\frac{H}{2}\cos(-\sigma t)=\frac{H}{2}\cos\sigma t$



Small-amplitude wave theory assumes

$$L \sim O(d)$$
$$H/L \quad = \text{wave steepness} << 1$$
$$\eta \sim O(H)$$
$$t \sim O(T)$$
$$x \sim O(L)$$
$$z \sim O(d) \sim O(L)$$
$$u = \frac{\partial\phi}{\partial x} \sim O\left(\frac{H}{T}\right)$$
$$\phi \sim O\left(\frac{HL}{T}\right)$$



<u>DFSBC</u>

$$\frac{1}{2}\left(u^2 + w^2\right) + gz + \frac{p}{\rho} + \frac{\partial\phi}{\partial t} = 0 \qquad \text{at} \quad z = \eta$$

$$\frac{1}{2}\left(u^2 + w^2\right) + g\eta + \frac{\partial\phi}{\partial t} = 0 \qquad \text{at} \quad z = \eta$$

$$\left(\frac{H^2}{T^2}\right) \qquad (gH) \quad \left(\frac{HL}{T^2}\right)$$

$$\left(\frac{H}{L}\right) << 1 \quad \left(\frac{gT^2}{L}\right) \quad (1)$$

$$g\eta + \frac{\partial\phi}{\partial t} = 0 \qquad \text{at} \quad z = \eta$$

Taylor series expansion about $z=0$ gives

$$\left(g\eta + \frac{\partial\phi}{\partial t}\right)_{z=\eta} = \left(g\eta + \frac{\partial\phi}{\partial t}\right)_{z=0} + \eta\frac{\partial}{\partial z}\left(g\eta + \frac{\partial\phi}{\partial t}\right)_{z=0} + \cdots = 0$$

$$(gH) \quad \left(\frac{HL}{T^2}\right) \quad \left(g\frac{H^2}{L}\right) \quad \left(\frac{H^2}{T^2}\right)$$

$$\left(\frac{gT^2}{L}\right) \quad (1) \quad \left(\frac{gT^2}{L}\frac{H}{L}\right) \quad \left(\frac{H}{L}\right)$$

Now

$$g\eta + \frac{\partial\phi}{\partial t} = 0 \qquad \text{at} \quad z=0 \quad \text{(LDFSBC)}$$

Assume

$$\phi(x,z,t) = Z(z)\sin(kx - \sigma t)$$

Substitution into the Laplace equation and applying BBC and LDFSBC give

$$\phi(x,z,t) = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx-\sigma t)$$

<u>KFSBC</u>

$$w = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} \qquad \text{at} \quad z=\eta$$

Order of magnitude analysis and Taylor series expansion about $z=0$ gives

$$w = \frac{\partial\eta}{\partial t} \qquad \text{at} \quad z=0 \quad \text{(LKFSBC)}$$

Using LDFSBC,

$$w = \frac{\partial \eta}{\partial t} = -\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2} \qquad \text{at} \quad z = 0$$

Using $w = \partial \phi / \partial z$,

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \qquad \text{at} \quad z = 0 \quad \text{(LCFSBC)}$$

<u>Dispersion Relationship</u>

Substitution of ϕ into LKFSBC gives

$$\sigma^2 = gk \tanh kd \qquad \text{(dispersion relationship)}$$

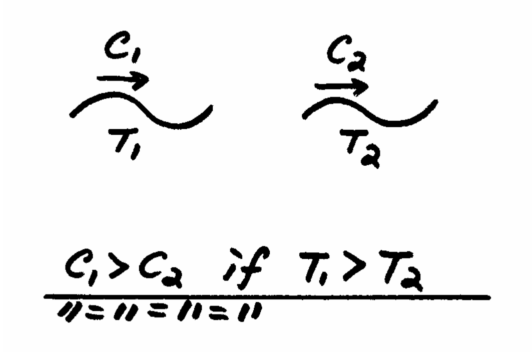
$$C = \frac{L}{T} = \frac{\sigma}{k} = \frac{\sqrt{gk \tanh kd}}{k} = \sqrt{\frac{g}{k}\tanh kd} = \sqrt{\frac{gL}{2\pi}\tanh\frac{2\pi d}{L}}$$

$$\frac{(2\pi)^2}{T^2} = g\frac{2\pi}{L}\tanh\frac{2\pi d}{L}$$

$$\frac{L}{T^2} = \frac{g}{2\pi}\tanh\frac{2\pi d}{L}$$

$$C = \frac{gT}{2\pi}\tanh\frac{2\pi d}{L}$$

$$L = \frac{gT^2}{2\pi}\tanh\frac{2\pi d}{L} \quad \leftarrow \text{proves} \quad \frac{gT^2}{L} \sim O(1) \quad \text{in previous order of magnitude analysis}$$



Waves disperse due to difference of $T(= 2\pi / \sigma)$ $\Rightarrow$ frequency dispersion

The wave period (T) is constant as a wave propagates from deep to shallow water (i.e., d varies). But L and C vary as d varies.

The dispersion relationship should be solved for k (or L) for given d and σ (or T) by iteration using Newton-Raphson method. Approximate formulas are available.

Eckart (1951):

$$\sigma^2 = gk\sqrt{\tanh\left(\frac{\sigma^2}{g}d\right)}$$

$$L = \frac{gT^2}{2\pi}\sqrt{\tanh\left(\frac{4\pi^2}{T^2}\frac{d}{g}\right)}$$

Hunt (1979):

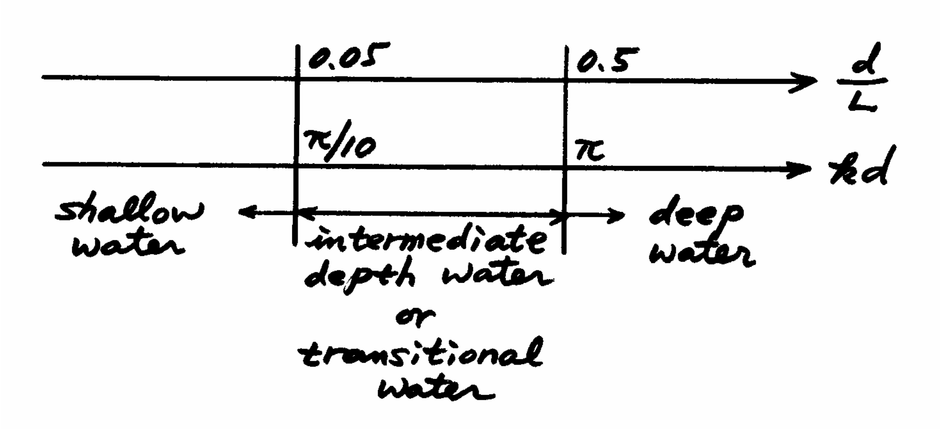
$$(kd)^2 = y^2 + \frac{y}{1+\sum_{n=1}^{6} c_n y^n}$$

where

$y = \sigma^2 d / g$ and $c_1 = 0.6666666666$, $c_2 = 0.3555555555$, $c_3 = 0.1608465608$, $c_4 = 0.0632098765$, $c_5 = 0.0217540484$, $c_6 = 0.0065407983$.

