TIME-DEPENDENT CONDUCTION

- **Lumped Thermal Capacitance Method**
- **Analytical Method:**

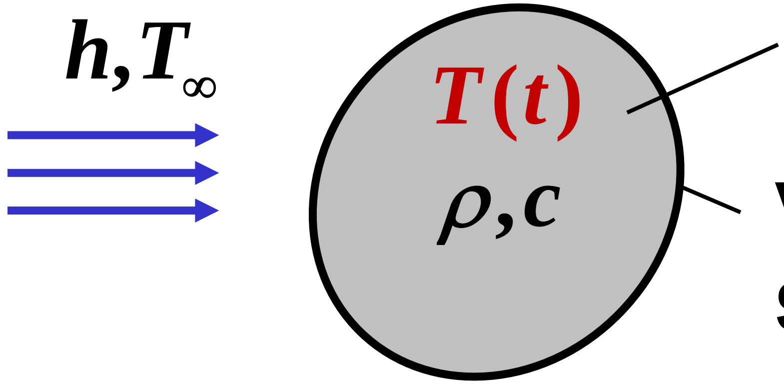
  Separation of Variables
- **Semi-Infinite Solid:**

  Similarity Solution
- **Numerical Method:**

  Finite Difference Method

Lumped Thermal Capacitance Method

negligible spatial effect $T(x,y,z,t) \approx T(t)$



$$\cancel{\dot{E}_{\text{in}}} + \cancel{\dot{E}_{\text{g}}} - \dot{E}_{\text{out}} = \dot{E}_{\text{st}} \quad \rightarrow \dot{E}_{\text{st}} + \dot{E}_{\text{out}} = 0$$

$$\dot{E}_{\text{out}} = hA_s\left[T(t) - T_\infty\right], \qquad \dot{E}_{\text{st}} = \rho V c \frac{dT}{dt}$$

$$\rho V c \frac{dT}{dt} + hA_s\left[T(t) - T_\infty\right] = 0$$

excess temperature: $\theta(t) \equiv T(t) - T_\infty$

$$\rho V c \frac{dT}{dt} + hA_s \left[T(t) - T_\infty \right] = 0$$

$$\frac{d\theta}{dt} + \frac{hA_s}{\rho V c}\theta = 0, \qquad \theta(t) = C \exp\left[-\left(\frac{hA_s}{\rho V c} \right) t \right]$$

initial condition: $\theta(0) = T(0) - T_\infty = T_i - T_\infty \equiv \theta_i$

$$\theta(0) = \theta_i = C$$

$$\frac{\theta(t)}{\theta_i} = \frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c} \right) t \right]$$

thermal time constant

$$\frac{\theta(t)}{\theta_i} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right]$$

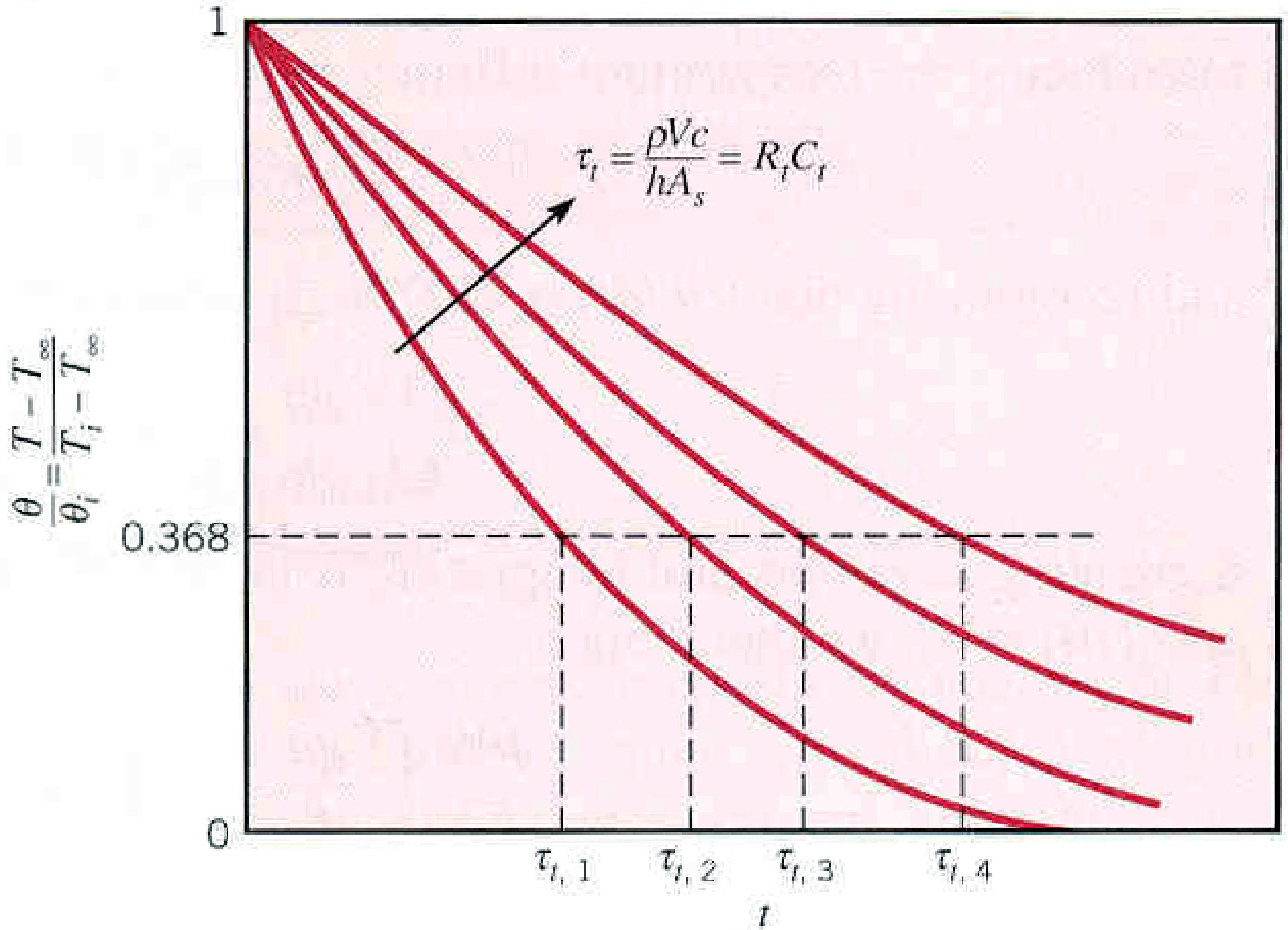
$$\tau_t = \frac{1}{hA_s}\rho Vc = R_t C_t$$

$R_t = \dfrac{1}{hA_s}$: convection resistance

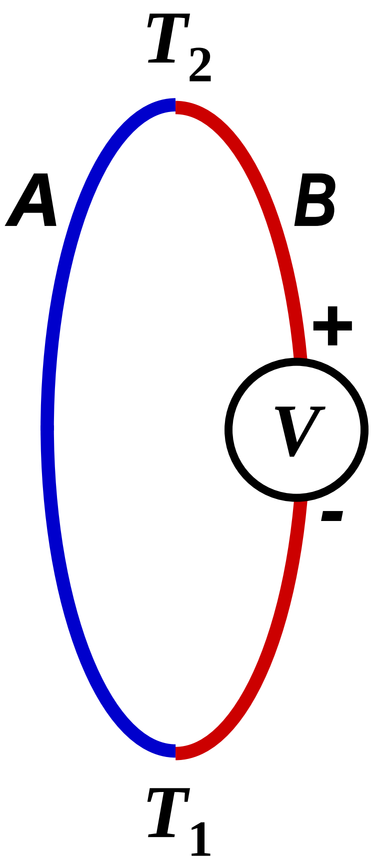
$C_t = \rho Vc$: lumped thermal capacitance

Thermocouple?

