INTRODUCTION TO CONDUCTION

- **Thermal Transport Properties**
- **Heat Diffusion (Conduction) Equation**
- **Boundary and Initial Conditions**

Thermal Properties of Matter

transport properties: physical structure of matter, atomic and molecular

- **thermal conductivity:** $$k_x = -\frac{q''_x}{\partial T / \partial x}$$

  isotropic medium: $k_x = k_y = k_z = k$

- **thermal diffusivity:** α [m²/s]

$$\alpha = \frac{k}{\rho c_p} = \frac{\text{ability to conduct thermal energy}}{\text{ablity to store thermal energy}}$$

Kinetic Theory

$$q''_x = -\frac{1}{3} C v \lambda \frac{dT}{dx}$$

C : volumetric specific heat [J/m^{3}K]

λ : mean free path [m]

v : characteristic velocity of particles [m/s]

Fourier law of heat conduction

$$q''_x = -k \frac{dT}{dx} \rightarrow k = \frac{C v \lambda}{3}$$

$$k = \frac{1}{3}\left[(C v \lambda)_l + (C v \lambda)_e \right]$$

First term : lattice (phonon) contribution

Second term : electron contribution

Solids

$$k = k_e + k_l$$

Pure metal: $k_e >> k_l$

Alloys: $k_e \sim k_l$

Non-metallic solids : $k_l > k_e$

Fluids

$$k = Cv\lambda/3 \rightarrow k \propto nv\lambda$$

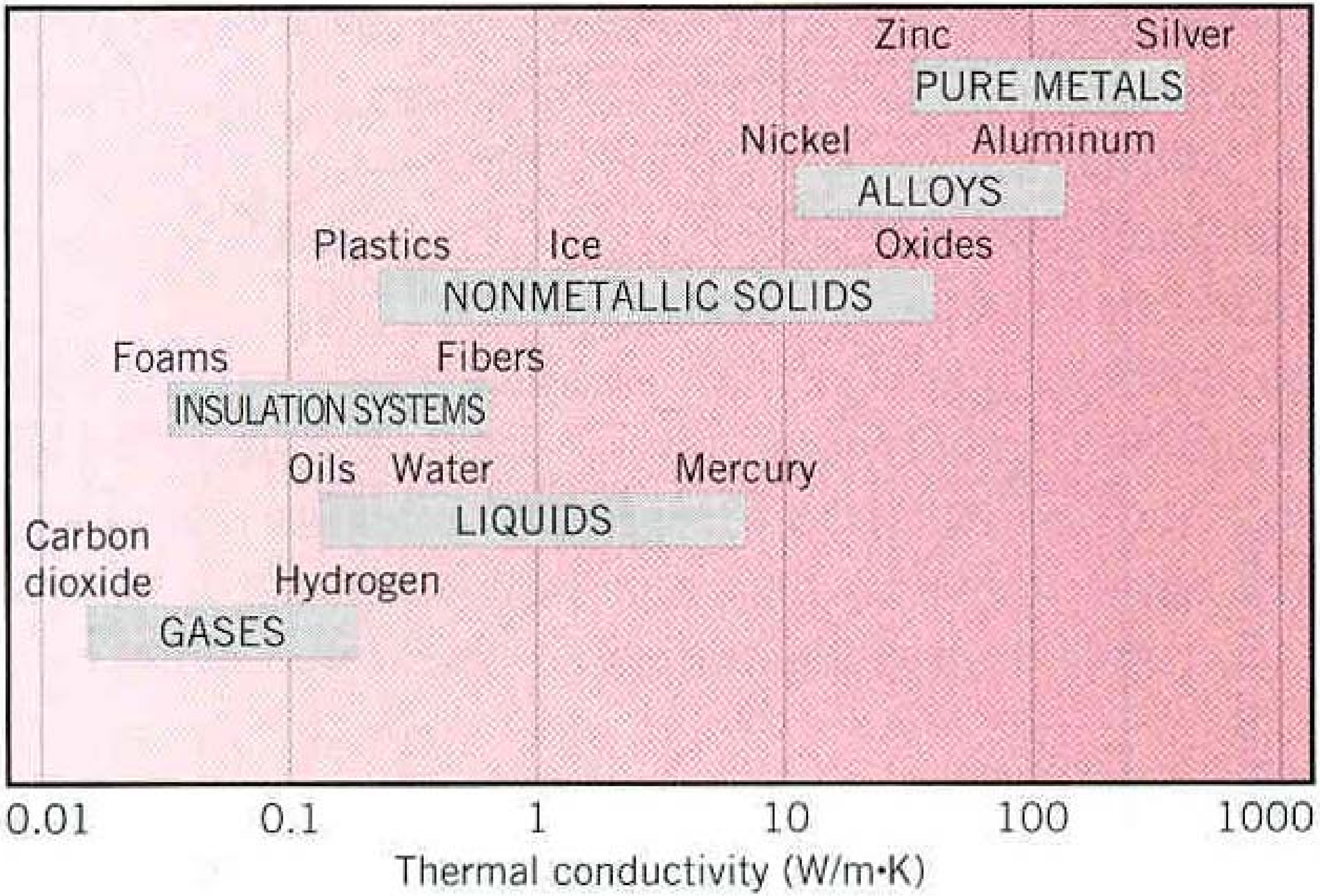
n : number of particles per unit volume

weak dependence on pressure

As p goes up, n increases

but λ becomes shorten $\rightarrow k = k\,(T)$

Range of thermal conductivity for various of states of matter at normal temperature and pressure