2.3 Wave Classification

Relative depth = $\frac{d}{L}$ or $kd\left(=\frac{2\pi d}{L}\right)$



In deep water, $\tanh kd \cong 1$, so that

$$C_0 = \sqrt{\frac{gL_0}{2\pi}} = \frac{gT}{2\pi}$$

$$L_0 = \frac{gT^2}{2\pi}$$

No dependence on d

In shallow water, $\tanh kd \cong kd$, so that

$C = \sqrt{gd}$ $\rightarrow$ No dependence on T (Shallow water wave is non-dispersive)

$$L = \sqrt{gd}T$$

In intermediate depth water,

$$\frac{C}{C_0} = \frac{L}{L_0} = \tanh\frac{2\pi d}{L} \quad \Rightarrow \quad C, L \downarrow \text{ as } d \downarrow$$

2.4 Wave Kinematics and Pressure

Particle velocity

$$u = \frac{\partial \phi}{\partial x} = \frac{H}{2}\frac{kg}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\cos(kx - \sigma t)$$
$$= \frac{\pi H}{T}\frac{\cosh k(d+z)}{\sinh kd}\cos(kx - \sigma t)$$
$$w = \frac{\partial \phi}{\partial z} = \frac{\pi H}{T}\frac{\sinh k(d+z)}{\sinh kd}\sin(kx - \sigma t)$$

Acceleration

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \cong \frac{\partial u}{\partial t} = \frac{\sigma \pi H}{T}\frac{\cosh k(d+z)}{\sinh kd}\sin(kx - \sigma t)$$
$$= \frac{2\pi^2 H}{T^2}\frac{\cosh k(d+z)}{\sinh kd}\sin(kx - \sigma t)$$
$$\frac{dw}{dt} \cong \frac{\partial w}{\partial t} = -\frac{\sigma \pi H}{T}\frac{\sinh k(d+z)}{\sinh kd}\cos(kx - \sigma t)$$
$$= -\frac{2\pi^2 H}{T^2}\frac{\sinh k(d+z)}{\sinh kd}\cos(kx - \sigma t)$$

Acceleration is $90°$ out of phase with velocity so that

$$\left(\frac{du}{dt}\right)_{\max} \quad \text{at} \quad w_{\max}$$
$$-\left(\frac{dw}{dt}\right)_{\max} \quad \text{at} \quad u_{\max}$$

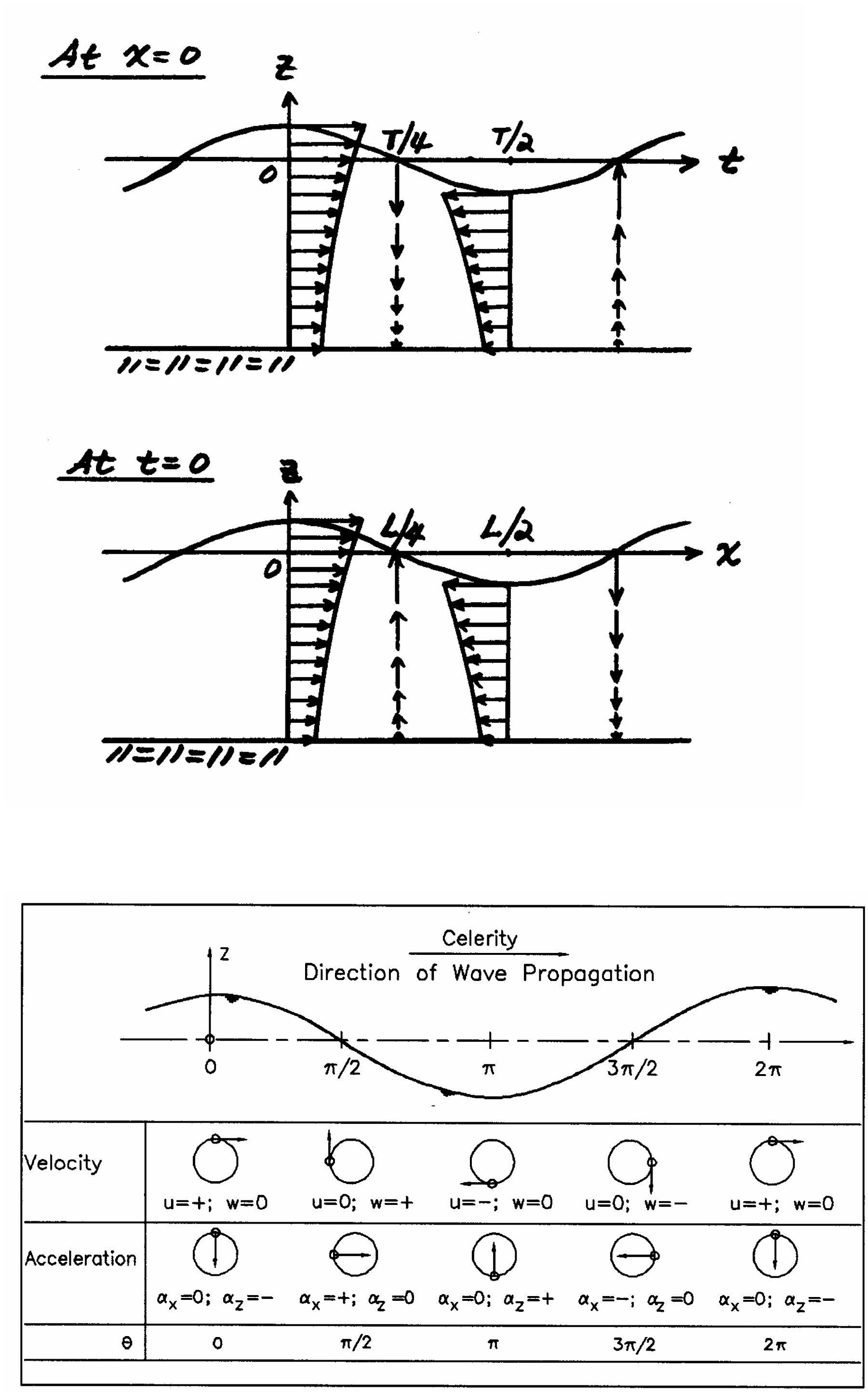


Figure II-1-2. Local fluid velocities and accelerations

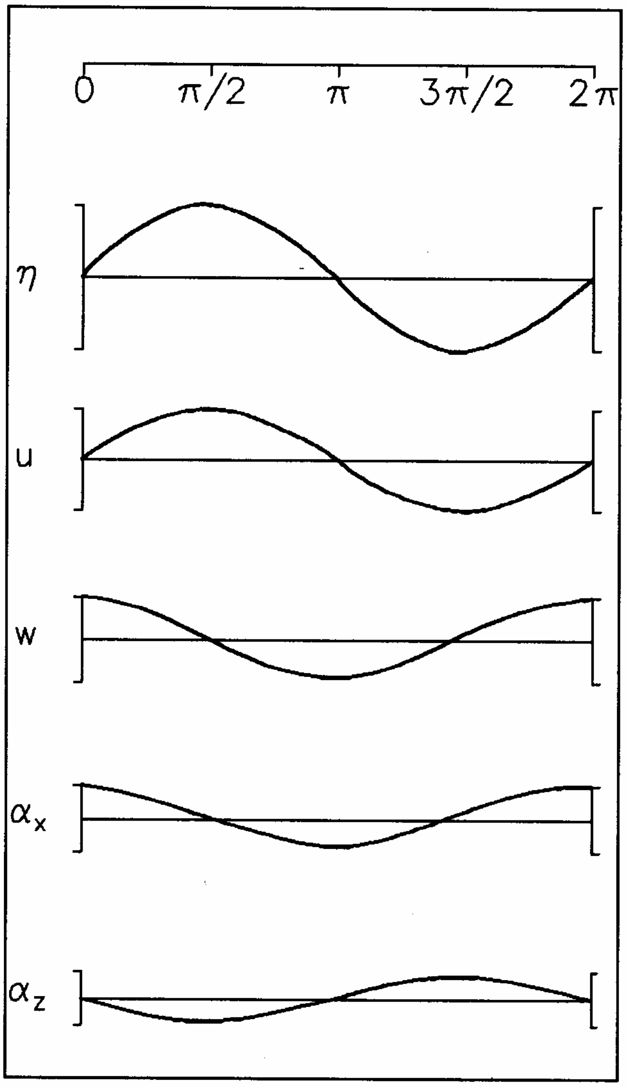
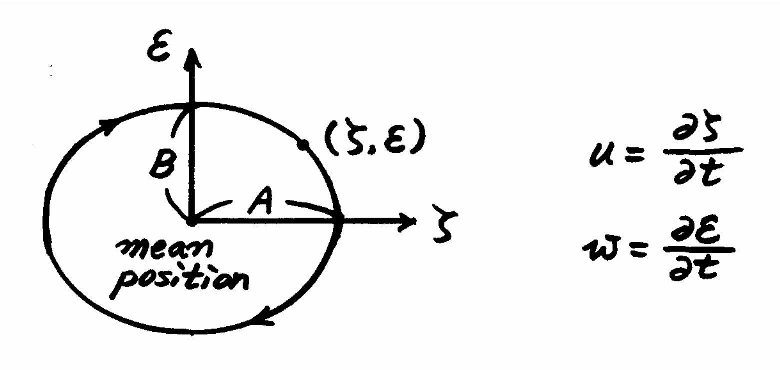