Transient temperature response of lumped capacitance solids for different thermal time constant τ_t



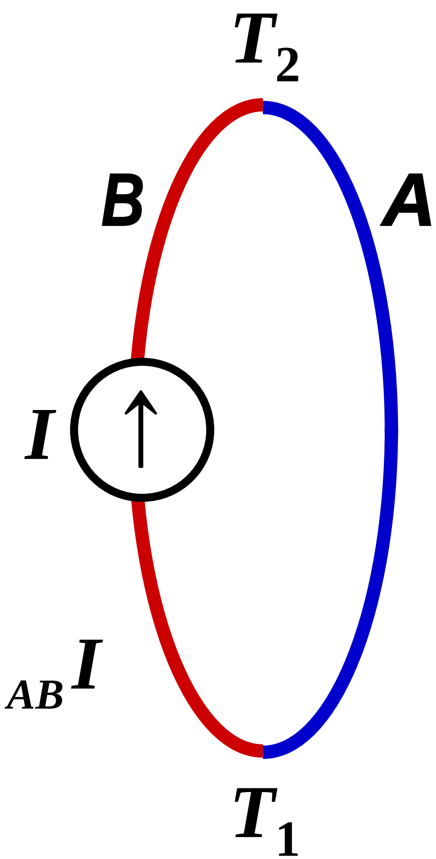
Seebeck effect and Peltier effect



$$V = \int_{T_1}^{T_2} \left(S_B(T) - S_A(T) \right) dT$$

$$= \left(S_B - S_A \right)\left(T_2 - T_1 \right)$$

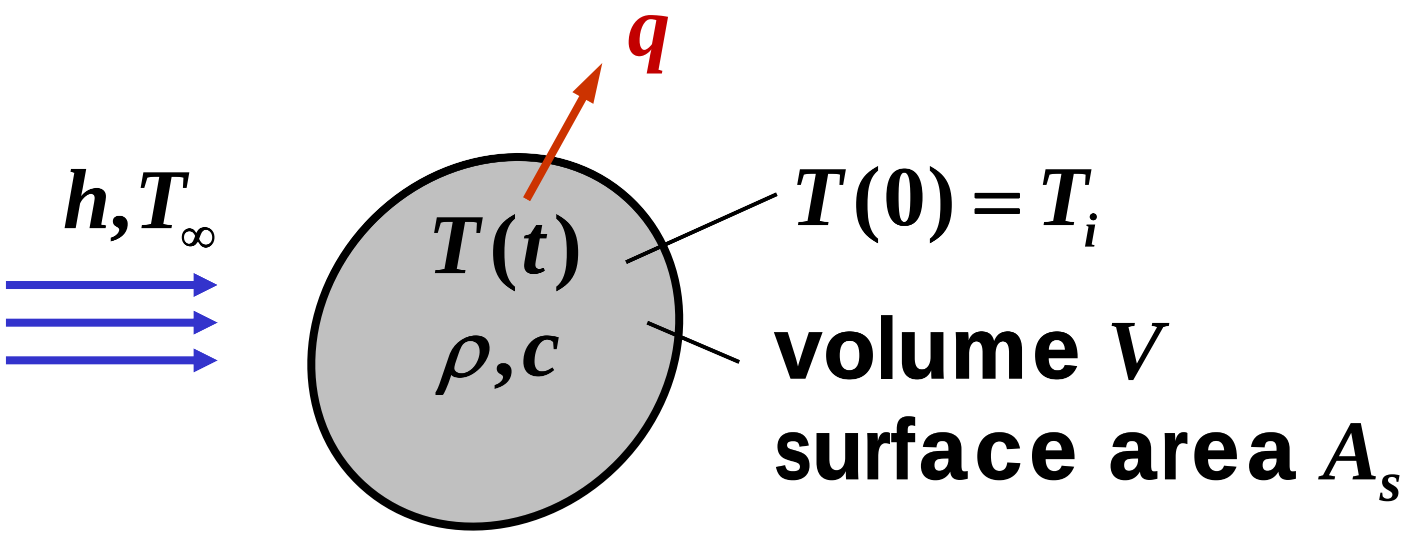
$$\dot{Q} = \left(\Pi_B - \Pi_A \right) I = \Pi_{AB} I$$



Seebeck effect (1821)

Peltier effect (1834)

total energy transfer in time t



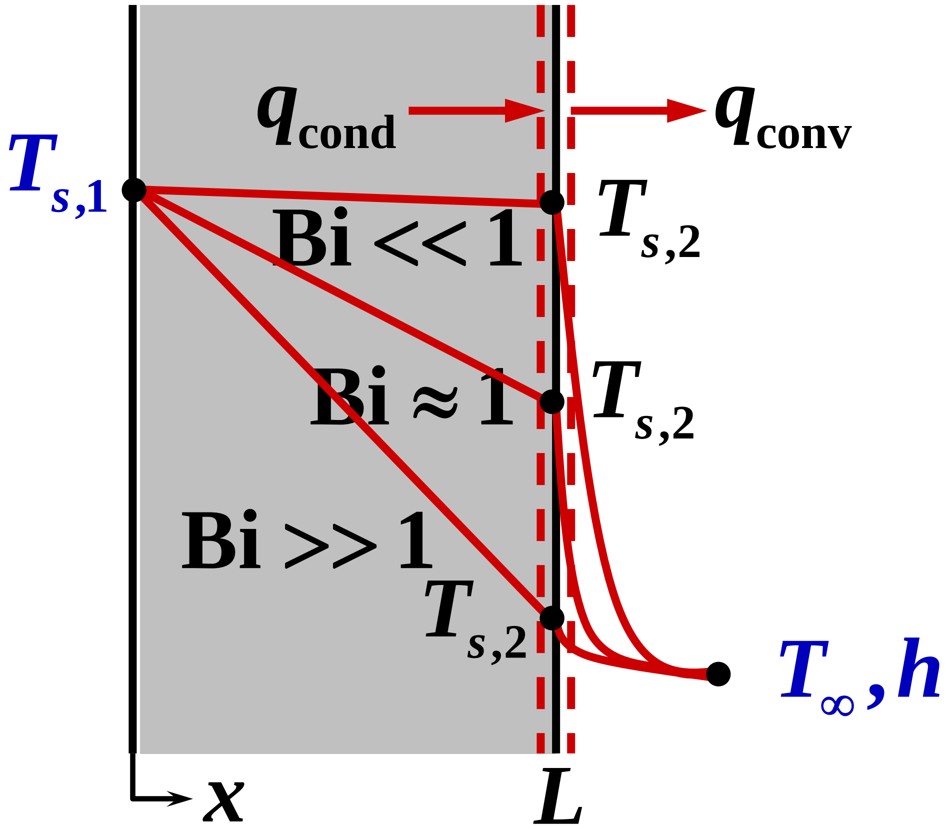
$$q(t) = hA_s\left[T(t) - T_\infty\right] = hA_s\theta(t)$$

$$Q = \int_0^t q\,dt = hA_s\int_0^t \theta(t)\,dt, \quad \frac{\theta(t)}{\theta_i} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right]$$

$$= \left(\rho Vc\right)\theta_i\left[1 - \exp\left(\frac{t}{\tau_t}\right)\right]$$

