Lecture Notes 6-1

Stiffness Matrix for the Euler Beam Theory (Finite element analysis – “very” basics)

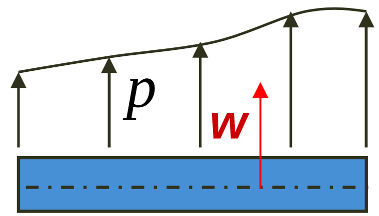
Lecture material for Topology Optimization Design #1

Contents

Euler Beam Theory

Euler-Bernoulli Beam Equation

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = p$$



- **To set up the Euler Beam Theory, need to know**
  - Kinematics
  - Constitutive
  - Equilibrium
  - Resultants

Kinematics (Euler Beam Theory)

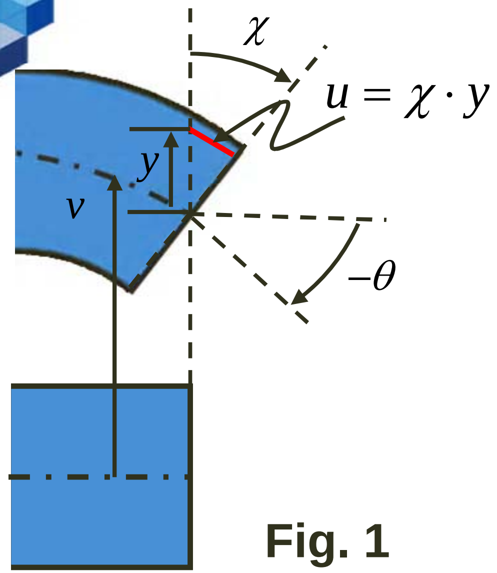


Fig. 1

$$\varepsilon = \frac{du}{dx} \quad (1) \xrightarrow{u(x,y)=\chi(x)\cdot y} \varepsilon(x,y) = \frac{d\chi}{dx}\cdot y \quad (2)$$

and

$$\chi = -\theta = -\frac{dv}{dx} \quad (3)$$

Kirchhoff Assumptions for Normals

- remain straight (do not bend)
- remain unstretched (keep the same length)
- remain normal (always 90° to neutral plane)

nstitutive and Equilibrium (Euler Beam Theory)

Constitutive

By 1-D Hooke's Equation

$$\sigma(x,y) = E \cdot \varepsilon(x,y) \quad (4)$$

Equilibrium

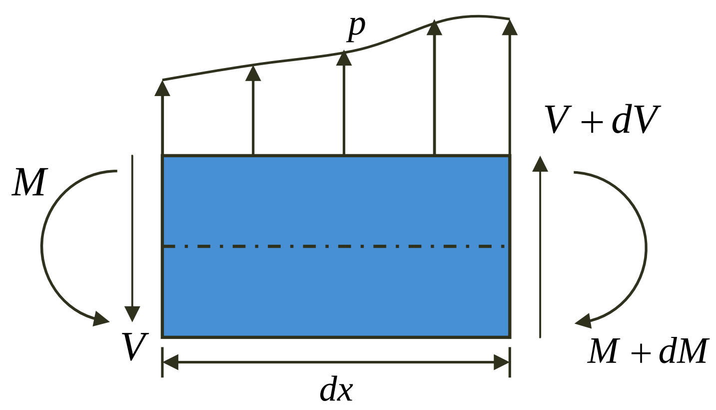


Fig. 2

$$\sum F_y = 0 \Rightarrow \frac{dV}{dx} = -p \quad (5)$$

$$\sum M_z = 0 \Rightarrow \frac{dM}{dx} = V \quad (6)$$

Resultants (Euler Beam Theory)

Moment

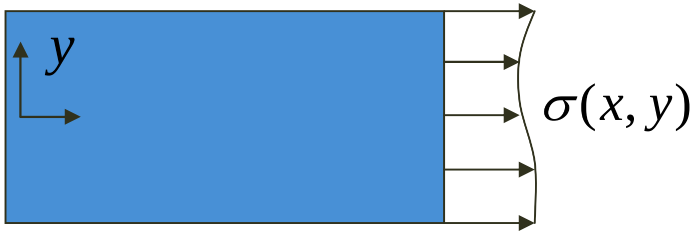


Fig. 3

$$M(x) = \int\int y \cdot \sigma(x, y) \cdot dy \cdot dz \quad (7)$$

using $I = \int\int y^2 \cdot dy \cdot dz$

$$\sigma = \frac{My}{I} \quad (8)$$

Shear

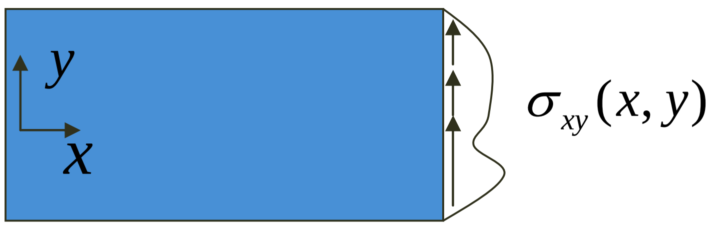


Fig. 4

$$V(x) = \int\int \sigma_{xy}(x, y) \cdot dy \cdot dz \quad (9)$$

Derivation (Euler Beam Theory)