Figure II-1-3. Profiles of particle velocity and acceleration by Airy theory in relation to the surface elevation

Particle orbit



$$\varsigma = \int u dt = -\frac{1}{\sigma}\frac{\pi H}{T}\frac{\cosh k(d+z)}{\sinh kd}\sin(kx-\sigma t)$$

$$= -\frac{H}{2}\frac{\cosh k(d+z)}{\sinh kd}\sin(kx-\sigma t) = -A\sin(kx-\sigma t)$$

$$\varepsilon = \int w dt = \frac{H}{2}\frac{\sinh k(d+z)}{\sinh kd}\cos(kx-\sigma t) = B\cos(kx-\sigma t)$$

$$\left(\frac{\varsigma}{A}\right)^2 + \left(\frac{\varepsilon}{B}\right)^2 = \sin^2(kx-\sigma t) + \cos^2(kx-\sigma t) = 1 \qquad \therefore \text{ Ellipse}$$

In deep water,

$$A = \frac{H}{2}\frac{e^{k(d+z)}}{e^{kd}} = \frac{H}{2}e^{kz}$$

$$B = \frac{H}{2}\frac{e^{k(d+z)}}{e^{kd}} = \frac{H}{2}e^{kz} = A \qquad \therefore \text{ Circle}$$

Diameter of the circle decreases exponentially with depth

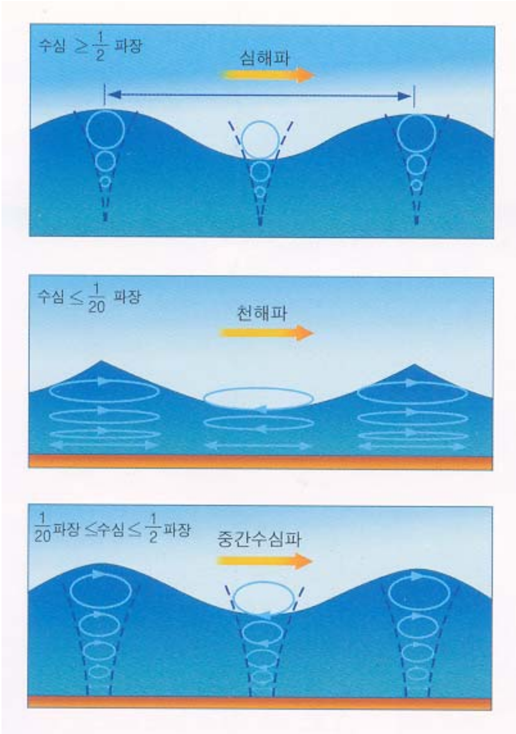
$\rightarrow$ No wave effect on sea bed (See Figure 2.2 in textbook)

In shallow water,

$$A = \frac{H}{2}\frac{1}{kd} \neq f(z) \;\leftarrow\; A \text{ is constant over depth}$$

$$B = \frac{H}{2}\frac{k(d+z)}{kd} = \frac{H}{2}\left(1+\frac{z}{d}\right) = \begin{cases} \dfrac{H}{2} & \text{at} \quad z=0 \\ 0 & \text{at} \quad z=-d \end{cases}$$

The ellipse becomes flatter and flatter with depth (see Figure 2.2 in textbook)



Water particle motions in deep, shallow, and intermediate-depth waters

<u>Wave pressure</u>

The unsteady Bernoulli equation for wave motion is

$$\underbrace{\frac{1}{2}\left(u^2+w^2\right)}_{\text{small(nonlinear)}} + \frac{p}{\rho} + gz + \frac{\partial \phi}{\partial t} = 0$$

Using

$$\phi = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx - \sigma t)$$

we have

$$
\begin{aligned}
p &= -\rho g z - \rho \frac{\partial \phi}{\partial t} \\
&= -\rho g z + \rho g \frac{H}{2} \frac{\cosh k(d+z)}{\cosh kd} \cos(kx - \sigma t)
\end{aligned}
$$

The first term is the hydrostatic pressure and the second term is the dynamic pressure due to wave motion. The dynamic pressure is positive under the wave crest but it is negative under the trough. See Figure 2.3 of textbook. The above equation is valid below the still water level. Above the still water level, the pressure is assumed to be hydrostatic so that

$$p = \rho g(\eta - z) \quad \text{for} \quad 0 < z < \eta$$

Since

$$\eta = \frac{H}{2} \cos(kx - \sigma t)$$

the dynamic pressure is related to the free surface elevation by

$$
\begin{aligned}
p_d &= \rho g \frac{H}{2} \frac{\cosh k(d+z)}{\cosh kd} \cos(kx - \sigma t) \\
&= \rho g \frac{\cosh k(d+z)}{\cosh kd} \eta
\end{aligned}
$$

$$\therefore \ \eta = \frac{p_d}{\rho g \dfrac{\cosh k(d+z)}{\cosh kd}}$$

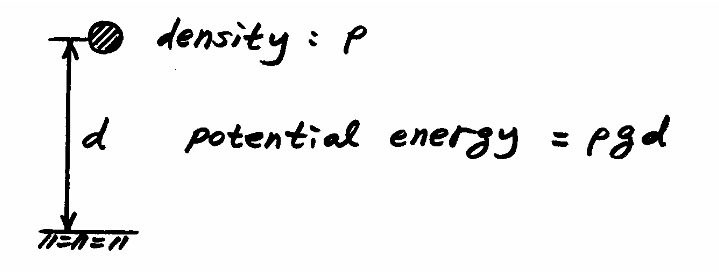
Therefore the surface waves can be measured using a pressure transducer.

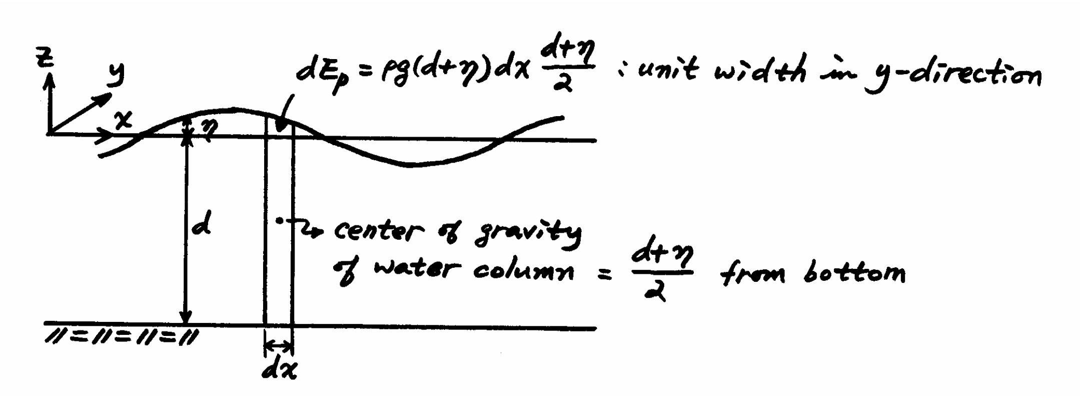
2.5 Energy, Power, and Group Velocity

Total energy (E) = potential energy (E_p) due to displacement of free surface

+ kinetic energy (E_k) due to water particle movement

Potential energy





$$E_p \text{ per one wavelength} = \int_0^L dE_p - \rho g d L\left(\frac{d}{2}\right)$$