Validation of Lumped Capacitance Method

$$\frac{kA}{L}\left(T_{s,1}-T_{s,2}\right)=hA\left(T_{s,2}-T_{\infty}\right)$$

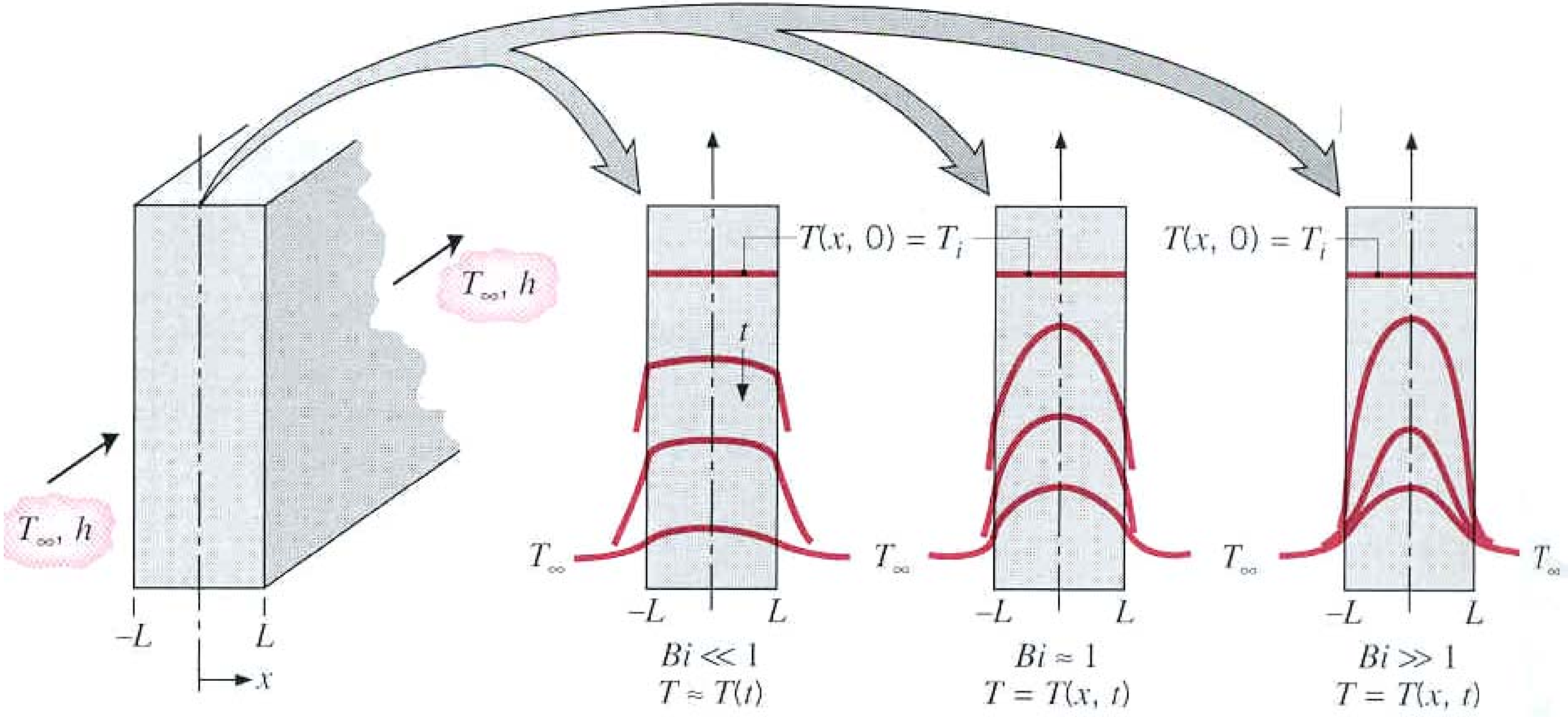


$$\frac{T_{s,1}-T_{s,2}}{T_{s,2}-T_{\infty}}=\frac{L/kA}{1/hA}$$

$$=\frac{hL}{k}\equiv \text{Bi}$$

Bi : Biot number

Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection



When $\mathrm{Bi} = \frac{hL_c}{k} < 0.1$, spatial effect is negligible.

L_c : characteristic length $L_c \equiv \frac{V}{A_s}$

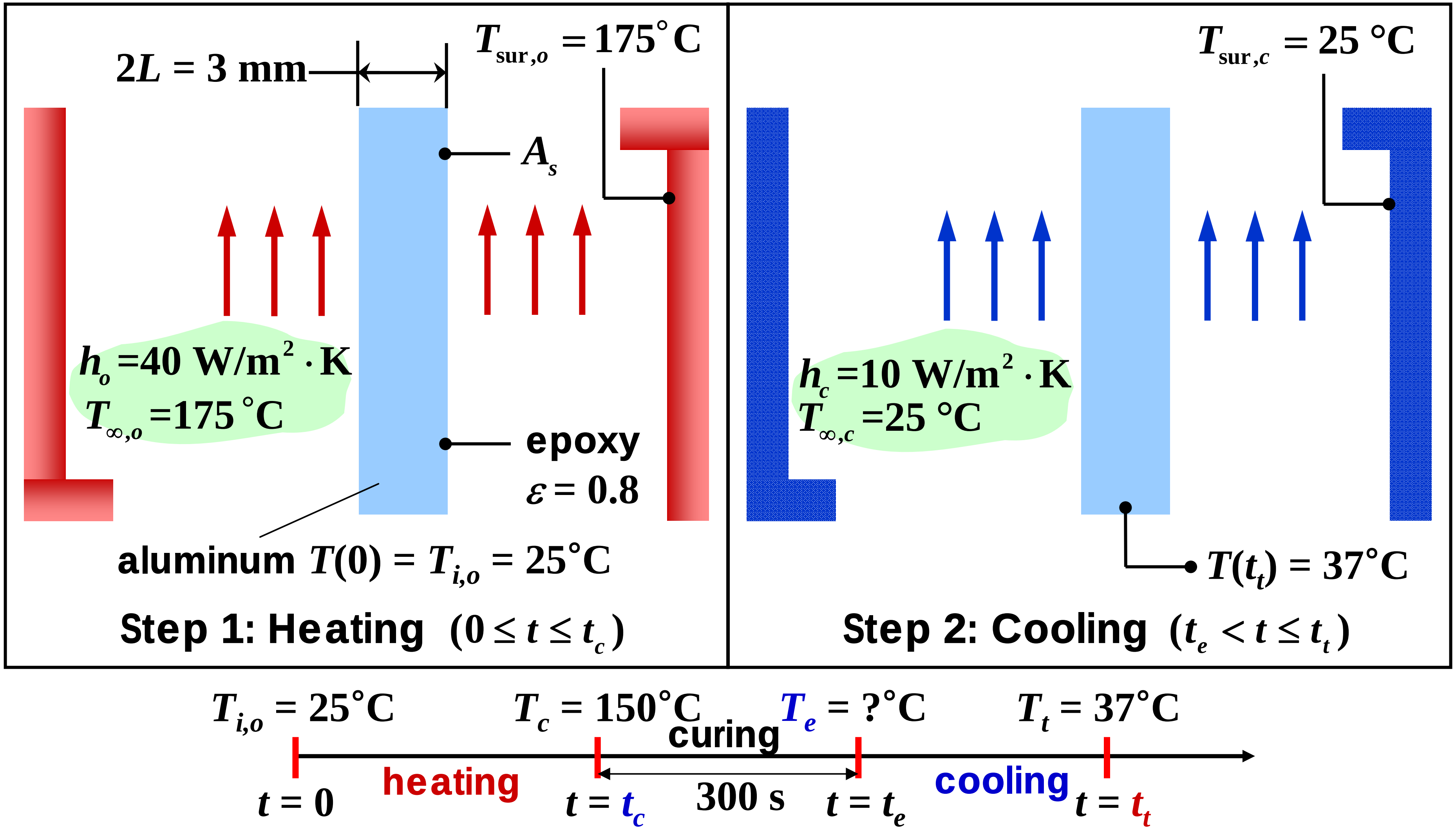
$$\frac{\theta(t)}{\theta_i} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right]$$

$$\frac{hA_s t}{\rho Vc} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{kt}{\rho c L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2} \equiv \mathrm{Bi}\cdot\mathrm{Fo}$$