Combine (5) and (6) → $$\frac{d^2M}{dx^2} = -p \quad (10)$$

(7)→ (10) → $$\frac{d^2}{dx^2}\left(\int\int y \cdot \sigma \cdot dy \cdot dz\right) = -p \quad (11)$$

(4)→ (11) → $$\frac{d^2}{dx^2}\left(E\int\int y \cdot \varepsilon \cdot dy \cdot dz\right) = -p \quad (12)$$

(2)→ (12) → $$\frac{d^2}{dx^2}\left(E\frac{d\chi}{dx}\int\int y^2 \cdot dy \cdot dz\right) = -p \quad (13)$$

(3)→ (13) → $$\frac{d^2}{dx^2}\left(E\frac{d^2v}{dx^2}\int\int y^2 \cdot dy \cdot dz\right) = p \quad (14)$$

Recall $I = \int\int y^2 \cdot dy \cdot dz$ → $$\frac{d^2}{dx^2}\left[EI\frac{d^2v}{dx^2}\right] = p \quad (15)$$

Multiscale Design Center [lab] Idea

iffness Matrix for Euler Beam Theory

DOF of a Beam (2D)

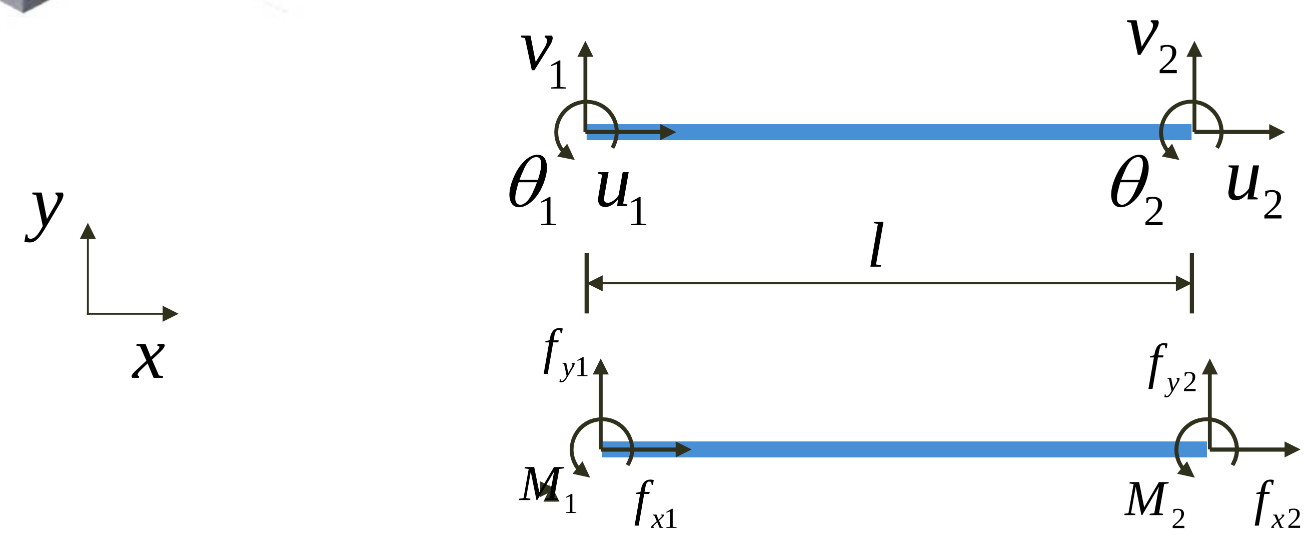


Fig. 5a

Fig. 5b

An Euler beam has 3 DOFs in each node to make 6 DOFs in 2D, corresponding with 6 external loads. A stiffness matrix which shows the relations between these DOFs should be defined.

Stiffness Matrix for Euler Beam

From (8)

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} = \chi E \Rightarrow M = -EI\frac{d^2v}{dx^2} \quad (16)$$

The bending moment in Fig. 5 will be

$$M = M_1 - f_{y1}x \quad (17)$$

(17)→(16)

$$EI\frac{d^2v}{dx^2} = f_{y1}x - M_1 \quad (18)$$

Integrating (18) twice

$$EI\frac{d^2v}{dx^2} = f_{y1}x - M_1$$

$$EI\frac{dv}{dx} = f_{y1}\frac{x^2}{2} - M_1x + C_1$$

$$EIv = f_{y1}\frac{x^3}{6} - M_1\frac{x^2}{2} + C_1x + C_2 \quad (19)$$

Stiffness Matrix for Euler Beam

Boundary Conditions:

$$\text{at } x = 0 \quad v = v_1 \quad \text{and} \quad \frac{dv}{dx} = \theta_1 \qquad (20a)$$

$$\text{at } x = l \quad v = v_2 \quad \text{and} \quad \frac{dv}{dx} = \theta_2 \qquad (20b)$$