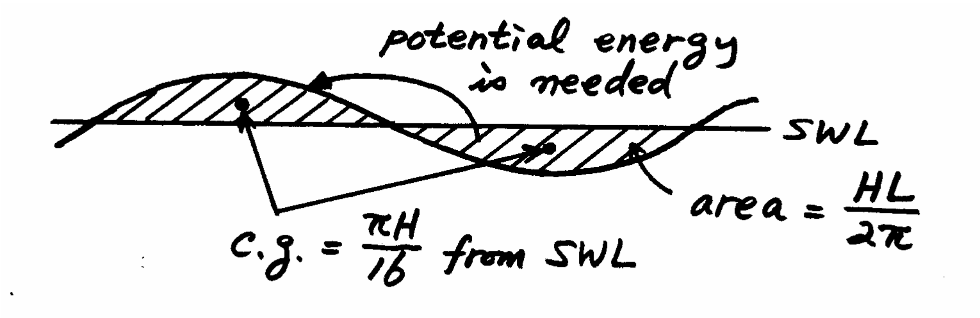
$$= \int_0^L \frac{\rho g}{2}(d+\eta)^2 dx - \frac{\rho g}{2} d^2 L$$

$$= \frac{\rho g}{16} H^2 L$$

Potential energy per unit surface area is

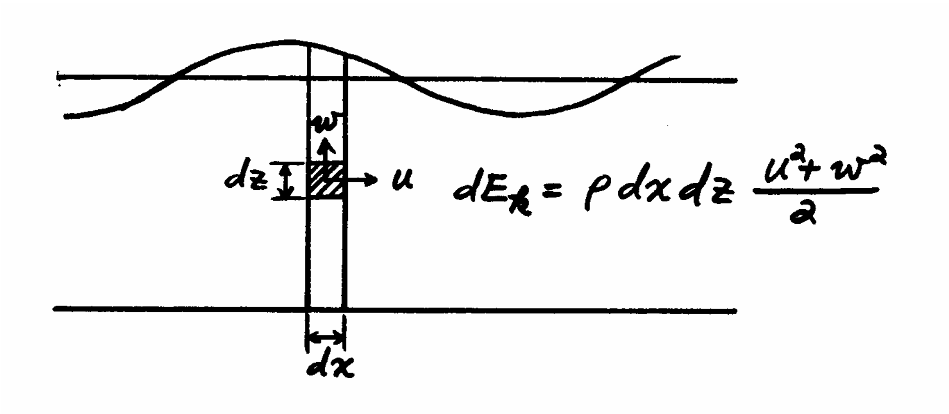
$$\overline{E}_p = \frac{E_p}{L} = \frac{\rho g}{16} H^2$$

Another view



$$E_p = \rho g \frac{HL}{2\pi} \times \frac{\pi H}{16} \times 2 = \frac{\rho g}{16} H^2 L$$

<u>Kinetic energy</u>



$$E_k \cong \int_0^L \int_{-d}^0 \frac{\rho}{2}\left(u^2 + w^2\right) dz dx = \frac{\rho g}{16} H^2 L$$

$$\overline{E}_k = \frac{E_k}{L} = \frac{\rho g}{16} H^2$$

$$\therefore \ \overline{E}_p = \overline{E}_k$$

Now the total energy per unit surface area is

$$\overline{E} = \overline{E}_p + \overline{E}_k = \frac{\rho g}{8} H^2 = \frac{\rho g}{2} a^2 \quad \left(\frac{\mathrm{N \cdot m}}{\mathrm{m^2}} \ \text{or} \ \frac{\mathrm{Joule}}{\mathrm{m^2}} \right)$$

Wave power

Wave power (P) = rate of work done = energy flux (F) = $\int_{-d}^{0} p_d u dz \quad \frac{\mathrm{N \cdot m}}{\mathrm{s}}$ or $\frac{\mathrm{J}}{\mathrm{s}}$

Average power over one wave period

$$\begin{aligned}\overline{P} &= \frac{1}{T}\int_0^T \int_{-d}^{0} p_d u dz dt \\ &= \frac{1}{8}\rho g H^2 \frac{\sigma}{k}\left[\frac{1}{2}\left(1+\frac{2kd}{\sinh 2kd}\right)\right] \\ &= \overline{E} C n\end{aligned}$$

where

$$n = \frac{1}{2}\left(1+\frac{2kd}{\sinh 2kd}\right)$$

Defining the group velocity as

$$C_g = nC$$

we have

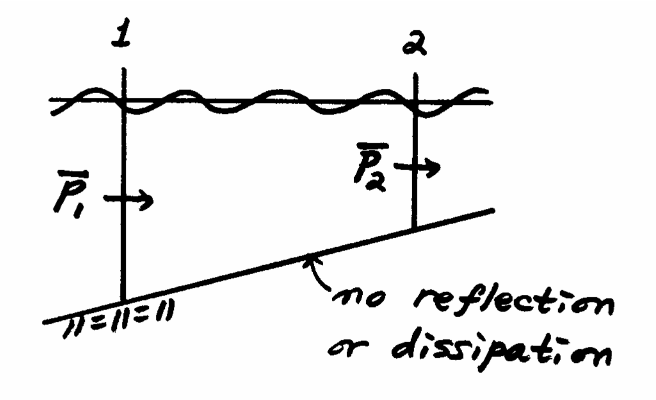
$$\overline{P} = \overline{F} = \overline{E} C_g$$

Now, the group velocity is the velocity at which wave energy is transferred. When we generate waves using a wavemaker in a wave flume, the leading edge of the generated waves propagates at the speed of group velocity.

In deep water, $n \to \frac{1}{2}, \quad \therefore \; C_{g_0} = \frac{C_0}{2}$

In shallow water, $n \to 1, \quad \therefore \; C_g = C$

Wave shoaling



In steady state, energy conservation equation is

$$\overline{P}_1 = \overline{P}_2 = \text{constant}$$

$$\left(\overline{E}Cn\right)_1 = \left(\overline{E}Cn\right)_2$$

$$\frac{1}{8}\rho g H_1^2 C_1 n_1 = \frac{1}{8}\rho g H_2^2 C_2 n_2$$

$$\frac{H_2}{H_1} = \sqrt{\frac{C_1 n_1}{C_2 n_2}} = \sqrt{\frac{L_1 n_1}{L_2 n_2}}$$

If the location 1 is in deep water,

$$\frac{H_2}{H_0} = \sqrt{\frac{C_0 n_0}{C_2 n_2}} = \sqrt{\frac{C_0}{2 C_2 n_2}} = \sqrt{\frac{C_0}{2 C_{g_2}}}$$

$$H_2 = H_0 \sqrt{\frac{C_0}{2 C_{g_2}}} = H_0 K_s$$

where K_s is the shoaling coefficient given by

$$K_s = \sqrt{\frac{C_0}{2C_g}} = \frac{H}{H_0}$$

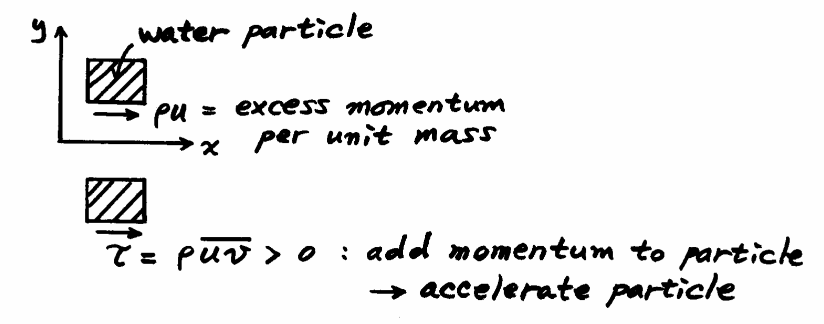
The shoaling coefficient is shown in Figure 2.5 of textbook as a function of the relative depth, d/L.