Fo : Fourier number (dimensionless time)

$$\frac{\theta(t)}{\theta_i} = \exp\left[-\mathrm{Bi}\cdot\mathrm{Fo}\right]$$

Example 5.3



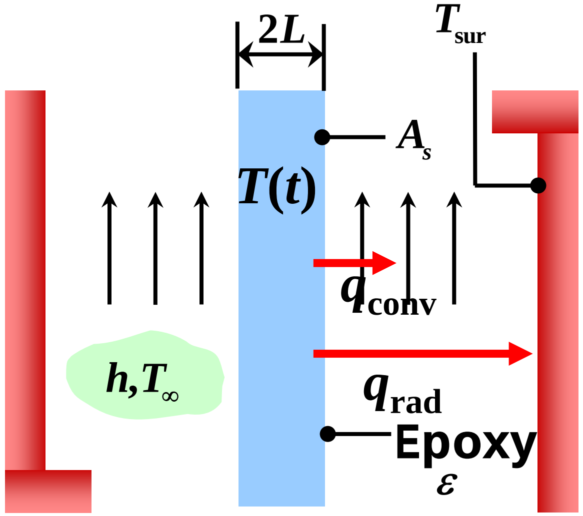
Find: Total time t_t required for the two-step process

Assumption: Thermal resistance of epoxy is negligible.

Biot numbers for the heating and cooling processes

Aluminum: $k = 177\ \mathrm{W/m\cdot K},\ c = 875\ \mathrm{J/kg\cdot K},\ \rho = 2770\ \mathrm{kg/m^3}$

$$\mathrm{Bi}_h = \frac{h_o L}{k} = 3.4\times10^{-4},\quad \mathrm{Bi}_c = \frac{h_c L}{k} = 8.5\times10^{-5}$$



Thus, lumped capacitance approximation can be applied.

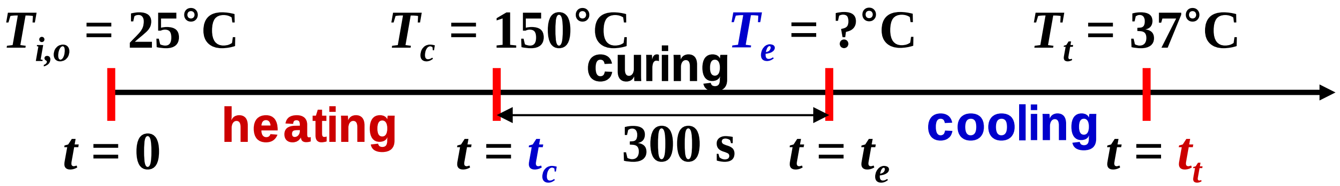
$$\dot{E}_{st} = \cancel{\dot{E}_{in}} - \dot{E}_{out} + \cancel{\dot{E}_g}$$

$$\rho c\left(2LA_s\right)\frac{dT}{dt}$$

$$= -h\left(2A_s\right)\left[T(t) - T_\infty\right] - \varepsilon\sigma\left(2A_s\right)\left[T^4(t) - T_{sur}^4\right]$$

$$dt = \frac{\rho c L dT}{h\left[T_\infty - T(t)\right] + \varepsilon\sigma\left[T_{sur}^4 - T^4(t)\right]}$$

$$dt = \frac{\rho c L dT}{h\left[T_\infty - T(t)\right] + \varepsilon\sigma\left[T_{\text{sur}}^4 - T^4(t)\right]}$$



Heating process

$$\int_0^{t_c} dt = \int_{T_{i,o}}^{T_c} \frac{\rho c L dT}{h\left[T_\infty - T(t)\right] + \varepsilon\sigma\left[T_{\text{sur}}^4 - T^4(t)\right]}$$

t_c = 124 s

Curing process

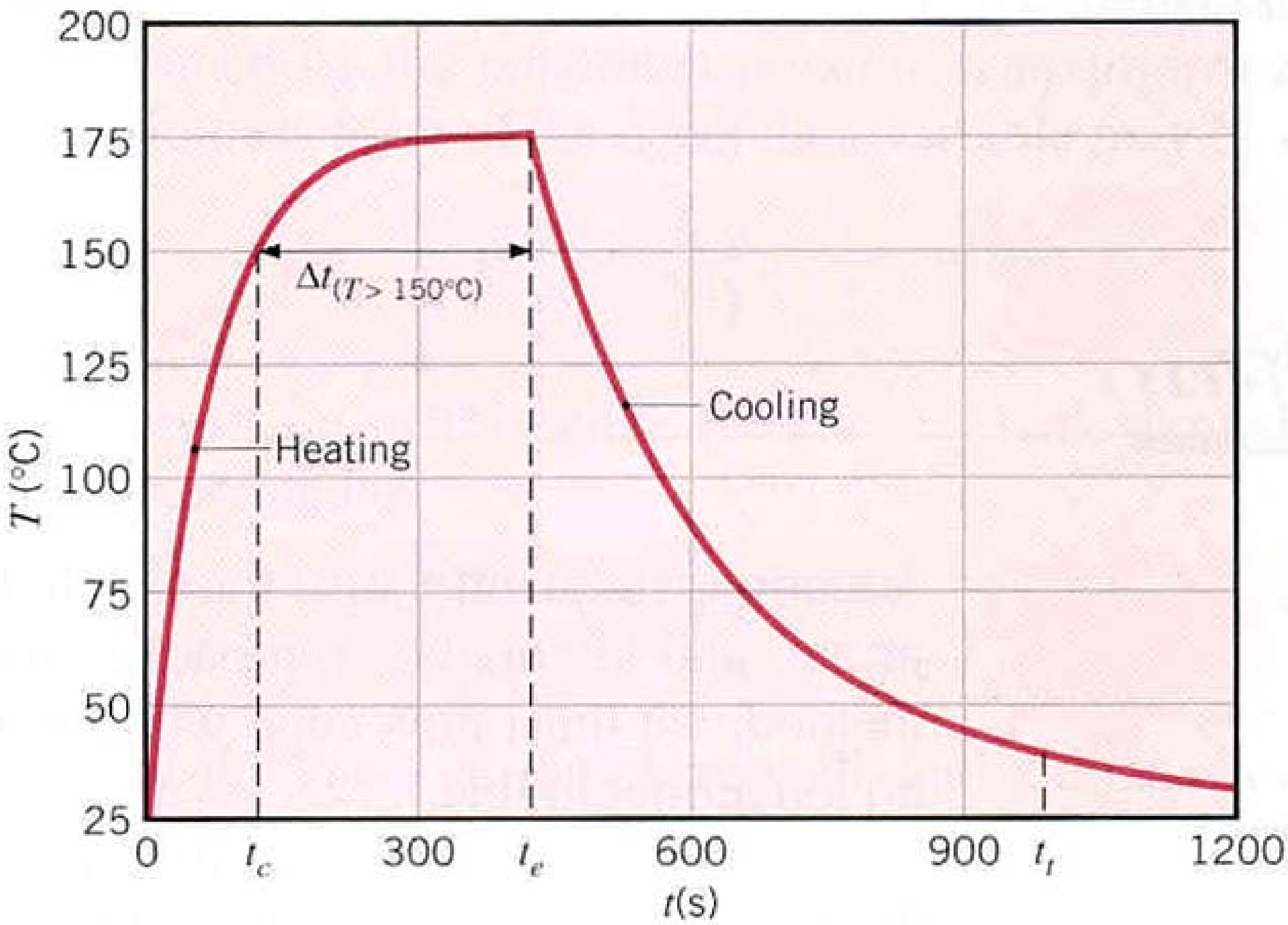
$$\int_{t_c}^{t_e} dt = \int_{T_c}^{T_e} \frac{\rho c L dT}{h\left[T_\infty - T(t)\right] + \varepsilon\sigma\left[T_{\text{sur}}^4 - T^4(t)\right]}$$