Temperature dependence of thermal conductivity of soilds

Solids

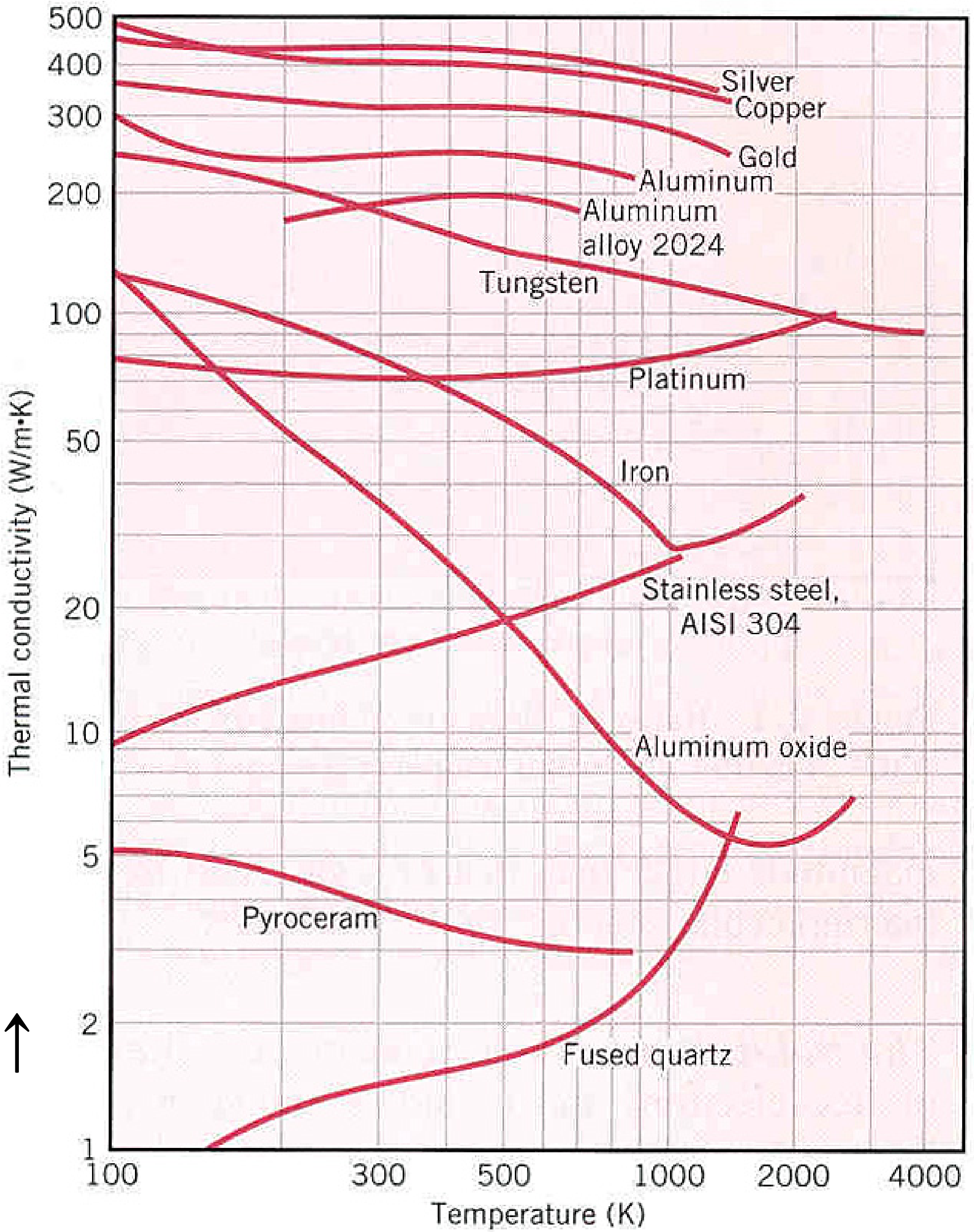
$k = k_e + k_l$

Pure metal: $k_e >> k_l$

Alloys: $k_e \sim k_l$

Non-metal : $k_l > k_e$

In general, $k \downarrow$ **as** $T \uparrow$

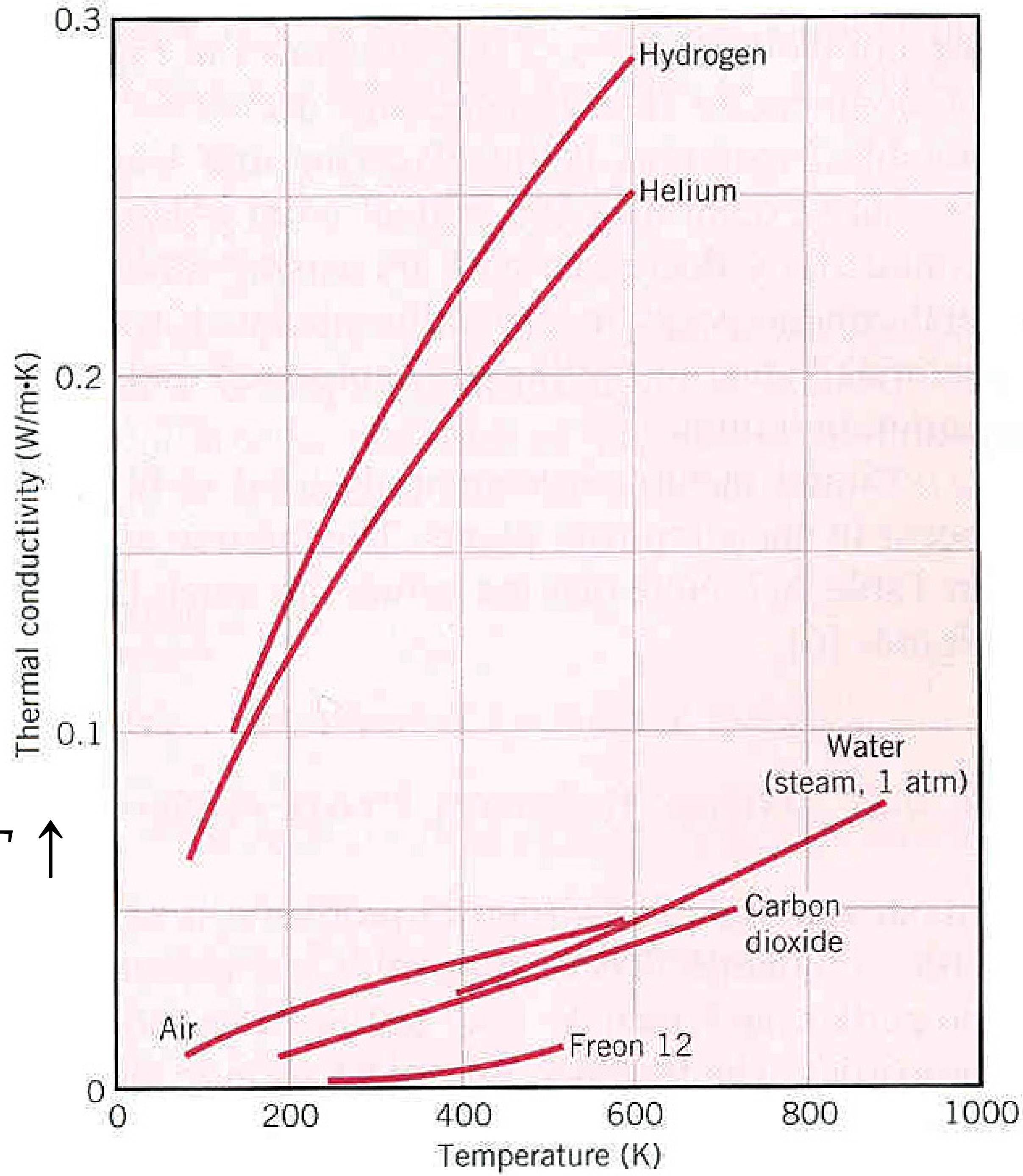


Temperature dependence of thermal conductivity of gases