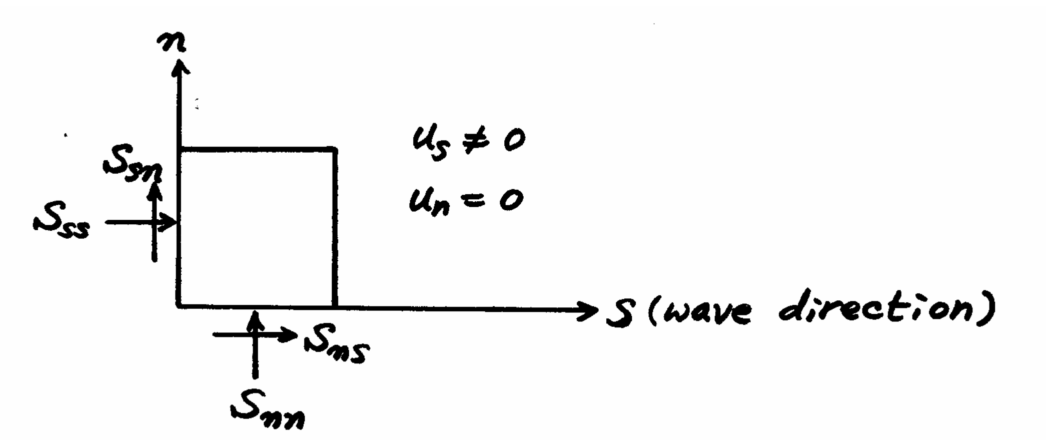
2.6 Radiation Stress and Wave Setup

Radiation stress = excess momentum flux due to waves

= pressure force (due to dynamic pressure)

\+ momemtum transfer (analogous to Reynolds stress of turbulent flow)





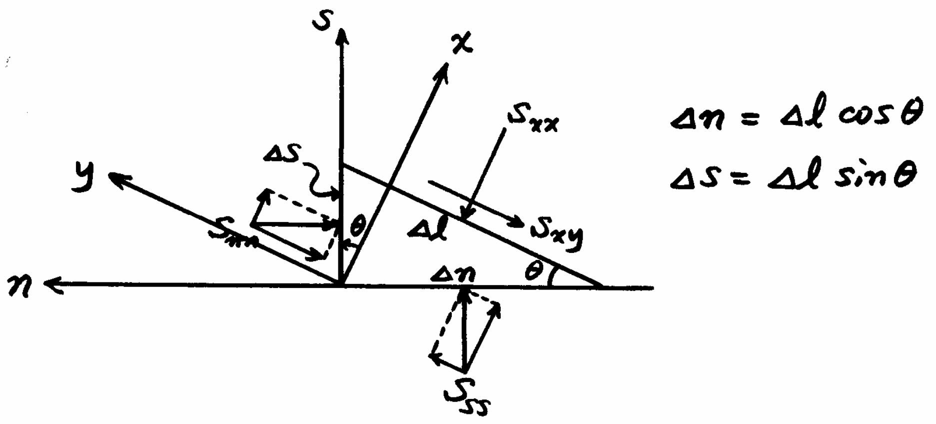
$$[S] = \begin{bmatrix} S_{ss} & S_{sn} \\ S_{ns} & S_{nn} \end{bmatrix} \qquad \begin{matrix} \text{1st subscript} = \text{plane} \\ \text{2nd subscript} = \text{direction} \end{matrix}$$

$$S_{ss} = \overline{\int_{-d}^{\eta} \left(p + \rho u_s^2\right) dz} - \int_{-d}^{0} (-\rho g z) dz = \frac{1}{8}\rho g H^2 \left(\frac{1}{2} + \frac{2kd}{\sinh 2kd}\right) = \overline{E}\left(2n - \frac{1}{2}\right)$$

$$\left.\begin{aligned} S_{sn} &= \overline{\int_{-d}^{\eta}(\rho u_n)u_s dz} = 0 \\ S_{ns} &= \overline{\int_{-d}^{\eta}(\rho u_s)u_n dz} = 0 \end{aligned}\right\} \quad \because \; u_n = 0, \quad \text{pressure force} \perp \text{plane}$$

$$S_{nn} = \overline{\int_{-d}^{\eta}\left(p + \rho u_n^2\right)dz} - \int_{-d}^{0}(-\rho g z)dz = \frac{1}{8}\rho g H^2 \frac{kd}{\sinh 2kd} = \overline{E}\left(n - \frac{1}{2}\right)$$

When a wave is propagating with an angle θ with respect to x-axis,



$$\Delta l S_{xx} = S_{ss}\cos\theta\Delta n + S_{nn}\sin\theta\Delta s = S_{ss}\cos^2\theta\Delta l + S_{nn}\sin^2\theta\Delta l$$

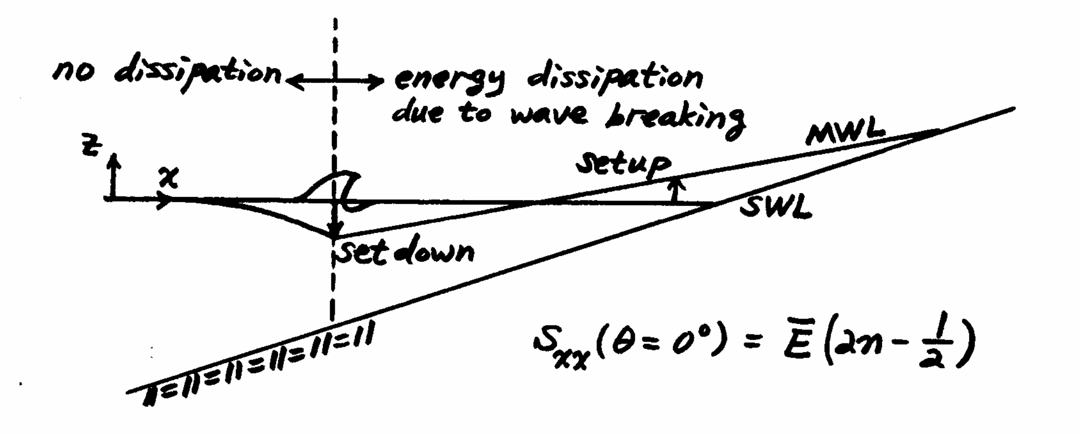
$$\begin{aligned} S_{xx} &= S_{ss}\cos^2\theta + S_{nn}\sin^2\theta \\ &= \overline{E}\left(2n - \frac{1}{2}\right)\cos^2\theta + \overline{E}\left(n - \frac{1}{2}\right)\sin^2\theta \\ &= \overline{E}\left[\left(2n - \frac{1}{2}\right)\cos^2\theta + \left(n - \frac{1}{2}\right)\left(1 - \cos^2\theta\right)\right] \\ &= \overline{E}\left[n\left(\cos^2\theta + 1\right) - \frac{1}{2}\right] \end{aligned}$$