T_e = 175°C

Cooling process

$$\int_{t_e}^{t_t} dt = \int_{T_e}^{T_t} \frac{\rho c L dT}{h\left[T_\infty - T(t)\right] + \varepsilon\sigma\left[T_{\text{sur}}^4 - T^4(t)\right]}$$

t_t = 989 s



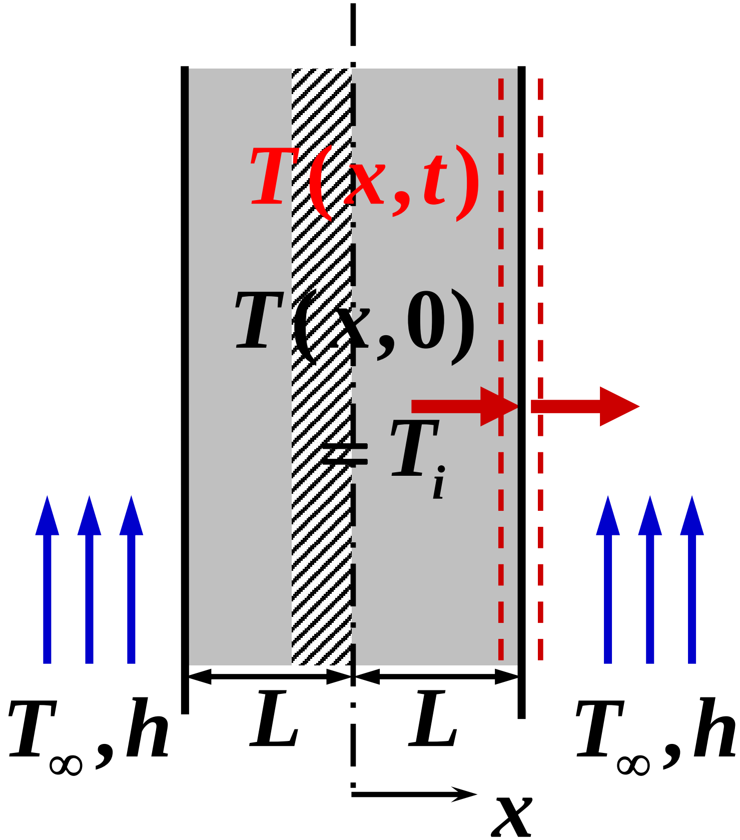
Total time for the two-step process : $t_t = 989$ s

Intermediate times : $t_c = 124$ s $\quad t_e = 424$ s

Analytical Method

Separation of Variables

Plane wall with convection



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

i.c. $T(x,0) = T_i$

b.c. $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$$

$$T = T(x, t, \alpha, T_i, k, L, h, T_\infty)$$

Dimensional analysis

$$T = T(x, t, \alpha, T_i, k, L, h, T_\infty)$$

$$T, T_i, T_\infty : \mathrm{K}\ [D],\quad x : \mathrm{m}\ [L],\quad t : \mathrm{s}\ [T]$$

$$\alpha : \mathrm{m^2/s}\ [L^2 T^{-1}],\quad L : \mathrm{m}\ [L]$$

$$k : \mathrm{W/m \cdot K = kg \cdot m/s^3 \cdot K}\ [LMT^{-3}D^{-1}]$$

$$h : \mathrm{W/m^2 \cdot K = kg/s^3 \cdot K}\ [MT^{-3}D^{-1}]$$

dimensionless variables

$$\theta^* = \frac{T - T_\infty}{T_i - T_\infty},\quad x^* = \frac{x}{L},\quad t^* = \frac{\alpha t}{L^2} = \mathrm{Fo},\quad \mathrm{Bi} = \frac{hL}{k}$$

Fo: Fourier number, Bi: Biot number

$$\theta^* = \theta^*(x^*, t^*; \mathrm{Bi})$$

Equation in dimensionless form

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2},\ \theta^*=\frac{T-T_\infty}{T_i-T_\infty},\ x^*=\frac{x}{L},\ t^*=\frac{\alpha t}{L^2},\ \mathrm{Bi}=\frac{hL}{k}$$

$$\frac{\partial T}{\partial t}=\left(T_i-T_\infty\right)\frac{\partial\theta^*}{\partial t}=\left(T_i-T_\infty\right)\frac{\partial\theta^*}{\partial t^*}\frac{\partial t^*}{\partial t}$$

$$=\cancel{\left(T_i-T_\infty\right)}\frac{\alpha}{L^2}\frac{\partial\theta^*}{\partial t^*}$$

$$\alpha\frac{\partial^2 T}{\partial x^2}=\alpha\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)=\alpha\frac{\partial}{\partial x^*}\left[\left(T_i-T_\infty\right)\frac{\partial\theta^*}{\partial x^*}\frac{\partial x^*}{\partial x}\right]\frac{\partial x^*}{\partial x}$$

$$=\frac{\alpha\cancel{\left(T_i-T_\infty\right)}}{L^2}\frac{\partial^2\theta^*}{\partial x^{*2}}$$

$$\frac{\partial\theta^*}{\partial t^*}=\frac{\partial^2\theta^*}{\partial x^{*2}}$$

$$\theta^* = \frac{T - T_\infty}{T_i - T_\infty},\; x^* = \frac{x}{L},\; t^* = \frac{\alpha t}{L^2},\; \mathrm{Bi} = \frac{hL}{k}$$

initial condition $T(x,0) = T_i \rightarrow \theta^*(x^*,0) = 1$

boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{T_i - T_\infty}{L}\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0 \rightarrow \left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$$

$$\rightarrow -\frac{k\cancel{\left(T_i - T_\infty\right)}}{L}\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = h\cancel{\left(T_i - T_\infty\right)}\theta^*(1,t^*)$$

$$\rightarrow \left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} + \mathrm{Bi}\,\theta^*(1,t^*) = 0$$