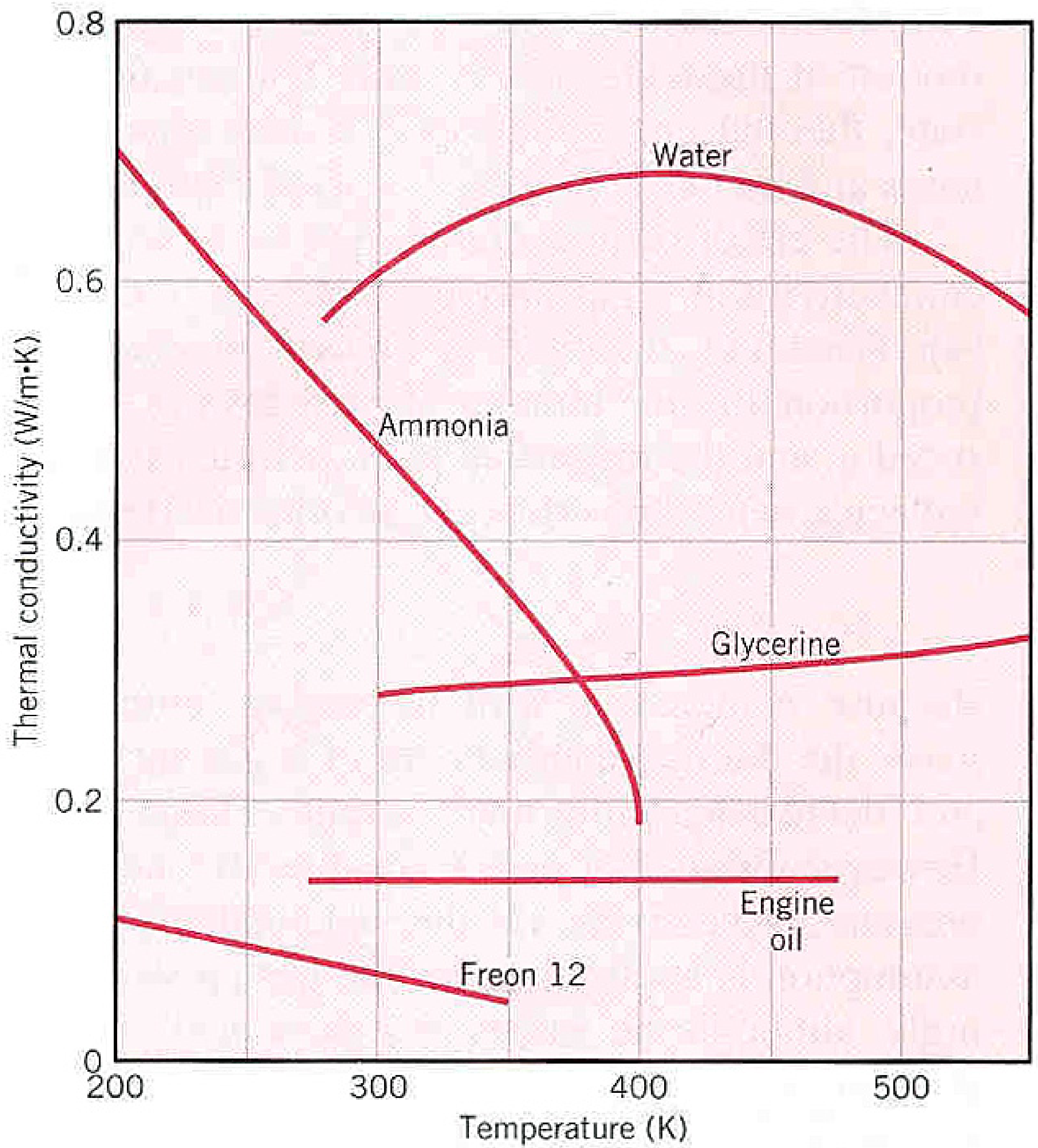
Gases

$k = Cv\lambda/3$

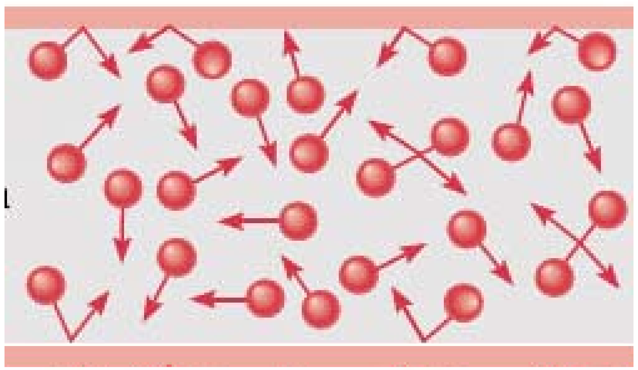
In general, $k \uparrow$ as $T \uparrow$



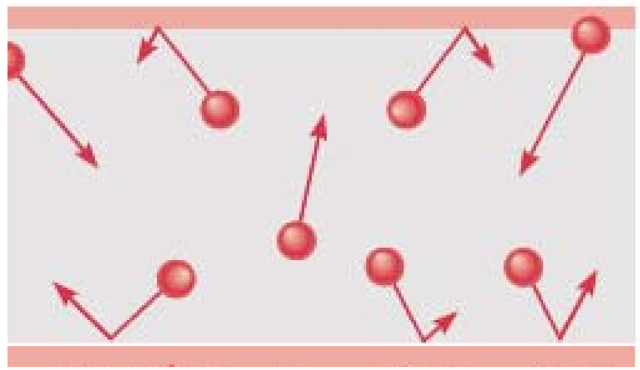
Temperature dependence of thermal conductivity of nonmetallic liquids under saturated conditions