Likewise,

$$S_{yy} = \overline{E}\left[n\left(\sin^2\theta + 1\right) - \frac{1}{2}\right]$$

$$S_{xy} = \overline{E}n\sin\theta\cos\theta$$

Wave setup and setdown



Outside surf zone,

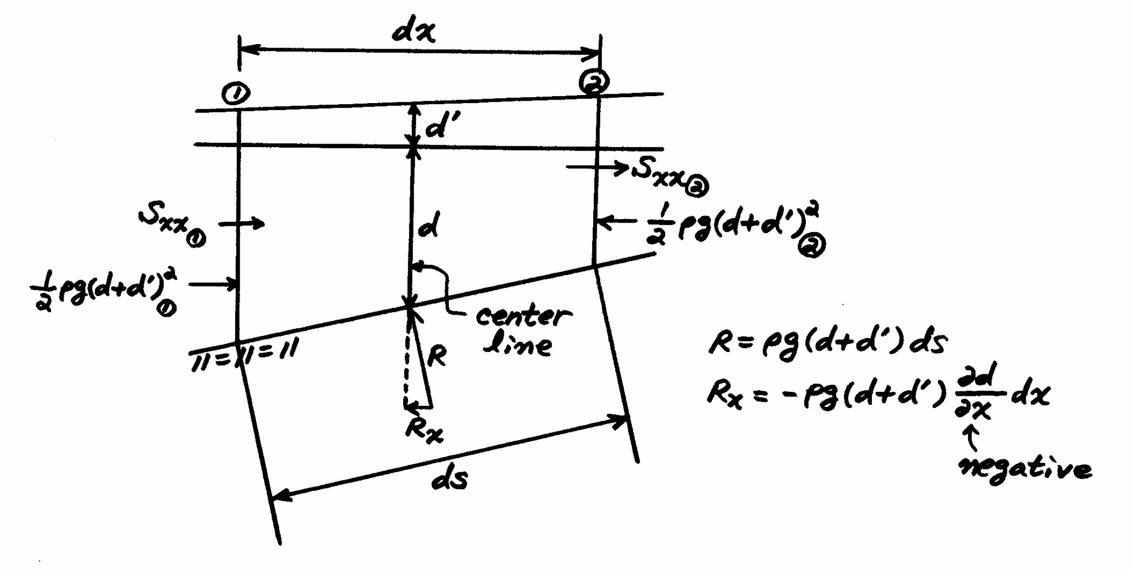
$\bar{E}\uparrow,\ n\uparrow$ as $d\downarrow$

$\therefore\ S_{xx}\uparrow$ as $d\downarrow\ \rightarrow$ setdown

Inside surf zone,

$\bar{E}\downarrow,\ n\uparrow$ as $d\downarrow$ but $\bar{E}\downarrow >> n\uparrow$

$\therefore\ S_{xx}\downarrow$ as $d\downarrow\ \rightarrow$ setup



Force-momentum flux balance gives

$$S_{xx1} + \frac{1}{2}\rho g(d+d')^2{}_1 = S_{xx2} + \frac{1}{2}\rho g(d+d')^2{}_2 + R_x$$

Taylor series expansion about center line gives

$$S_{xx} - \frac{\partial S_{xx}}{\partial x}\frac{dx}{2} + \frac{1}{2}\rho g(d+d')^2 - \frac{\partial}{\partial x}\left[\frac{1}{2}\rho g(d+d')^2\right]\frac{dx}{2}$$

$$= S_{xx} + \frac{\partial S_{xx}}{\partial x}\frac{dx}{2} + \frac{1}{2}\rho g(d+d')^2 + \frac{\partial}{\partial x}\left[\frac{1}{2}\rho g(d+d')^2\right]\frac{dx}{2} - \rho g(d+d')\frac{\partial d}{\partial x}dx$$

$$\frac{\partial S_{xx}}{\partial x}dx + \frac{\partial}{\partial x}\left[\frac{1}{2}\rho g(d+d')^2\right]dx - \rho g(d+d')\frac{\partial d}{\partial x}dx = 0$$

$$\frac{\partial S_{xx}}{\partial x} + \frac{1}{2}\rho g 2(d+d')\frac{\partial(d+d')}{\partial x} - \rho g(d+d')\frac{\partial d}{\partial x} = 0$$

$$\frac{\partial S_{xx}}{\partial x} + \rho g(d+d')\frac{\partial d'}{\partial x} = 0 \quad \leftarrow \text{Eq. (2.52) except } \; d \rightarrow d+d'$$

Wave setdown outside surf zone is given by

$$d' = -\frac{1}{8}\frac{H^2 k}{\sinh 2kd}$$

Inside surf zone, wave height is assumed to be proportional to water depth, that is

$$H = \gamma(d + d')$$

where γ is a constant of about 0.9. Also, inside surf zone where the shallow water condition is satisfied, $n = 1$ so that

$$S_{xx} = \overline{E}\left(2n - \frac{1}{2}\right) \cong \frac{3}{2}\overline{E} = \frac{3}{2}\frac{1}{8}\rho g H^2 = \frac{3}{16}\rho g \gamma^2 (d + d')^2$$

Substituting in (2.52),

$$\frac{3}{8}\rho g \gamma^2 (d + d')\frac{\partial(d + d')}{\partial x} + \rho g(d + d')\frac{\partial d'}{\partial x} = 0$$

$$\left(1 + \frac{3}{8}\gamma^2\right)\frac{\partial d'}{\partial x} = -\frac{3}{8}\gamma^2 \frac{\partial d}{\partial x}$$

$$\frac{\partial d'}{\partial x} = -\frac{\frac{3}{8}\gamma^2}{1 + \frac{3}{8}\gamma^2}\frac{\partial d}{\partial x} = -\left(1 + \frac{8}{3\gamma^2}\right)^{-1}\frac{\partial d}{\partial x} \quad \leftarrow \text{Eq. (2.54)}$$

Integration gives

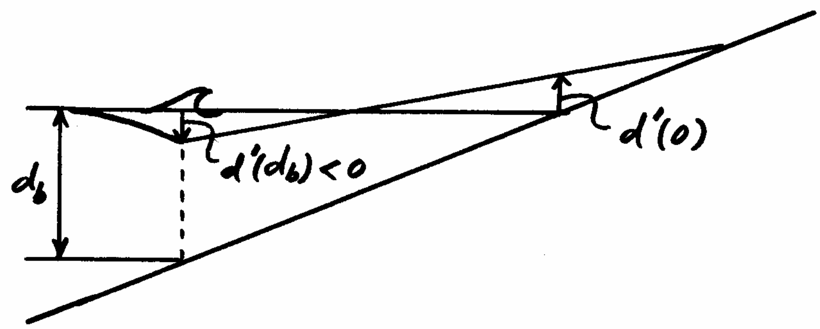
$$d' = -\left(1 + \frac{8}{3\gamma^2}\right)^{-1} d + C_1$$

$$- \qquad d'(d_b) = -\left(1 + \frac{8}{3\gamma^2}\right)^{-1} d_b + C_1$$

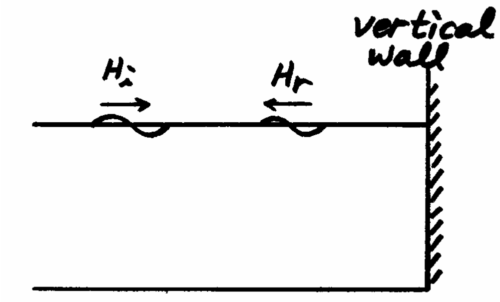
$$d' = d'(d_b) + \left(1 + \frac{8}{3\gamma^2}\right)^{-1}(d_b - d)$$

where $d'(d_b)$ is the wave setdown at breaking point, given by Eq. (2.53). Now, the wave setup at the still water line contour is given by