(20)→(19)

$$EI\theta_2 = f_{y1}\frac{l^2}{2} - M_1 l + EI\theta_1 \qquad (21a)$$

$$EIv_2 = f_{y1}\frac{l^3}{6} - M_1\frac{l^2}{2} + EI\theta_1 l + EIv_1 \qquad (22b)$$

Solving for f_{y1} and M_1

$$f_{y1} = \frac{EI}{l^3}\left(12v_1 - 12v_2 + 6l\theta_1 + 6l\theta_2\right) \qquad (23a)$$

$$M_1 = \frac{EI}{l^2}\left(6v_1 - 6v_2 + 4l\theta_1 + 2l\theta_2\right) \qquad (23b)$$

Stiffness Matrix for Euler Beam

Using equilibrium in Fig. 5b $V_2 = -V_1$ and $M_2 = V_1 l - M_1$ (24)

(24)→(23)

$$f_{y2} = \frac{EI}{l^3}\left(-12v_1 + 12v_2 - 6l\theta_1 - 6l\theta_2\right) \qquad (25a)$$

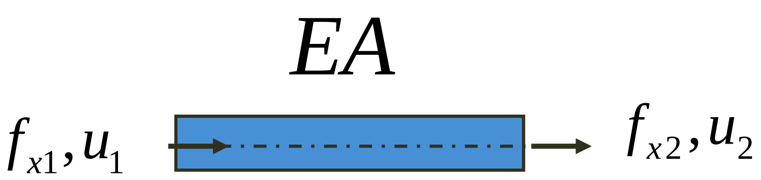
$$M_2 = \frac{EI}{l^2}\left(6v_1 - 6v_2 + 2l\theta_1 + 4l\theta_2\right) \qquad (25b)$$

In matrix form,

$$\begin{Bmatrix} f_{y1} \\ M_1 \\ f_{y2} \\ M_2 \end{Bmatrix}^e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}^e \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}^e \qquad (26)$$

Stiffness Matrix for Euler Beam

For axial force/Disp,



$$f_{x1} = \frac{EA}{l}(u_1 - u_2) \qquad (27a)$$

and

$$f_{x2} = \frac{EA}{l}(-u_1 + u_2) \qquad (27b)$$

In matrix form,

$$\begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}^e = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^e \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^e \qquad (28)$$

Stiffness Matrix for Euler Beam

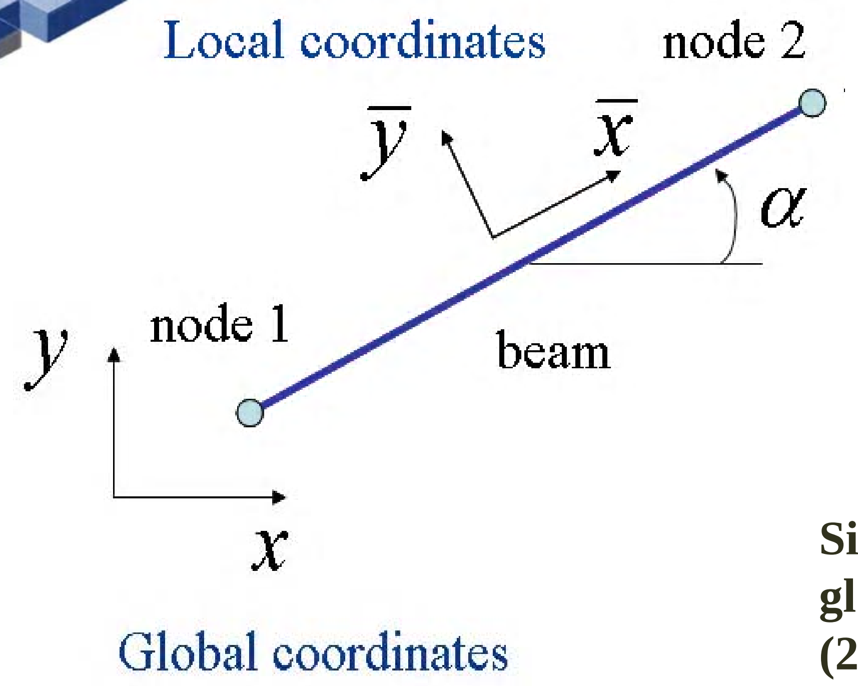
Combining (26) and (28),

$$\mathbf{K}^e\mathbf{u}^e = \mathbf{F}^e \Rightarrow \begin{Bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x2} \\ f_{y2} \\ M_2 \end{Bmatrix}^e = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix}^e \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}^e \quad (29)$$

(e: e^{th} element)

where $C_1 = \dfrac{AE}{L}$ and $C_2 = \dfrac{EI}{L^3}$

Transformation Matrix



Since not every beam is parallel to the global coordinates, the stiffness matrix in (29) should be modified according to its inclination angle, α