Nanoscale Structure

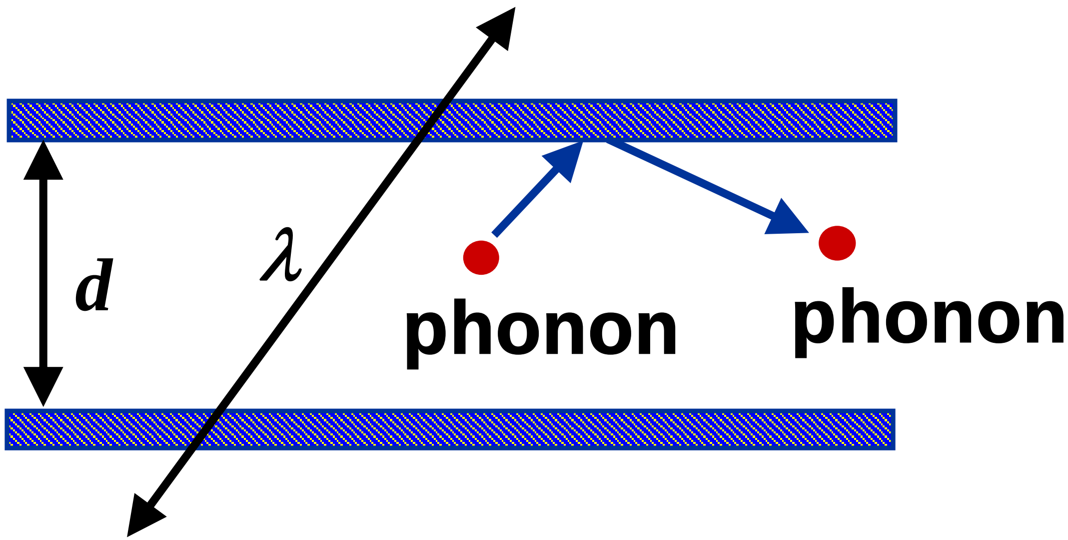


$d >> \lambda$



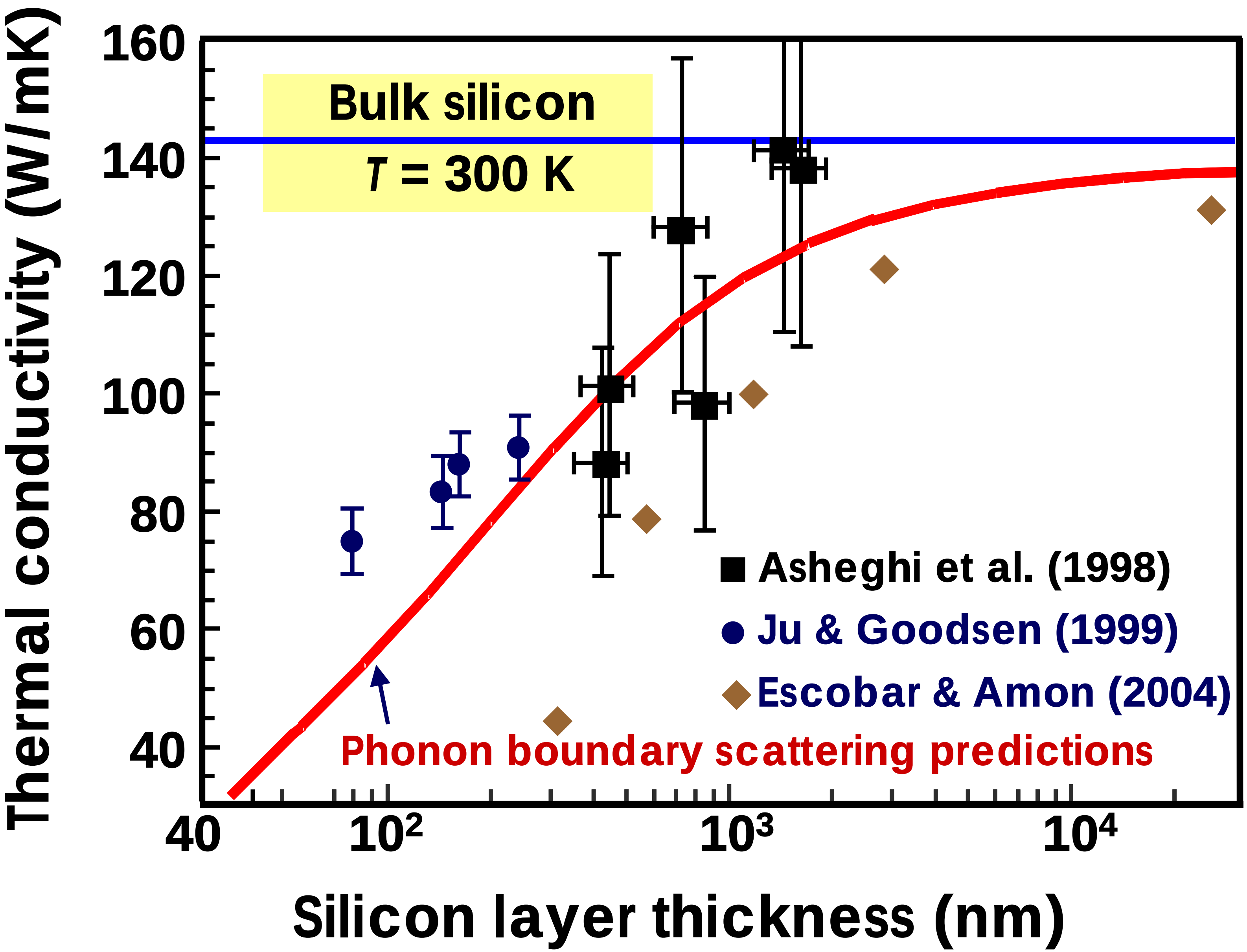
$d \sim \lambda$ or less

Phonon-Boundary Scattering

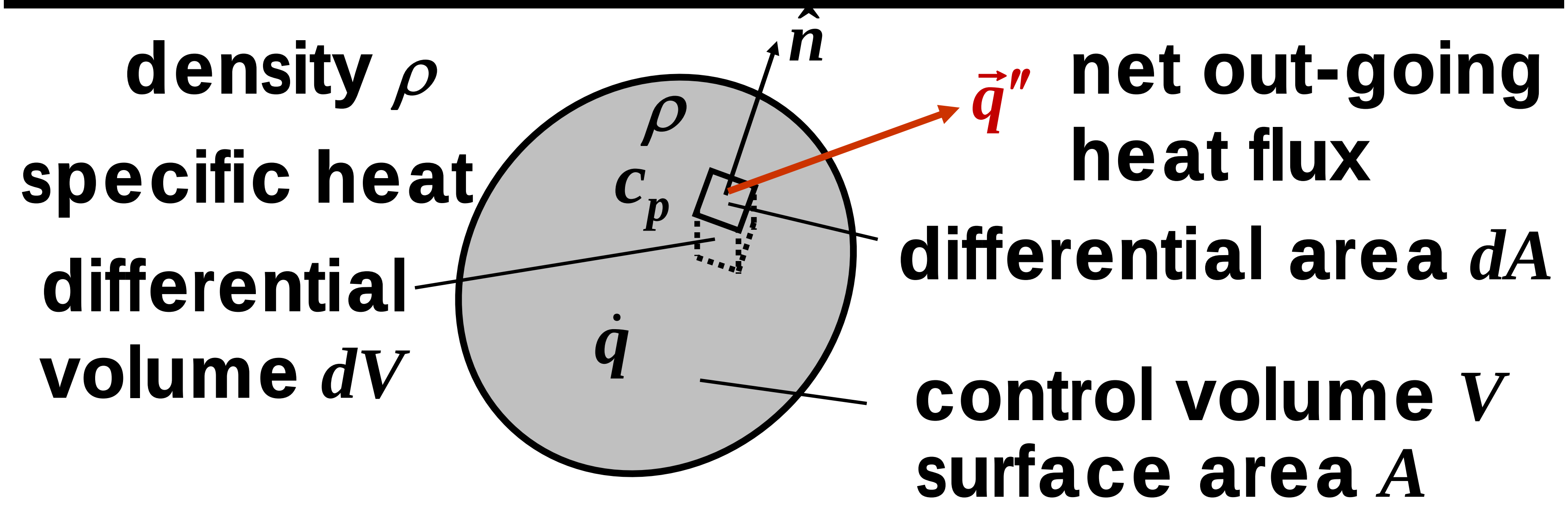


$\lambda_{Si@300K}$ = 300 nm

Thermal Conductivity of Silicon



Heat Diffusion (Conduction) Equation



$\dot{q}$: internal heat generation rate per unit volume [W/m^3]

$$\dot{E}_{\text{in}} + \dot{E}_{\text{g}} - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = -\int_A \vec{q}'' \cdot \hat{n} dA, \quad \dot{E}_{\text{g}} = \int_V \dot{q} dV$$

$$\dot{E}_{\text{st}} = \int_V \rho dV c_p \frac{\partial T}{\partial t} = \int_V \rho c_p \frac{\partial T}{\partial t} dV$$

$$\int_V \rho c_p \frac{\partial T}{\partial t} dV = -\int_A \vec{q}'' \cdot \hat{n} dA + \int_V \dot{q} dV$$

Divergence theorem:

$$\int_A \vec{q}'' \cdot \hat{n} dA = \int_V \nabla \cdot \vec{q}'' dV$$

Thus, $$\int_V \left(\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot \vec{q}'' - \dot{q} \right) dV = 0$$

Since V can be chosen arbitrary, the integrand should be zero.