Drop out * for convenience afterwards

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}$$

$$\theta(x,0)=1,\ \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\ \left.\frac{\partial\theta}{\partial x}\right|_{x=1}+\mathrm{Bi}\,\theta(1,t)=0$$

$$\theta(x,t)=X(x)\tau(t)$$

$$X\tau'=X''\tau,\quad \frac{X''}{X}=\frac{\tau'}{\tau}=-\zeta^2$$

$$X''+\zeta^2X=0,\quad \tau'+\zeta^2\tau=0$$

boundary conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\ \left.\frac{\partial\theta}{\partial x}\right|_{x=1}+\mathrm{Bi}\,\theta(1,t)=0$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=X'(0)\tau(t)=0\ \rightarrow X'(0)=0$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=1}+\mathrm{Bi}\,\theta(1,t)\ =X'(1)\tau(t)+\mathrm{Bi}X(1)\tau(t)$$

$$=\left[X'(1)+\mathrm{Bi}X(1)\right]\tau(t)=0\rightarrow X'(1)+\mathrm{Bi}X(1)=0$$

$$X(x):\ X'' + \zeta^2 X = 0$$

b.c. $X'(0) = 0,\quad X'(1) + \mathrm{Bi}X(1) = 0$

$$X(x) = \cancel{C_1 \sin(\zeta x)} + C_2 \cos(\zeta x)$$

$$X'(x) = \cancel{C_1 \zeta \cos(\zeta x)} - C_2 \zeta \sin(\zeta x)$$

$$X'(0) = C_1 = 0$$

$$X'(1) + \mathrm{Bi}X(1) = -C_2 \zeta \sin\zeta + \mathrm{Bi} C_2 \cos\zeta$$

$$= C_2 \left(\mathrm{Bi}\cos\zeta - \zeta \sin\zeta\right) = 0 \rightarrow \zeta \tan\zeta = \mathrm{Bi}$$

$$X_n(x) = a_n \cos(\zeta_n x)$$

ζ_n such that $\zeta_n \tan\zeta_n = \mathrm{Bi},\quad n = 1, 2, 3, \cdots$

$$\tau(t):\quad \tau' + \zeta^2 \tau = 0 \quad \rightarrow \tau_n(t) = b_n \exp(-\zeta_n^2 t)$$

$$X_n(x) = a_n \cos(\zeta_n x)$$

$$\theta(x,t) = \sum_{n=1}^{\infty} c_n \exp(-\zeta_n^2 t) \cos(\zeta_n x)$$

initial condition

$$\theta(x,0) = 1 = \sum_{n=1}^{\infty} c_n \cos(\zeta_n x)$$

$$\rightarrow c_n = \frac{\int_0^1 \cos(\zeta_n x) dx}{\int_0^1 \cos^2(\zeta_n x) dx} = \frac{4 \sin \zeta_n}{2\zeta_n + \sin(2\zeta_n)}$$

Approximate solution

$$\theta(x,t)=\sum_{n=1}^{\infty} c_n \exp(-\zeta_n^2 t)\cos(\zeta_n x)$$

$$\begin{aligned}\theta(x,t) &= c_1 \exp(-\zeta_1^2 t)\cos(\zeta_1 x) \\ &\quad + c_2 \exp(-\zeta_2^2 t)\cos(\zeta_2 x) + \cdots \\ &\equiv \theta_1 + \theta_2 + \cdots\end{aligned}$$

When $\mathrm{Fo} = t \geq 0.2$, $\dfrac{c_2 \exp(-\zeta_2^2 t)\cos(\zeta_2 x)}{c_1 \exp(-\zeta_1^2 t)\cos(\zeta_1 x)} << 1$

at $x = 0$, **Bi = 1.0**

	Fo = 0.1	Fo = 1
θ_1	1.0393	0.5339
θ_2	-0.0469	$-1.22\ 10^{-5}$
θ_3	0.0007	$4.7\ 10^{-20}$

Approximate solution

When $\mathrm{Fo} = t \geq 0.2$,

$$\theta(x,t) = c_1 \exp(-\zeta_1^2 t)\cos(\zeta_1 x)$$

$$\theta(0,t) = c_1 \exp(-\zeta_1^2 t) \equiv \theta_0$$

$$\frac{\theta(x,t)}{\theta_0} = \cos(\zeta_1 x)$$

$\boxed{\text{See Table 5.1}}$

$$\zeta_n \tan \zeta_n = \mathrm{Bi}$$

total energy transfer (net out-going)

$$Q(t) = -\int \rho c\left[T(x,t) - T_i\right]dV$$

maximum amount of energy transfer

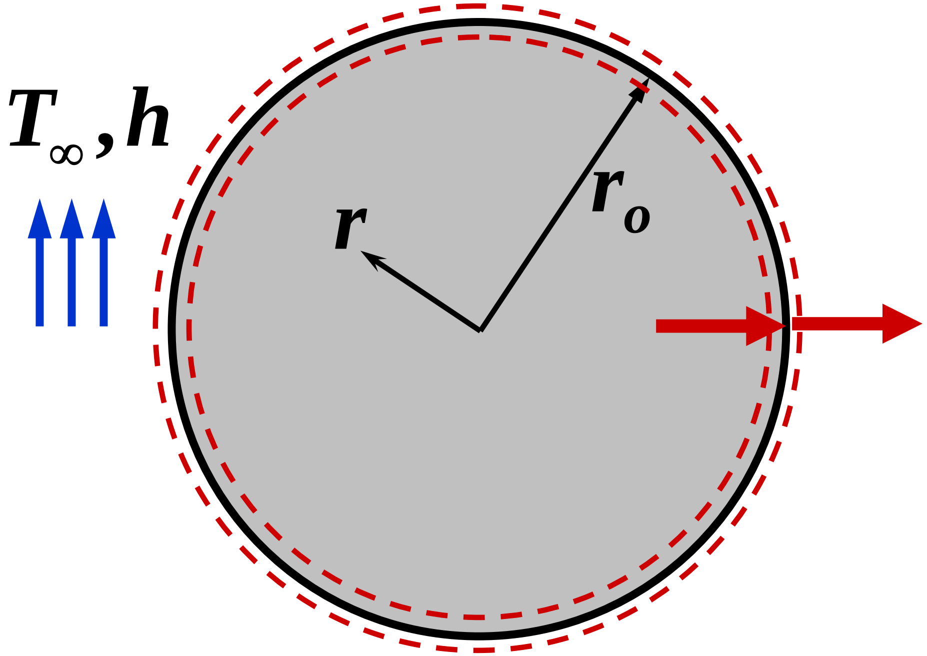
$$Q_0 = \rho c V\left(T_i - T_\infty\right)$$

$$\frac{Q}{Q_0} = \int \frac{-\cancel{\rho c}\left(T - T_i\right)}{\cancel{\rho c}V\left(T_i - T_\infty\right)}dV = -\frac{1}{V}\int \frac{T - T_\infty + T_\infty - T_i}{T_i - T_\infty}dV$$

$$= \frac{1}{V}\int\left(1-\theta\right)dV = \int_0^1\left[1-\theta_0\cos(\zeta_1 x)\right]dx$$

$$= 1 - \frac{\sin\zeta_1}{\zeta_1}\theta_0$$

Radial systems with convection



$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

i.c. $T(r,0) = T_i$

b.c. $\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$ **or** $T(0,t) = \text{finite}$

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = h\left[T(r_0,t) - T_\infty\right]$$

dimensionless variables

$$\theta^* = \frac{T - T_\infty}{T_i - T_\infty}, \; r^* = \frac{r}{r_0}, \; t^* = \frac{\alpha t}{r_0^2} = \mathrm{Fo}, \; \mathrm{Bi} = \frac{h r_0}{k}$$