Transformation Matrix

Let's define displacement and force vector of the e^{th} element in the global coordinate as as $\mathbf{u}^e$ and $\mathbf{F}^e$, and local ones as $\bar{\mathbf{u}}^e$ and $\bar{\mathbf{F}}^e$.
Also, global stiffness matrix and local one as $\mathbf{K}$ and $\bar{\mathbf{K}}^e$.
Then, the following relationship holds:

$$\bar{\mathbf{u}}^e = \mathbf{T}\mathbf{u}^e \qquad \bar{\mathbf{f}}^e = \mathbf{T}\mathbf{f}^e \qquad (30a)$$

$$\bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{f}}^e \qquad (30b)$$

Where $\mathbf{T}$ is a transformation matrix to adjust the inclination angle.

(30a)→(30b)

$$\bar{\mathbf{K}}^e \mathbf{T}\mathbf{u}^e = \mathbf{T}\mathbf{f}^e \quad (31a) \quad \text{and} \quad \mathbf{T}^{\mathrm{T}}\bar{\mathbf{K}}^e \mathbf{T}\mathbf{u}^e = \mathbf{f}^e \quad (31b)$$

Thus

$$\mathbf{T}^{\mathrm{T}}\bar{\mathbf{K}}^e \mathbf{T} = \mathbf{K}^e \quad (32)$$

Transformation Matrix

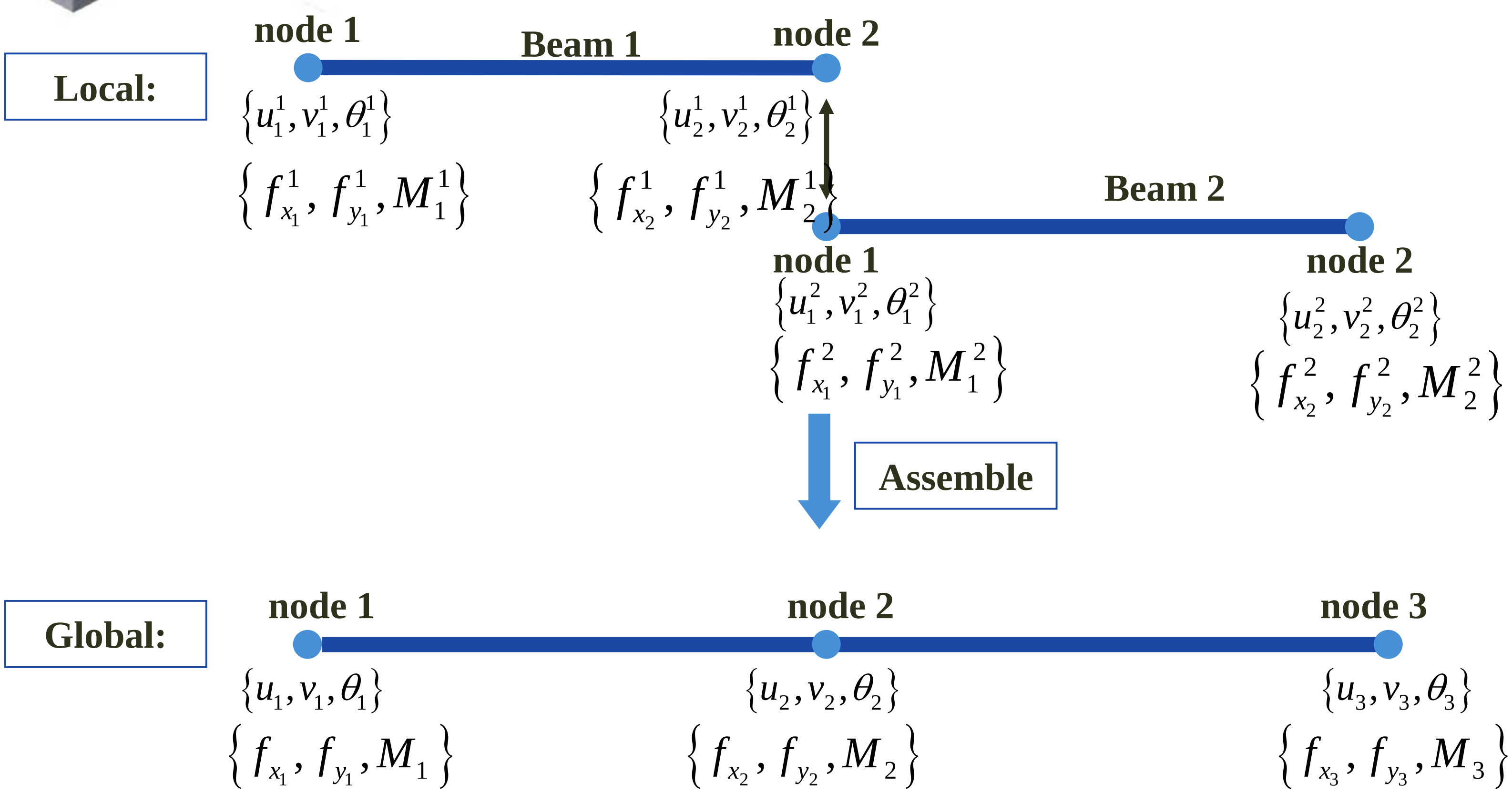
$$\mathbf{T}_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (33)$$

$$\mathbf{T}^T\mathbf{T} = \mathbf{I} \Leftrightarrow \mathbf{T}^{-1} = \mathbf{T}^T$$

Using this, every beam element can be analyzed in global coordinate.