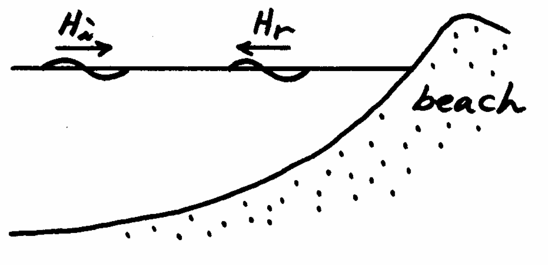
$$d'(0) = d'(d_b) + \left(1 + \frac{8}{3\gamma^2}\right)^{-1} d_b$$



2.7 Standing Waves, Wave Reflection

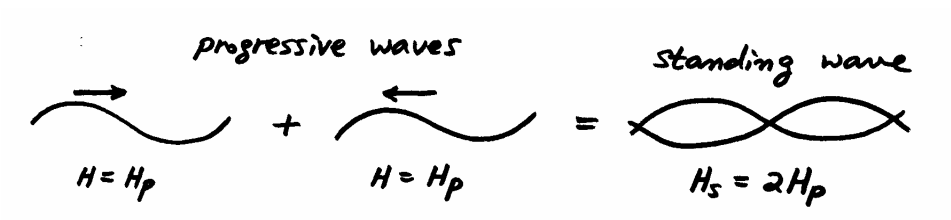


perfect reflection ($H_r = H_i$)



partial reflection ($H_r < H_i$)

$$\text{progressive wave}\begin{cases}\sin(kx-\sigma t) & \text{propagating in positive } x\text{ - direction}\\ \sin(kx+\sigma t) & \text{propagating in negative } x\text{ - direction}\end{cases}$$



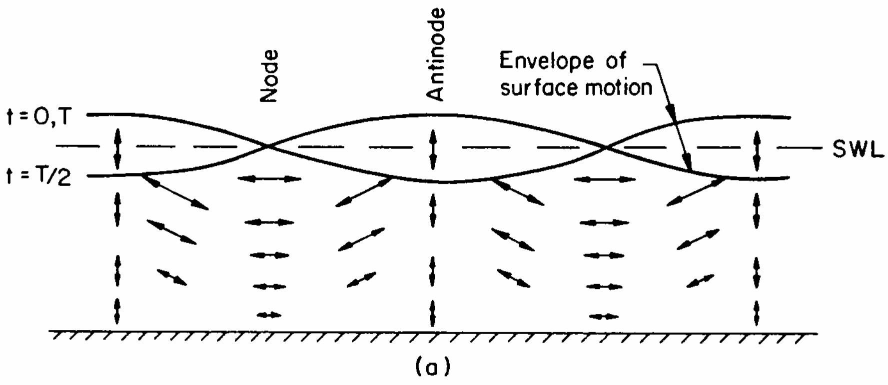
$$\begin{aligned}\eta_s &= \frac{H_p}{2}\cos(kx-\sigma t)+\frac{H_p}{2}\cos(kx+\sigma t)\\ &= \frac{H_p}{2}(\cos kx\cos\sigma t+\sin kx\sin\sigma t+\cos kx\cos\sigma t-\sin kx\sin\sigma t)\\ &= H_p\cos kx\cos\sigma t\\ &= \frac{H_s}{2}\cos kx\cos\sigma t\end{aligned}$$

$$\begin{aligned}\phi_s &= \frac{H_p}{2}\frac{g}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx-\sigma t)-\frac{H_p}{2}\frac{g}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\sin(kx+\sigma t)\\ &= \frac{H_p}{2}\frac{g}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}(\sin kx\cos\sigma t-\cos kx\sin\sigma t-\sin kx\cos\sigma t-\cos kx\sin\sigma t)\\ &= -H_p\frac{g}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\cos kx\sin\sigma t\\ &= -\frac{H_s}{2}\frac{g}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\cos kx\sin\sigma t\end{aligned}$$

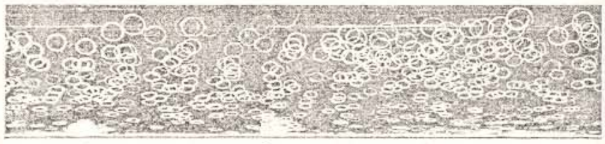
$$u_s = \frac{\partial \phi_s}{\partial x} = \frac{H_s}{2}\frac{gk}{\sigma}\frac{\cosh k(d+z)}{\cosh kd}\sin kx \sin \sigma t$$
$$= \frac{H_s}{2}\sigma\frac{\cosh k(d+z)}{\sinh kd}\sin kx \sin \sigma t$$

$$w_s = \frac{\partial \phi_s}{\partial z} = -\frac{H_s}{2}\frac{gk}{\sigma}\frac{\sinh k(d+z)}{\cosh kd}\cos kx \sin \sigma t$$
$$= -\frac{H_s}{2}\sigma\frac{\sinh k(d+z)}{\sinh kd}\cos kx \sin \sigma t$$

$$\eta_s = \frac{H_s}{2}\cos kx \cos \sigma t$$



Under the pure standing wave, at $t = 0$ or $T/2$, the water particle motion stops for an instant so that only potential energy exists, while at $t = T/4$ or $3T/4$, the water surface becomes flat so that only kinetic energy exists. Also, $w = 0$ at nodes, while $u = 0$ at antinodes.



$H_R/H_I = 0.00$, pure swell

$H_R/H_I = 0.24$

$H_R/H_I = 0.38$

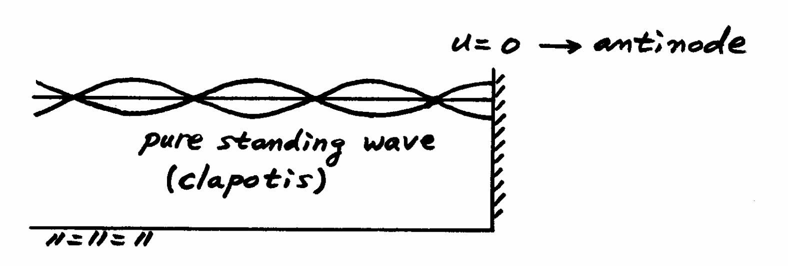
$H_R/H_I = 0.53$

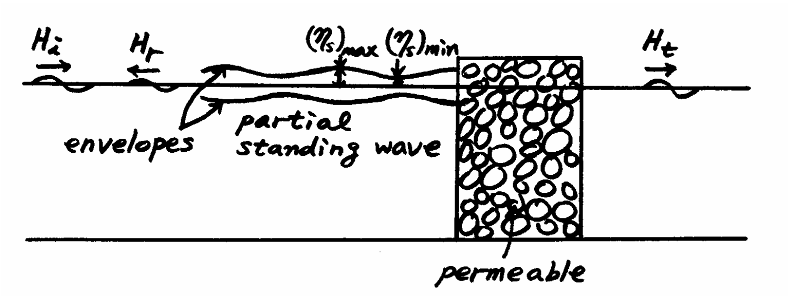
$H_R/H_I = 0.71$

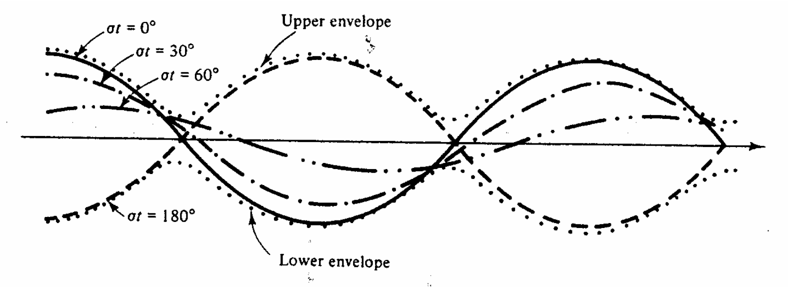
$H_R/H_I = 0.85$

$H_R/H_I = 1.00$, pure standing wave

Fig. 2.32. Water particle motions of pure swell, partially reflected waves, and pure standing wave (after Wallet and Ruellan, 1950)







$$H_i = (\eta_s)_{\max} + (\eta_s)_{\min}$$
$$H_r = (\eta_s)_{\max} - (\eta_s)_{\min}$$

Reflection coefficient, $C_r = \dfrac{H_r}{H_i}$.