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot \vec{q}'' - \dot{q} = 0 \quad \textbf{or} \quad \rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \vec{q}'' + \dot{q}$$

Fourier law : $\vec{q}'' = -k\nabla T$

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \vec{q}'' + \dot{q} \;\rightarrow\; \rho c_p \frac{\partial T}{\partial t} = \nabla \cdot \left(k\nabla T\right) + \dot{q}$$

For constant k : $\dfrac{\partial T}{\partial t} = \alpha \nabla^2 T + \dfrac{\dot{q}}{\rho c_p}$

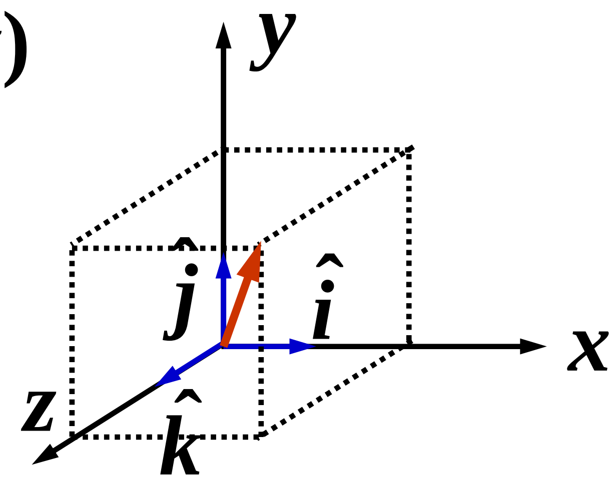
Steady-state : $\nabla \cdot \left(k\nabla T\right) + \dot{q} = 0$

Constant k : $\nabla^2 T + \dfrac{\dot{q}}{k} = 0$

No heat generation : $\nabla^2 T = 0$

Cartesian coordinates (x, y, z)

$$\vec{q}'' = -k\left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)$$



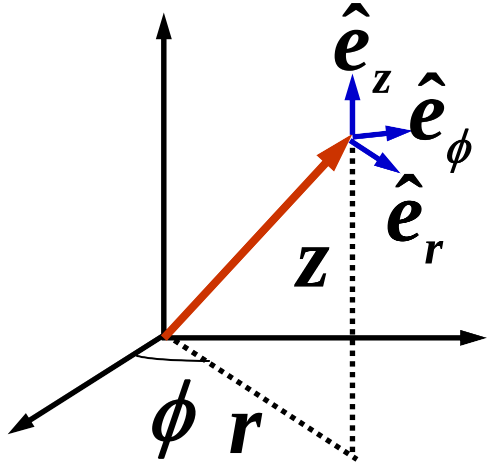
$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}$$

Steady-state, constant k, no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Cylindrical coordinates (r, ϕ, z)

$$\vec{q}'' = -k\left(\frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{e}_\phi + \frac{\partial T}{\partial z}\hat{e}_z\right)$$



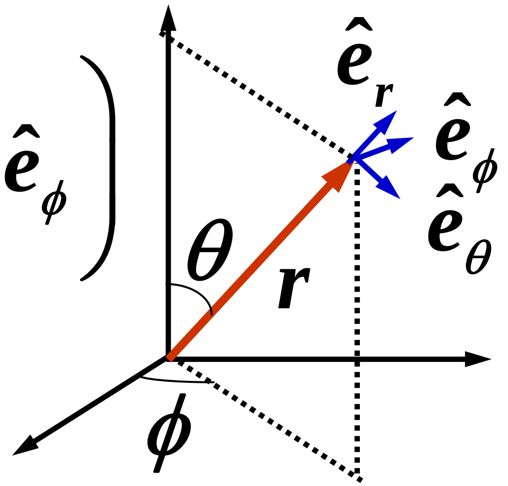
$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}$$

Steady-state, constant k, no heat generation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Spherical coordinates (r, ϕ, θ)

$$\vec{q}'' = -k\left(\frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{e}_\phi\right)$$



$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q}$$

Steady-state, constant *k*, no heat generation

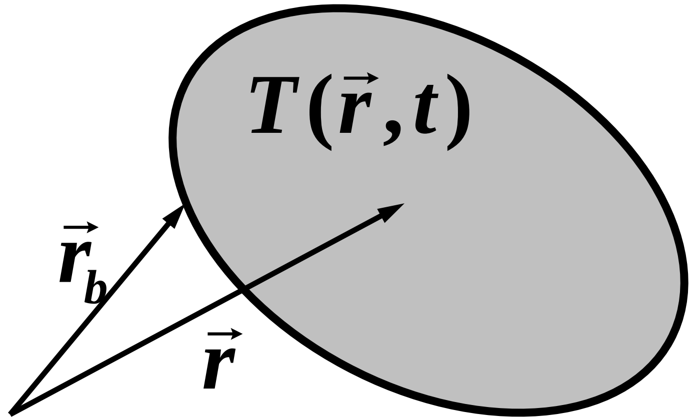
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) = 0$$

Boundary and Initial Conditions

Equations: Elliptic, Parabolic, Hyperbolic eq.

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot \left(k \nabla T \right) + \dot{q}$$

elliptic in space and parabolic in time

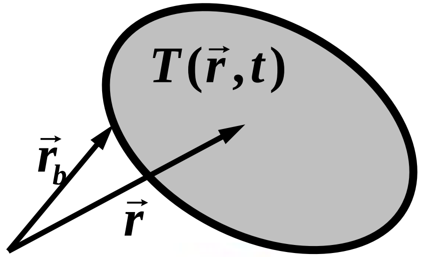


1. Initial condition

condition in the medium at $t = 0$:

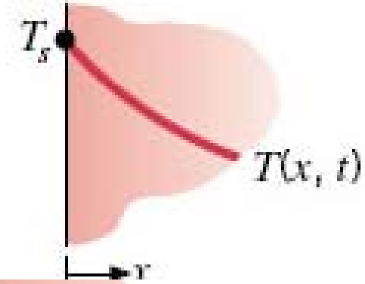
$T(\vec{r},0) = f(\vec{r})$ **or constant**

2. Boundary conditions



1) Dirichlet (first kind)