Assembly of Stiffness Matrices

u_i^j : displacement at i[th] local node of j[th] beam



How can this be done?

Assembly of Stiffness Matrices

<Detailed Procedure>

continuity: **local** → **global**

$$\begin{pmatrix} u_1^1 = u_1 \\ u_2^1 = u_1^2 = u_2 \\ u_2^2 = u_3 \end{pmatrix}$$

Same for v_i^j and θ_i^j

equilibrium: **local** → **global**

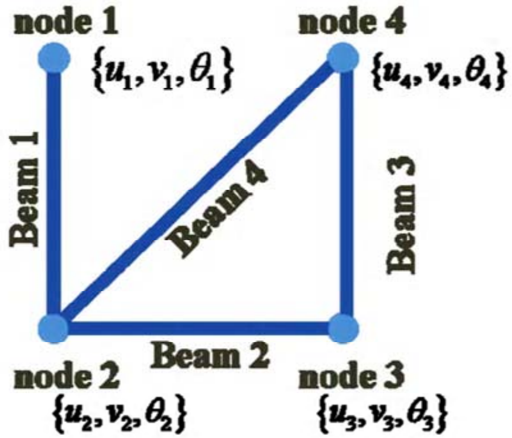
$$\begin{pmatrix} f_{x_1}^1 = f_{x_1} \\ f_{x_2}^1 + f_{x_1}^2 = f_{x_2} \\ f_{x_3}^2 = f_{x_3} \end{pmatrix}$$

Same for $f_{x_i}^j$ and M_i^j

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

The displacement of the i^{th} node is $\{3i-a\}^{th}$ $(a=1,2,3)$ element of the displacement vector

Assembly of Stiffness Matrices



Element Stiffness Matrices and vectors

Beam 1

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ M_1 \\ f_{x2} \\ f_{y2} \\ M_2 \end{Bmatrix} = \mathbf{T}_{\pi/2}^{\ T} \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix} \mathbf{T}_{\pi/2} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

Beam 2

$$\begin{Bmatrix} f_{x2} \\ f_{y2} \\ M_2 \\ f_{x3} \\ f_{y3} \\ M_3 \end{Bmatrix} = \mathbf{T}_{0}^{\ T} \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix} \mathbf{T}_{0} \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Beam 3

$$\begin{Bmatrix} f_{x3} \\ f_{y3} \\ M_3 \\ f_{x4} \\ f_{y4} \\ M_4 \end{Bmatrix} = \mathbf{T}_{\pi/2}^{\ T} \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix} \mathbf{T}_{\pi/2} \begin{Bmatrix} u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Beam 4

$$\begin{Bmatrix} f_{x2} \\ f_{y2} \\ M_2 \\ f_{x4} \\ f_{y4} \\ M_4 \end{Bmatrix} = \mathbf{T}_{\pi/4}^{\ T} \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix} \mathbf{T}_{\pi/4} \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