Drop out * for convenience afterwards

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \;\rightarrow\; \frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)$$

i.c. $\theta(r,0) = 1$

b.c. $\left.\frac{\partial \theta}{\partial r}\right|_{r=0} = 0$ **or** $\theta(0,t) = \text{finite}$

$$\left.\frac{\partial \theta}{\partial r}\right|_{r=1} + \mathrm{Bi}\,\theta(1,t) = 0$$

$$\theta(r,t) = R(r)\tau(t)$$

$$\frac{\partial\theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)$$

$$R\tau' = \frac{1}{r}\frac{\partial}{\partial r}(rR'\tau) = \frac{1}{r}\tau\left(R' + rR''\right)$$

$$\frac{\tau'}{\tau} = \frac{1}{r}\left(\frac{R'}{R} + r\frac{R''}{R}\right) = -\zeta^2$$

$$r\frac{R''}{R} + \frac{R'}{R} + r\zeta^2 = 0, \qquad \tau' + \zeta^2\tau = 0$$

$$R(r): \quad R'' + \frac{1}{r}R' + \zeta^2 R = 0$$

$$r^2 R'' + rR' + \zeta^2 r^2 R = 0$$

$$\left[x^2 y'' + xy' + m^2(x^2 - v^2)y = 0 \rightarrow y = AJ_v(mx) + BY_v(mx) \right]$$

$$R(r) = C_1 J_0(\zeta r) + \cancel{C_2 Y_0(\zeta r)}$$

$$\theta(0,t) = \text{finite} \rightarrow R(0) = \text{finite}$$

$$Y_0(0) \rightarrow -\infty, \text{ thus } C_2 = 0$$

$$\left.\frac{\partial \theta}{\partial r}\right|_{r=1} + \mathrm{Bi}\,\theta(1,t) = 0 \rightarrow R'(1) + \mathrm{Bi}R(1) = 0$$

$$C_1 \left.\frac{dJ_0(\zeta r)}{dr}\right|_{r=1} + C_1 \mathrm{Bi} J_0(\zeta) = 0$$

$$\left.\frac{dJ_0(\zeta r)}{dr}\right|_{r=1} + \mathrm{Bi}J_0(\zeta) = 0$$

Since $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$J_{n-1}(x) = \frac{2n}{x}J_n(x) - J_{n+1}(x)$$

$$\frac{d}{dx}\left[J_0(x)\right] = J_{-1}(x) = -J_1(x)$$

$$\left.\frac{dJ_0(\zeta r)}{dr}\right|_{r=1} + \mathrm{Bi}J_0(\zeta) = -\zeta J_1(\zeta) + \mathrm{Bi}J_0(\zeta) = 0$$

$R_n(r) = a_n J_0(\zeta_n r)$, ζ_n **such that** $\zeta_n \dfrac{J_1(\zeta_n)}{J_0(\zeta_n)} = \mathrm{Bi}$

$$\tau(t): \quad \tau' + \zeta^2 \tau = 0 \quad \rightarrow \tau_n(t) = b_n \exp(-\zeta_n^2 t)$$

$$R_n(r) = a_n J_0(\zeta_n r)$$

$$\theta(r,t) = \sum_{n=1}^{\infty} c_n \exp(-\zeta_n^2 t) J_0(\zeta_n r)$$

initial condition

$$\theta(r,0) = 1 = \sum_{n=1}^{\infty} c_n J_0(\zeta_n r)$$

$$\rightarrow c_n = \frac{\int_0^1 r J_0(\zeta_n r) dr}{\int_0^1 r J_0^2(\zeta_n r) dr}$$

Approximate solution

$$\theta(r,t) = c_1 \exp(-\zeta_1^2 t) J_0(\zeta_1 r)$$

$$\theta(0,t) = c_1 \exp(-\zeta_1^2 t) J_0(0) = c_1 \exp(-\zeta_1^2 t) \equiv \theta_0$$

$$\frac{\theta(x,t)}{\theta_0} = J_0(\zeta_1 x)$$

Total energy transfer (net out-going)

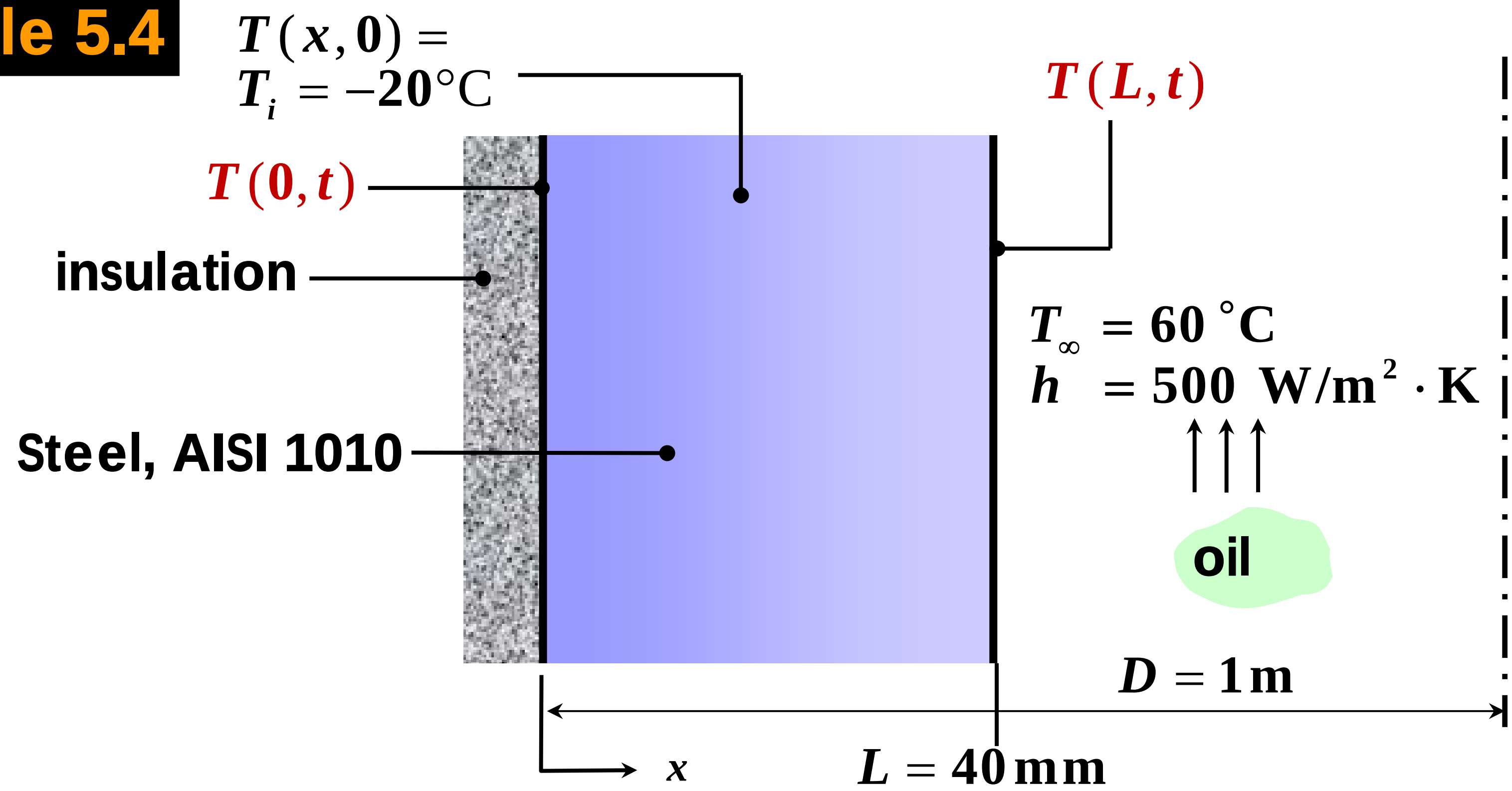
$$Q(t) = -\int \rho c \left[T(x,t) - T_i \right] dV, \;\; Q_0 = \rho c V \left(T_i - T_\infty \right)$$

$$\frac{Q}{Q_0} = \frac{1}{V} \int \left(1 - \theta\right) dV = \frac{1}{V} \int_0^1 \left[1 - \theta_0 J_0(\zeta_1 r)\right] dV$$

Since $V = \pi L$, $dV = 2\pi r dr L$

$$\frac{Q}{Q_0} = 2\int_0^1 \left[1 - \theta_0 J_0(\zeta_1 r)\right] r dr = 1 - \frac{2J_1(\zeta_1)}{\zeta_1}\theta_0$$

Example 5.4



Find:

1) **Biot** and **Fourier** numbers after 8 min
2) Temperature of exterior pipe surface after 8 min, $T(0, 8min)$
3) Heat flux to the wall at 8 min, $q''(8\,min)$
4) Energy transferred to pipe per unit length after 8 min, Q'

Assumption:

Pipe wall can be approximated as plane wall, since $L << D$.