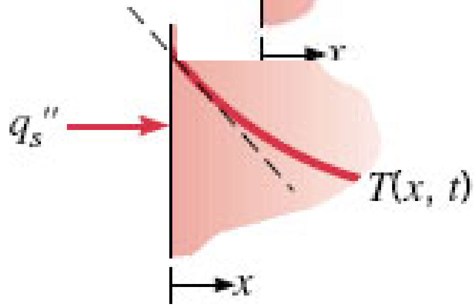
variable value specified at the boundaries



$$T(\vec{r}_b,t)=\alpha(t) \text{ or constant}$$

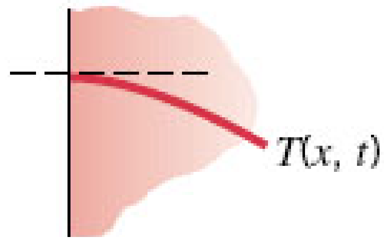
2) Neumann (second kind)

gradient specified at the boundaries



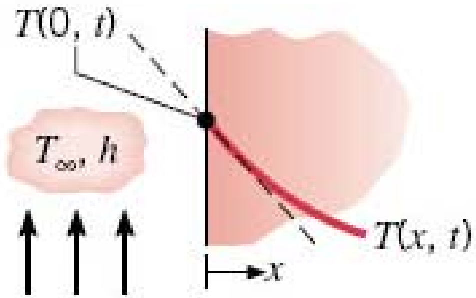
$$\frac{\partial T}{\partial n}(\vec{r}_b,t)=\beta(t) \text{ or constant}$$

when $\beta(t)=0$: adiabatic condition



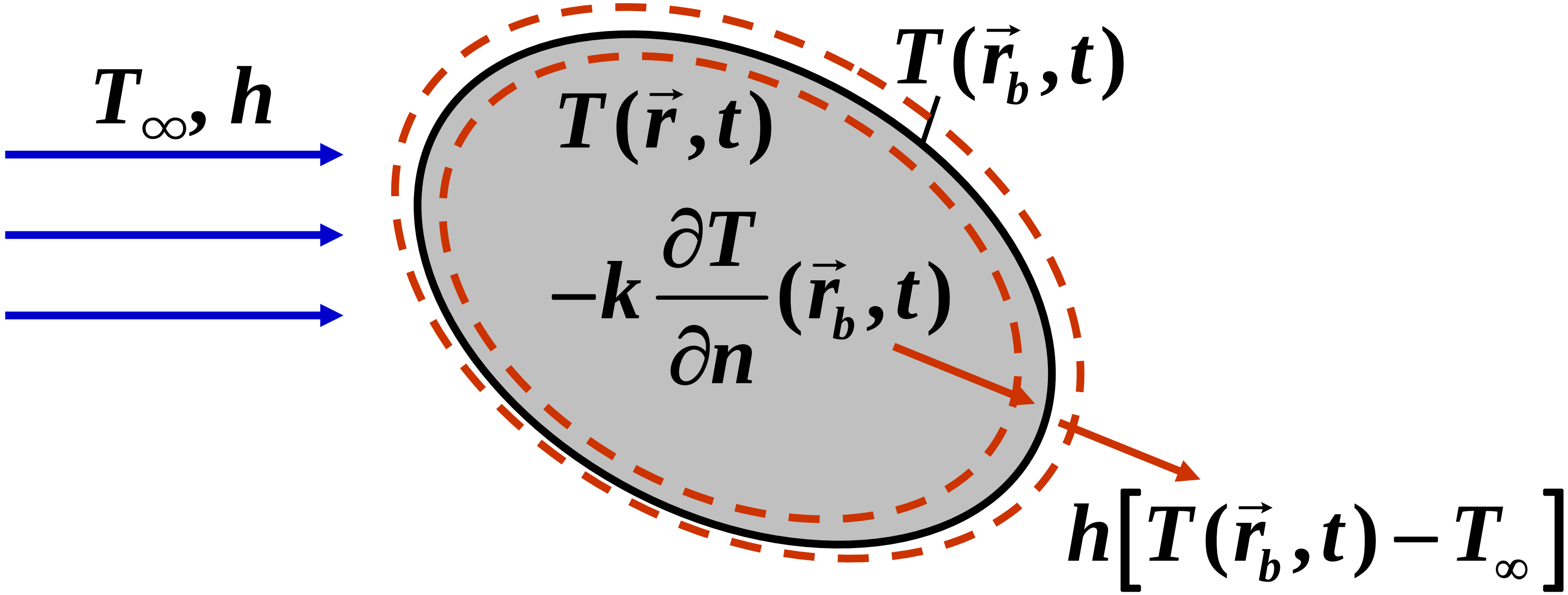
3) Cauchy (third kind)

mixed boundary condition



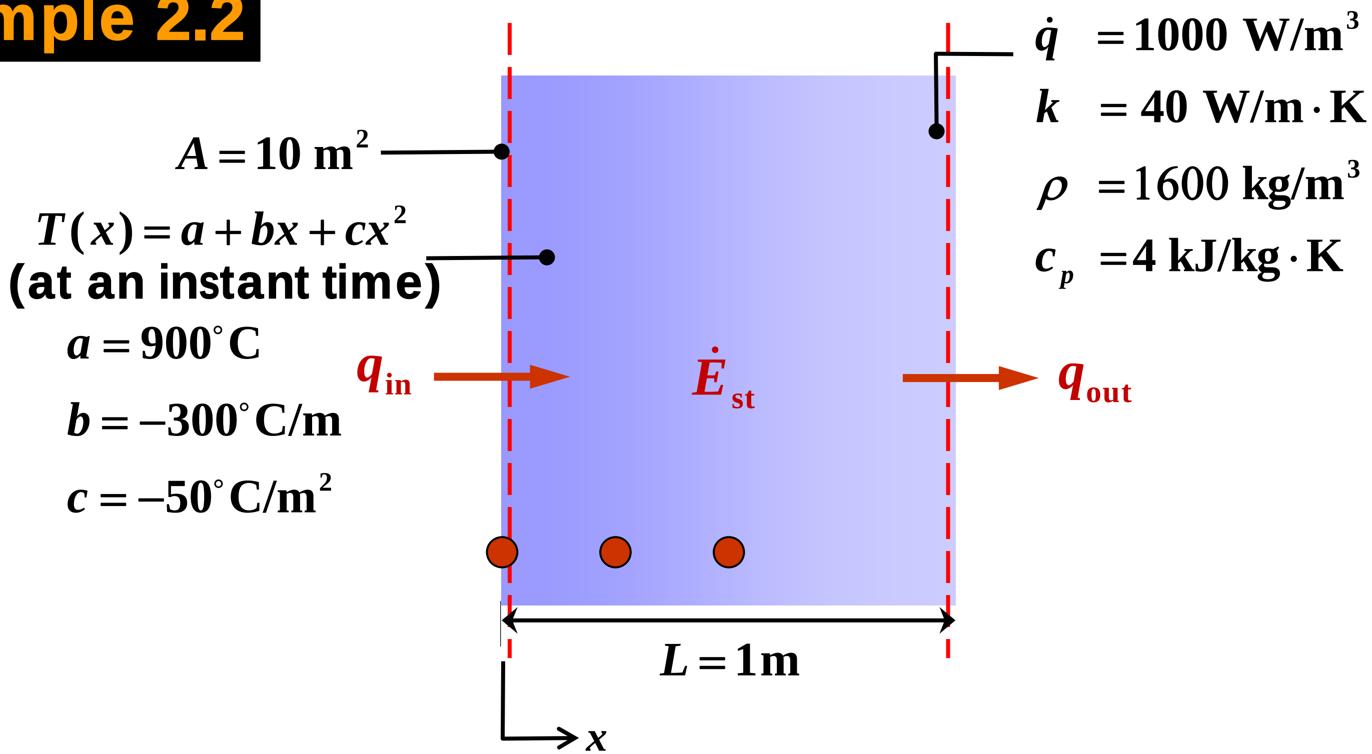
$$a\frac{\partial T}{\partial n}(\vec{r}_b,t)+bT(\vec{r}_b,t)=\gamma(t) \text{ or constant}$$

convection boundary condition



$$-k\frac{\partial T}{\partial n}(\vec{r}_b, t) = h\left[T(\vec{r}_b, t) - T_\infty\right]$$