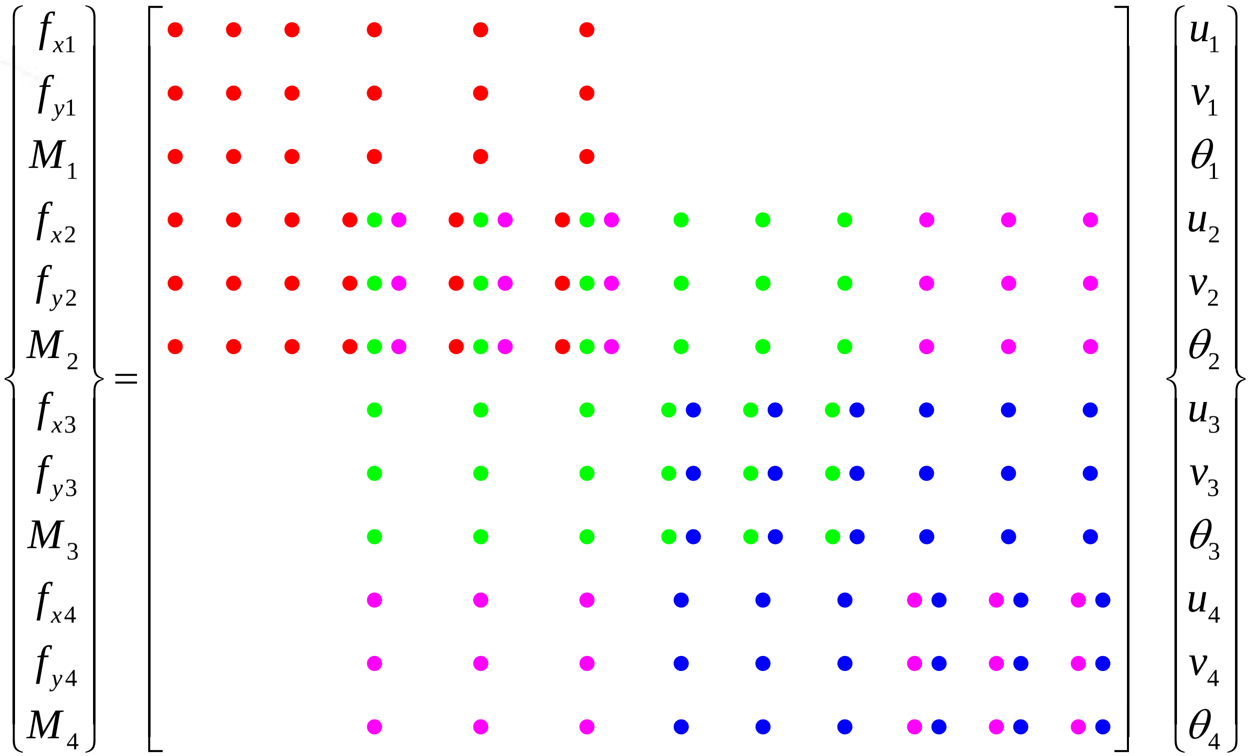
Assembly of Stiffness Matrices

$$\mathbf{K}^1\mathbf{u}^1 = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \qquad \mathbf{K}^3\mathbf{u}^3 = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ \theta_3 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

$$\mathbf{K}^2\mathbf{u}^2 = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{Bmatrix} \qquad \mathbf{K}^4\mathbf{u}^4 = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_4 \\ v_4 \\ \theta_4 \end{Bmatrix}$$

Each element stiffness matrix will be added to the global stiffness matrix with respect to the displacement DOFs it has, as seen in (34).

Assembly of Stiffness Matrices

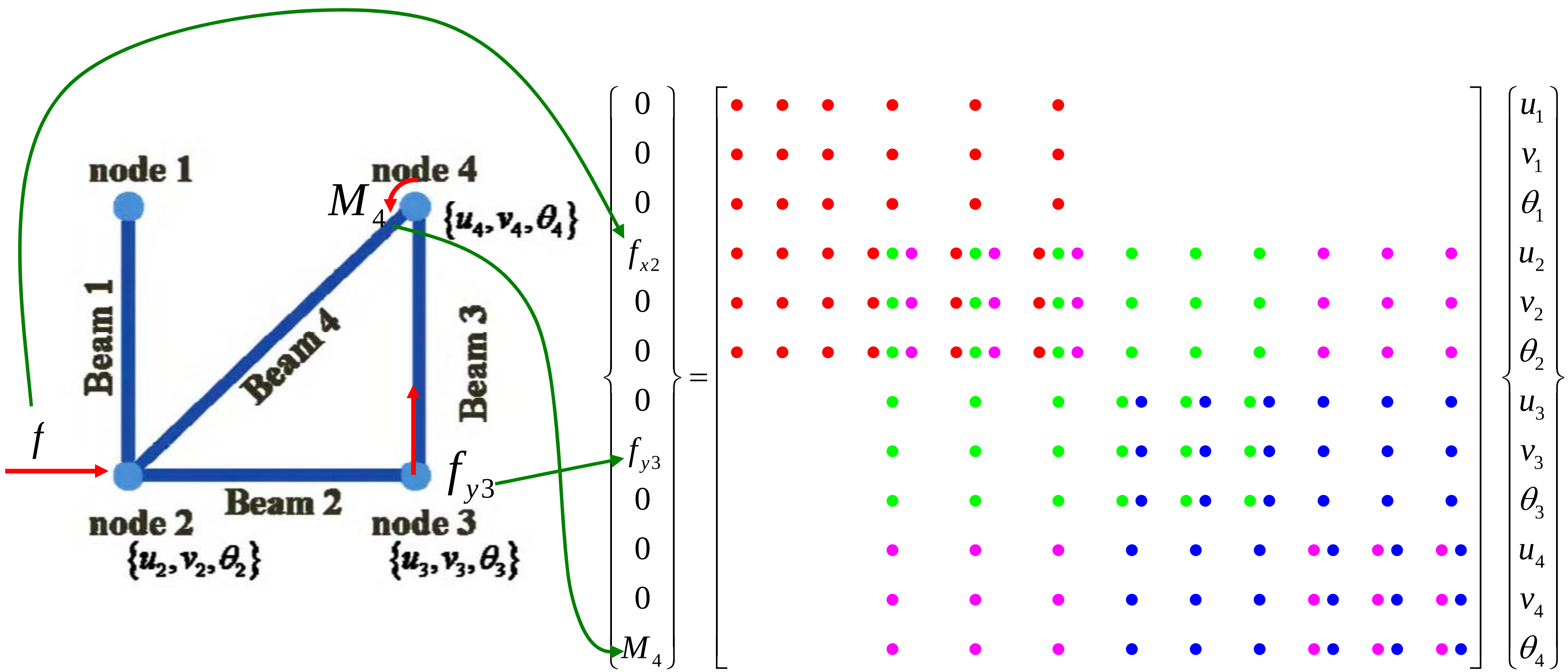


(34)