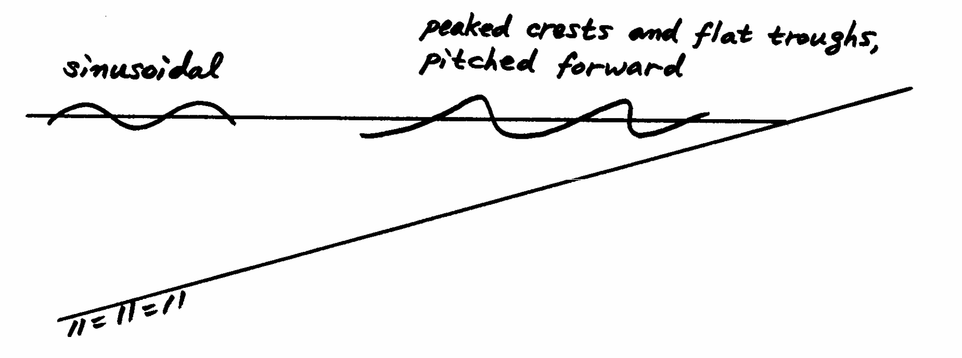
For pure standing wave, $(\eta_s)_{\min} = 0$, thus $H_i = H_r = (\eta_s)_{\max} = H_s / 2$ and $C_r = 1.0$ (perfect reflection).

Transmission coefficient, $C_t = \dfrac{H_t}{H_i}$.

If there is no energy dissipation through the structure, $C_r^2 + C_t^2 = 1$ or $H_r^2 + H_t^2 = H_i^2$ (energy is conserved)

2.8 Wave Profile Asymmetry and Breaking

Read text for profile asymmetry.



<u>Wave breaking</u>

In deep water,

$$u_c \propto H\,,\; C \neq f(H): \quad H\uparrow \;\rightarrow\; u_c\uparrow \;\rightarrow \text{wave breaking if } u_c > C$$

In shallow water,

$$u_c\uparrow \text{ and } C\downarrow \text{ as } d\downarrow \;\rightarrow \text{wave breaking}$$

Miche (1944) formula ← for any constant depth

$$\left(\frac{H}{L}\right)_{\max} = \frac{1}{7}\tanh\frac{2\pi d}{L}$$

In deep water, $(H_0/L_0)_{\max} = 1/7$, while in shallow water, $(H/L)_{\max} \cong (1/7)(2\pi d/L)$, thus $(H/d)_{\max} \cong (1/7)2\pi \cong 0.9$.

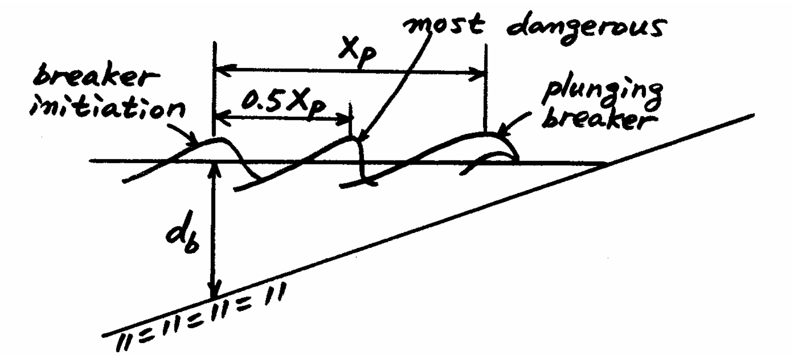
On sloping beach,

Spilling breaker: small H_0 / L_0 and m (beach slope)
Plunging breaker $\downarrow$
Surging breaker: large H_0 / L_0 and m

Given H_0', T, m $\rightarrow$ H_b (Fig. 2.11) $\rightarrow$ d_b (Fig. 2.12), where $H_0' = H / K_s$ is the unrefracted deepwater wave height, which is the deepwater wave height of the waves propagating normal to the shore with straight and parallel bottom contours where no wave refraction but only wave shoaling occurs. If wave refraction occurs as well, then H_0' should be

$$H_0' = \frac{H}{K_s} = \frac{H_0 K_s K_r}{K_s} = H_0 K_r$$

It is the most dangerous for coastal structures when plunging breaker hits the structures.

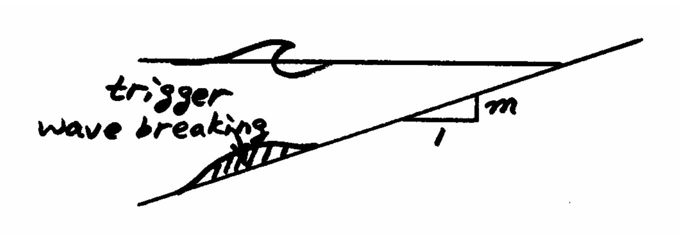


The most dangerous point is about $0.5X_p$ from the breaker initiation point, where X_p is the distance from breaker initiation to plunging point. Therefore, it is needed to design the structure for the waves breaking at $0.5X_p$ seaward of the structure. For plane slope,

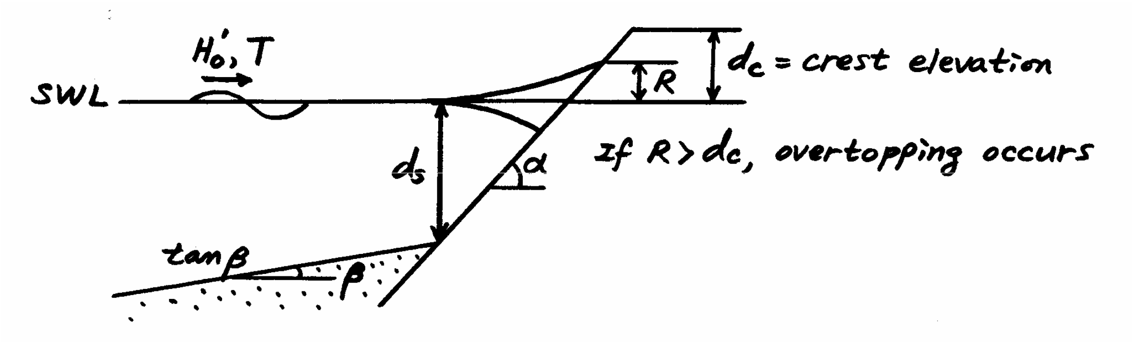
$$\frac{X_p}{H_b} = 3.95 \left(\frac{\sqrt{H_0 / L_0}}{m} \right)^{0.25}$$

For slope with submerged bar,

$$\frac{X_p}{H_b} = 0.63\left(\frac{\sqrt{H_0 / L_0}}{m}\right) + 1.81$$



2.9 Wave Runup



<u>Smooth, impermeable slope</u>

Saville (1957): Laboratory experiments with $\tan\beta = 0.1$

$$\frac{R}{H_0'} = fn\left(\cot\alpha, \frac{H_0'}{gT^2}, \frac{d_s}{H_0'}\right)$$ (Figure 2.15 of textbook)

$\frac{R}{H_0'}\uparrow$ as $\frac{H_0'}{gT^2}\downarrow$ (milder waves) and $\cot\alpha\downarrow$ (steeper structure slope)

If $\cot\alpha \le 1.0$ (or steeper than 1:1 slope), slope effect is negligible.

Rough, permeable slope

$$R_{rp} = rR_{si}$$

where R_{rp} is runup on rough, permeable slope, R_{si} is runup on smooth, impermeable slope, and $r < 1.0$ is the coefficient representing the effect of roughness and permeability of the slope (See Table 2.1 of textbook).