Example 2.2

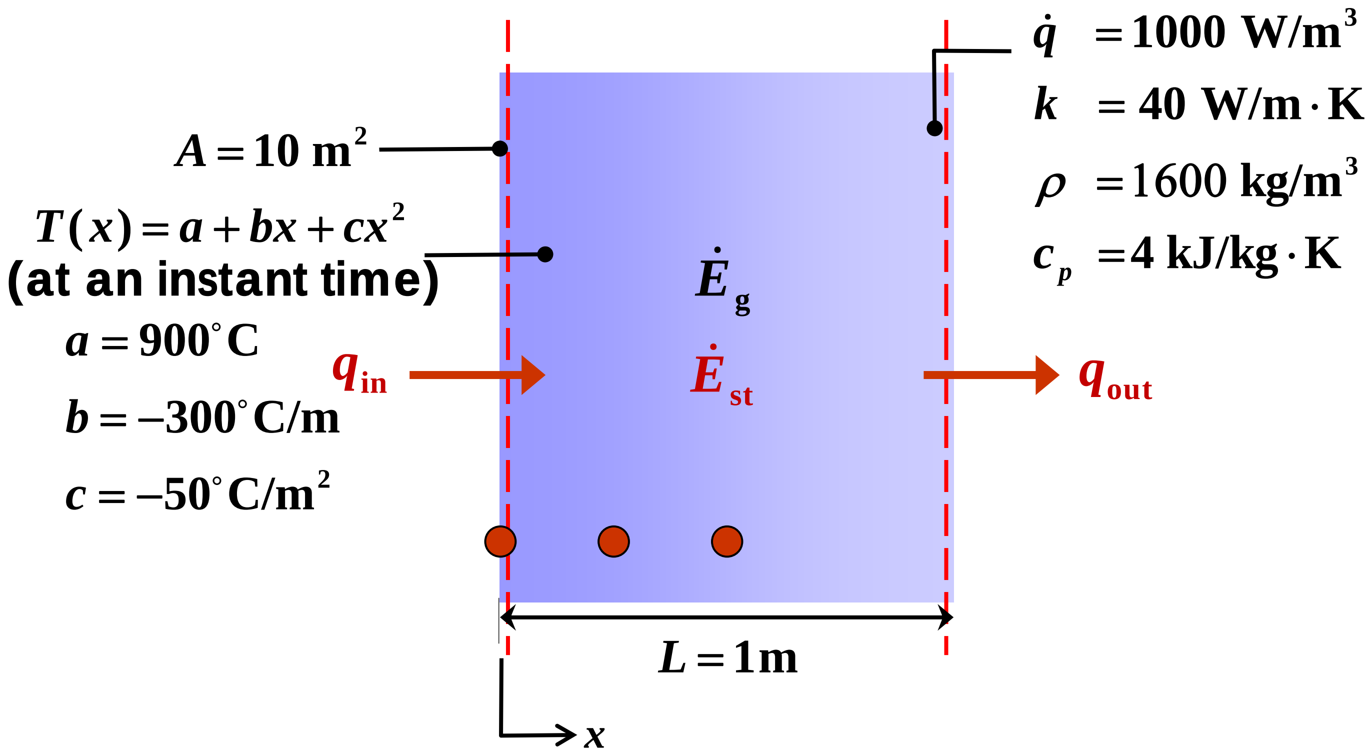


Find

1) Heat rates entering, $q_{in}(x=0)$, and leaving, $q_{out}(x=1)$, the wall
2) Rate of change of energy storage in the wall, $\dot{E}_{st}$
3) Time rate of temperature change $\dfrac{\partial T}{\partial t}$ at x = 0, 0.25, and 0.5 m

Assumptions :

1) 1-D conduction in the x-direction
2) Homogeneous medium with constant properties
3) Uniform internal heat generation



1) $q_{in} = q_x(0) = -kA\left.\dfrac{\partial T}{\partial x}\right|_{x=0}$, $q_{out} = q_x(L) = -kA\left.\dfrac{\partial T}{\partial x}\right|_{x=L}$ $\qquad \dfrac{\partial T}{\partial x} = b + 2cx$

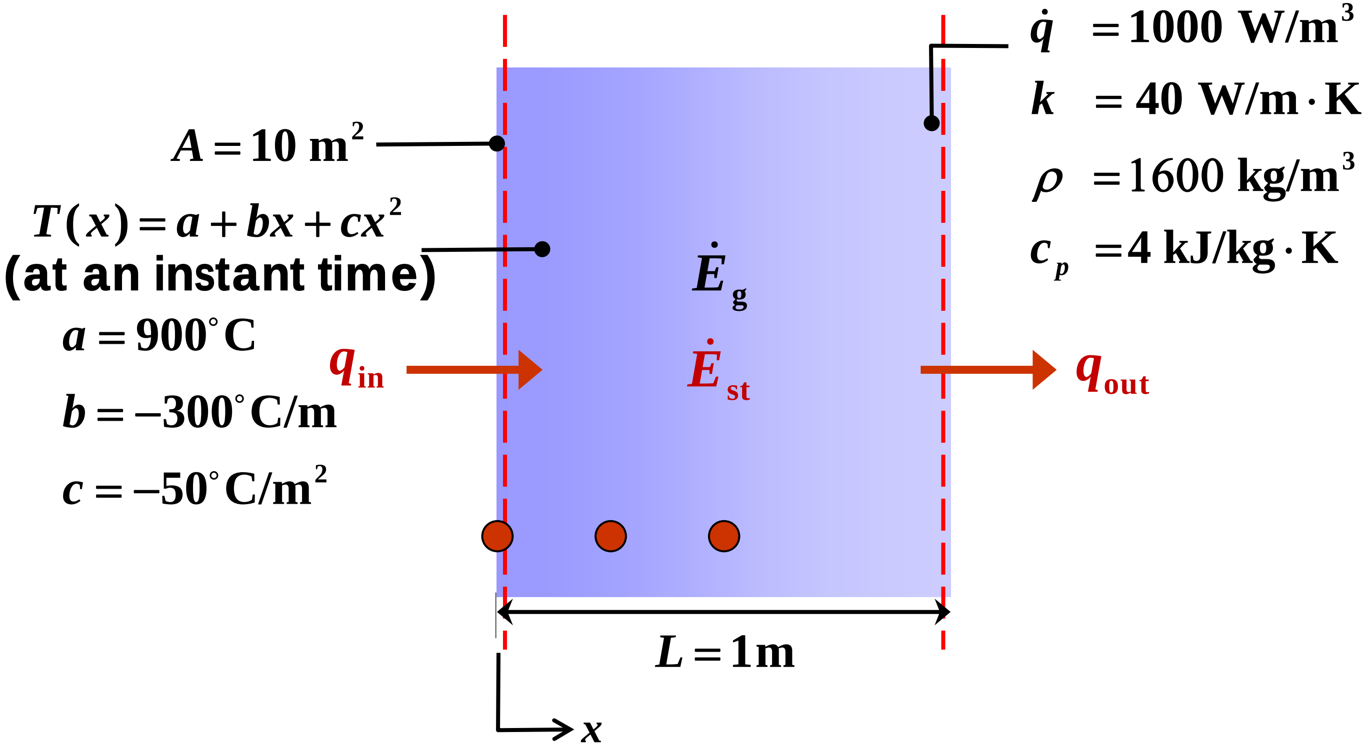
$$q_{in} = -kAb = -(40\ \mathrm{W/m \cdot K})(10\ \mathrm{m^2})(-300^\circ\mathrm{C/m}) = 120\ \mathrm{kW}$$