Boundary Conditions

Applying forces

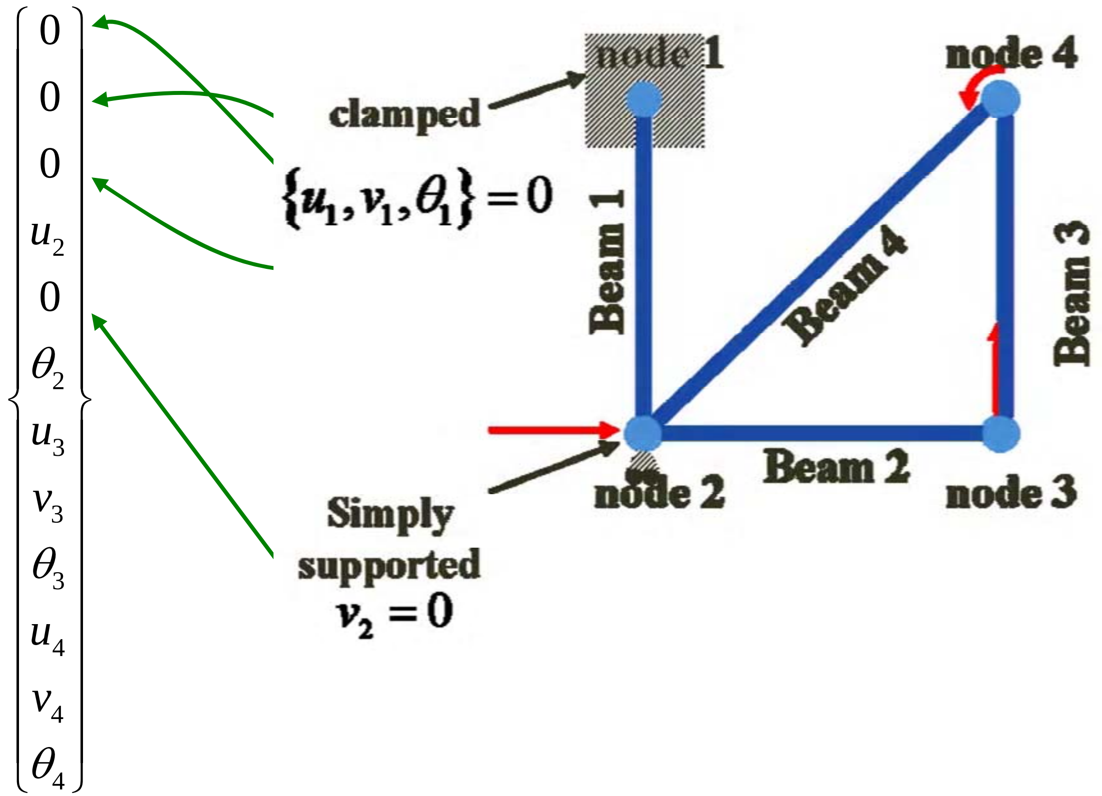
The external forces will be added to the force vector with respect to their type and the node on which they are applied.



