7.1 Conditional Distribution: Discrete Case

Recall for any two events E & F, P(F) > 0, the conditional probability of E given F is;

$$\mathrm{P(E|F)} = \frac{P(EF)}{P(F)}$$

If X and Y are discrete random variables, the conditional probability mass function of X given that Y=y is,

$$P_{X|Y}(x|y) = \mathrm{P\{X{=}x \mid Y{=}y\}}$$

$$= \frac{P\{X = x, Y = y\}}{P\{Y = y\}}$$

$$= \frac{p(x, y)}{P_Y(y)} \text{ , for all y such that } P_y(y) > 0$$

The conditional probability distribution function of X given that Y=y is,

$$F_{X|Y}(x|y) = \mathrm{P\{X{\leq}x \mid Y{=}y\}}$$

$$= \sum_{a \leq x} P_{X|Y}(x|y)$$

• The above definitions are exactly the same as in the unconditional case, except that everything is now conditional on the event that Y=y.

• If X is indept of Y, then

$$P_{X|Y}(x|y) = \mathrm{P\{X{=}x \mid Y{=}y\}}$$

$$= \frac{P\{X = x\}P\{Y = y\}}{P\{Y = y\}}$$

$$= \mathrm{P\{X{=}x\}}$$

∴ then the conditional mass/distribution functions are the same as the unconditional ones.

Example .

X, Y : indept Poisson random variables with parameters λ_1, λ_2
conditional distribution of X given that X+Y=n ?.

Solution.

$$\text{P\{X=K | X+Y=n\}} = \frac{P\{X=K, Y=n-K\}}{P\{X+Y=n\}} = \frac{P\{X=K\}P\{Y=n-K\}}{P\{X+Y=n\}}$$

Since X+Y is Poisson with $\lambda_1 + \lambda_2$,

$$\text{P\{X=K | X+Y=n\}} = \frac{\frac{e^{-\lambda_1}\lambda_1^K}{K!} \; \frac{e^{-\lambda_2}\lambda_2^{n-K}}{(n-K)!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{K} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^K \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-K}$$

$\therefore$ the conditional distribution of X, given that X + Y = n, is the binomial distribution with parameters n and $\frac{\lambda_2}{\lambda_1+\lambda_2}$

7.2 Conditional Distribution: Continuous Case

- If X & Y have a joint probability density function $f(x, y)$, then the conditional probability density function of X, given that Y=y, is defined for all values of y such that $f_Y(y) > 0$ by,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y)dx = \frac{f(x,y)dxdy}{f_Y(y)dy}$$

$$\approx \frac{P\{x \leq X \leq x+dx, y \leq Y \leq y+dy\}}{P\{y \leq Y \leq y+dy\}}$$

conditional probability that X is between x and $x + dx$, given that Y is between y and $y + dy$.

$$= P\{x \leq X \leq x+dx \mid y \leq Y \leq y+dy\}$$

- If X & Y are jointly continuous then for any set A,

$$\text{P\{X} \in \text{A | Y = y\}} = \int_A f_{X|Y}(x|y)dx$$

- In particular,

$$F_{\frac{x}{y}}(a|y) \;\equiv \mathrm{P}\{\mathrm{X} \leq \mathrm{a} \mid \mathrm{Y} = \mathrm{y}\} = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

Example .

$$f(x,y) = \begin{cases} \frac{15}{2}x(2-x-y) & 0 < x < 1,\, 0 < y < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Conditional density of X, given that Y=y, where $0 < y < 1$?

Solution.

For $0 < x,\, y < 1$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dx}$$

$$= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx}$$

$$= \frac{6x(2-x-y)}{4-3y}$$

- If X & Y are indept, continuous random variables then,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)}$$

$$= f_X(x) \qquad \text{(Just the unconditional density of X)}$$

- We can talk about conditional distributions when the random variables are neither jointly continuous nor jointly discrete.

Example .

X : Continuous random variable with density f.
N : Discrete random variable.
Then

$$\frac{P\{x \leq X \leq x+dx \mid N=n\}}{dx} = \frac{P\{N=n|x \leq X \leq x+dx\}}{P\{N=n\}} \frac{P\{x \leq X \leq x+dx\}}{dx}$$

$$\lim_{dx\to 0} \frac{P\{x \le X \le x+dx \mid N=n\}}{dx} = \underbrace{\frac{P\{N=n|X=x\}}{P\{N=n\}} f(x)}_{f_{X|N}(x|n)}$$

Example .

n+m trials with a common probability of success.
The success probability is not fixed, but uniform (0, 1).
The conditional distribution of the success probability given that the trial result in n successes ?

Solution.

Lets call it X $\sim$ U (0, 1)
N, the number of successes $\sim$ binomial with (n + m, x)
The conditional density of X given that N=n is

$$f_{X|N}(x|n) = \frac{P\{N=n|X=x\}}{P\{N=n\}} f(x)$$

$$= \frac{\binom{n+m}{n} x^n (1-x)^m . 1}{P\{N=n\}} \qquad 0 < x < 1$$

$$= \underbrace{\text{C } x^n (1-x)^m}_{(C \text{ does not depend on } x)} \qquad \text{Beta distribution with parameters n+1, m+1}$$