$$q_{out} = -kA(b + 2cL)$$

$$= -(40\ \mathrm{W/m \cdot K})(10\ \mathrm{m^2})\left[-300^\circ\mathrm{C/m} + 2 \times (-50^\circ\mathrm{C/m^2}) \times (1\ \mathrm{m})\right] = 160\ \mathrm{kW}$$

$$T(x) = a + bx + cx^2 = 900^\circ\mathrm{C} + (-300^\circ\mathrm{C/m})x + (-50^\circ\mathrm{C/m^2})x^2$$

$$= 1173\ \mathrm{K} + (-300\ \mathrm{K/m})x + (-50\ \mathrm{K/m^2})x^2$$



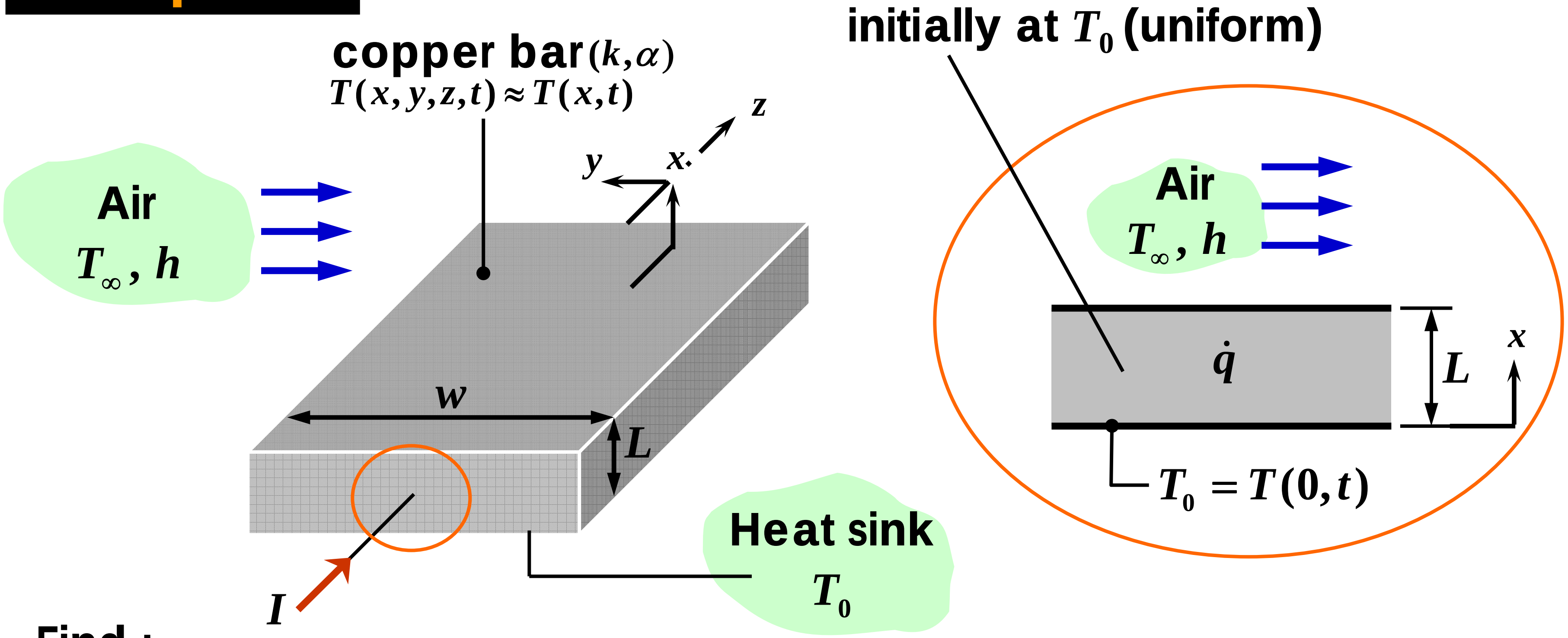
2) $$\dot{E}_{st} = \dot{E}_{in} + \dot{E}_{g} - \dot{E}_{out} = q_{in} + \dot{q}AL - q_{out}$$

$$= 120\ \text{kW} + \left(1000\ \text{W/m}^3\right)\left(10\ \text{m}^2\right)\left(1\ \text{m}\right) - 160\ \text{kW} = -30\ \text{kW}$$

3) $$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho C_p} \qquad \frac{\partial^2 T}{\partial x^2} = 2c$$

$$= -4.69 \times 10^{-4}\ ^\circ\text{C}$$

Example 2.3

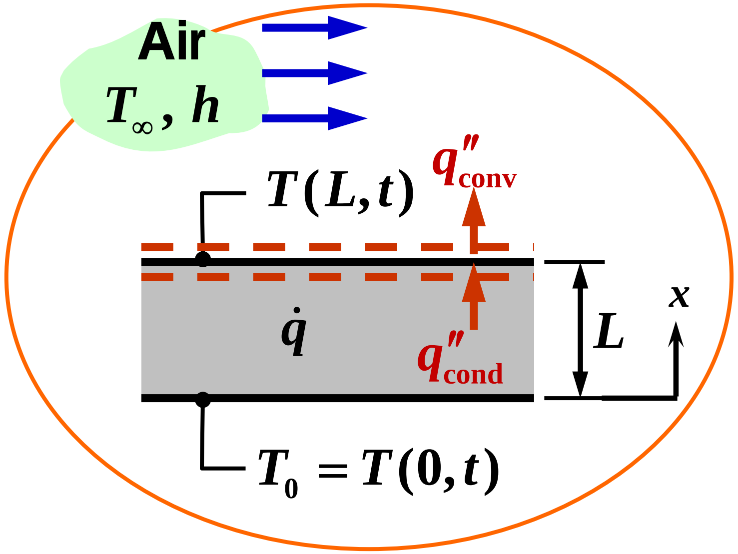


Find :

Differential equation and B.C. and I.C. needed to determine T as a function of position and time within the bar

Assumptions :

1) **Since $w >> L$, side effects are negligible and heat transfer within the bar is 1-D in the x-direction**
2) **Uniform volumetric heat generation**
3) **Constant properties**



governing equation

1-D unsteady with heat generation

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \dot{q}$$

B.C. at the bottom surface :

$$T(0,t) = T_0$$

B.C. at the top surface : convection boundary condition

$$q''_{cond} = q''_{conv} \rightarrow -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$$

I.C. $T(x,0) = T_0$