$(f_{x_2}, f_{y_3}, M_4 =$ known, applied forces$)$

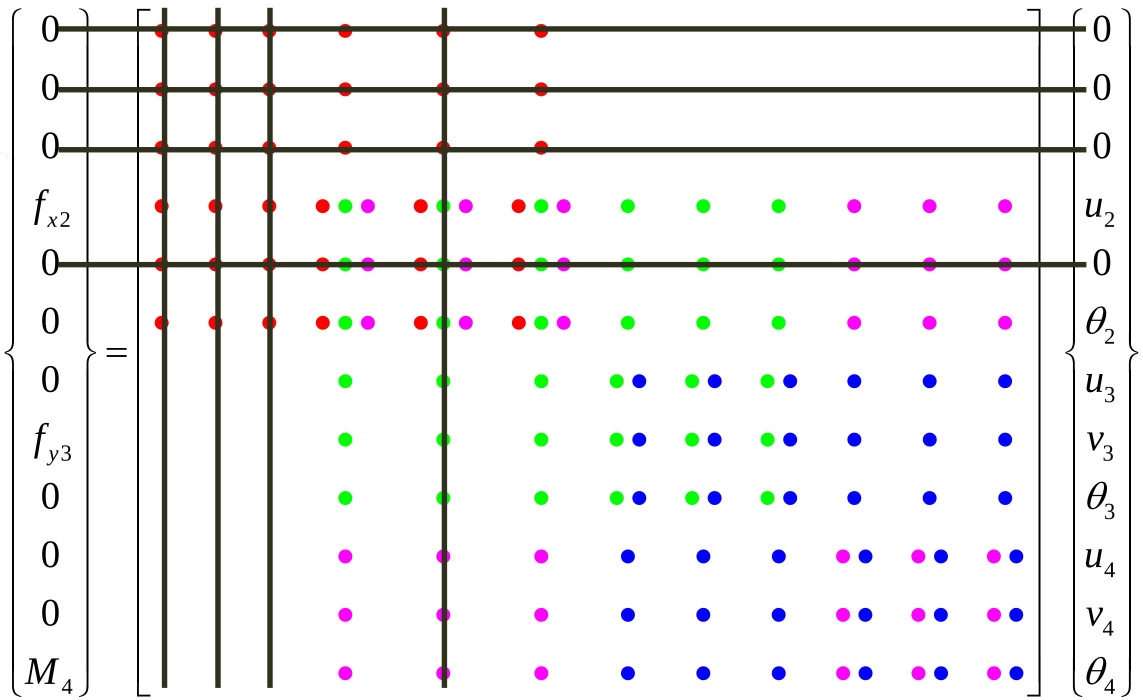
Boundary Conditions

Fixed DOFs



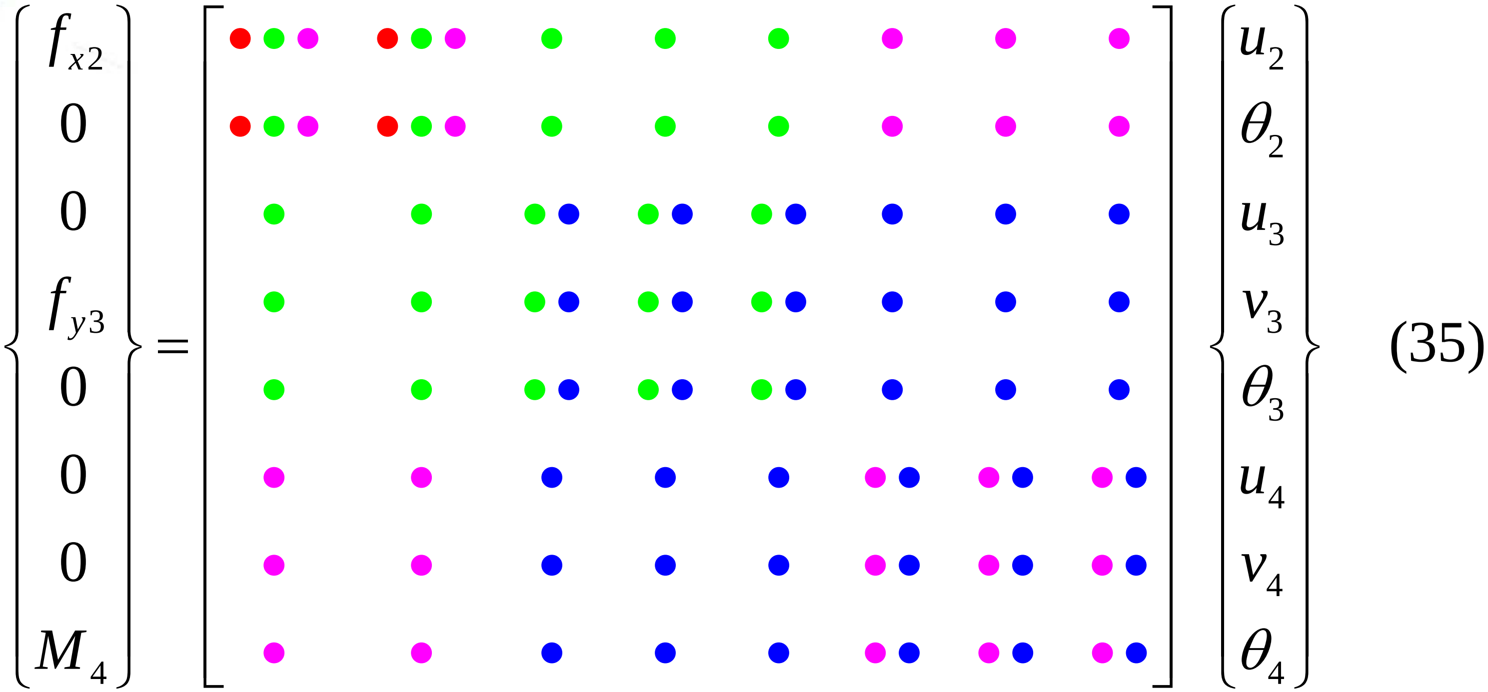
The restricted DOF in a structure will result in a value "0" in the displacement vector. This will cause the stiffness matrix to be reduced.

Boundary Conditions



Because there are 8 unknowns, only 8 equations are needed. Therefore, the underlined rows and columns do not need to be considered. The result becomes Eq. (35) in the next page.

Boundary Conditions



By solving $\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}$, we can obtain the displacements at nodes.