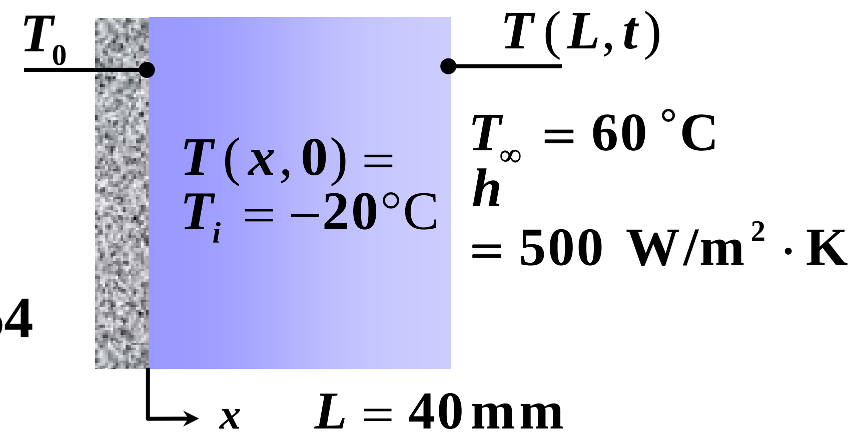
AISI 1010: $\rho = 7823\ \mathrm{kg/m^3},\ c = 434\ \mathrm{J/kg\cdot K},$

$k = 63.9\ \mathrm{W/m\cdot K},\ \alpha = 18.8\times10^{-6}\ \mathrm{m^2/s}$

1) Bi and Fo at t = 8 min

$$\mathrm{Bi} = \frac{hL}{k} = \frac{500\times0.04}{63.9} = 0.313$$

$$\mathrm{Fo} = \frac{\alpha t}{L^2} = \frac{18.8\times10^{-6}\times8\times60}{0.04^2} = 5.64$$



2) T(0, 8 min)

With Bi = 0.313, the lumped capacitance method is inappropriate.

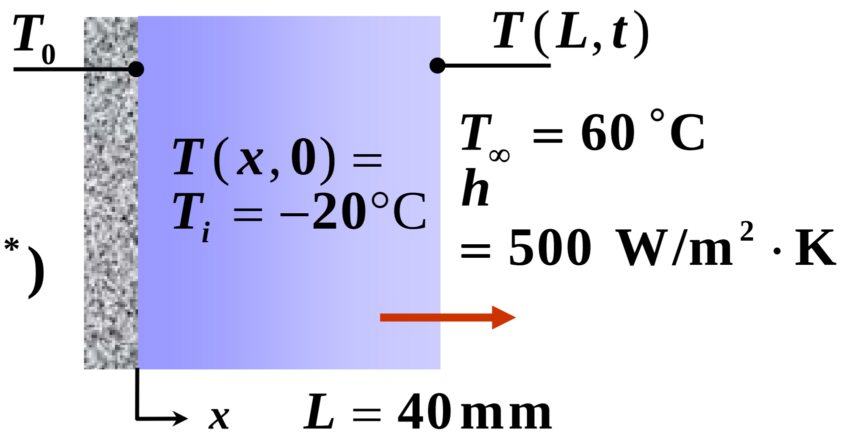
However, since Fo > 0.2, approximate solution can be applicable.

$$\theta_0^* = \frac{T_0 - T_\infty}{T_i - T_\infty} = c_1 \exp(-\zeta_1^2 \mathrm{Fo}) = 0.214$$

from Table 5.1 $\quad c_1 = 1.047,\ \zeta_1 = 0.531$

$$T(0, 8\,\mathrm{min}) = T_\infty + \theta_0^*(T_i - T_\infty) = 60 + 0.214\left(-20 - 60\right) = 42.9°\mathrm{C}$$

3) q''(480 s)



$$q''(480\text{ s}) = h\left[T(40\text{ mm}, 480\text{ s}) - T_\infty\right]$$

$$\theta^*(x^*, t^*) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = \theta_0^*(t^*)\cos(\zeta_1 x^*)$$

$$T(L,t) = T_\infty - \left(T_i - T_\infty\right)\theta_0^*\cos(\zeta_1)$$

$$T(40\text{ mm}, 480\text{ s}) = 60 - (-20 - 60)\times 0.214\times\cos(0.531) = 45.2$$

$$q'' = 500\left(45.2 - 60\right) = -7400\ \text{W/m}^2$$

4) The energy transfer to the pipe wall over the 8-min interval

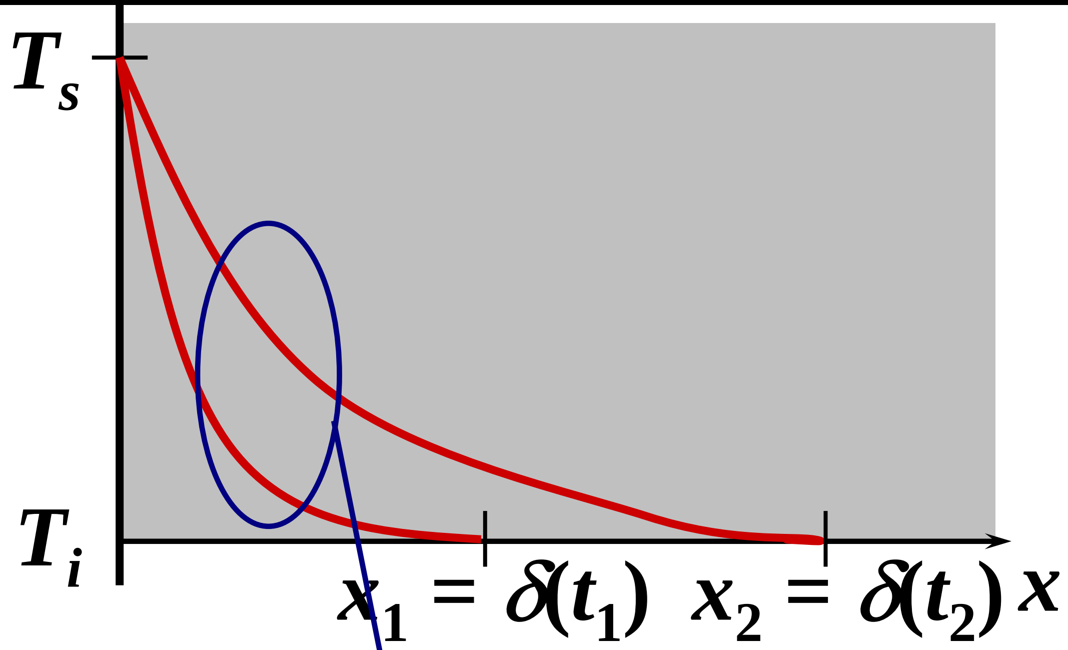
$$\frac{Q}{Q_0} = 1 - \frac{\sin(\zeta_1)}{\zeta_1}\theta_0^* = 1 - \frac{\sin(0.531)}{0.531}\times 0.214 = 0.80$$

$$Q = 0.80\,\rho c V\left(T_i - T_\infty\right)$$

$$Q' = 0.80\,\rho c\left(\pi D L\right)\left(T_i - T_\infty\right)$$

$$= -2.73\times 10^7\ \text{J/m}$$

Semi-Infinite Solid: Similarity Solution



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

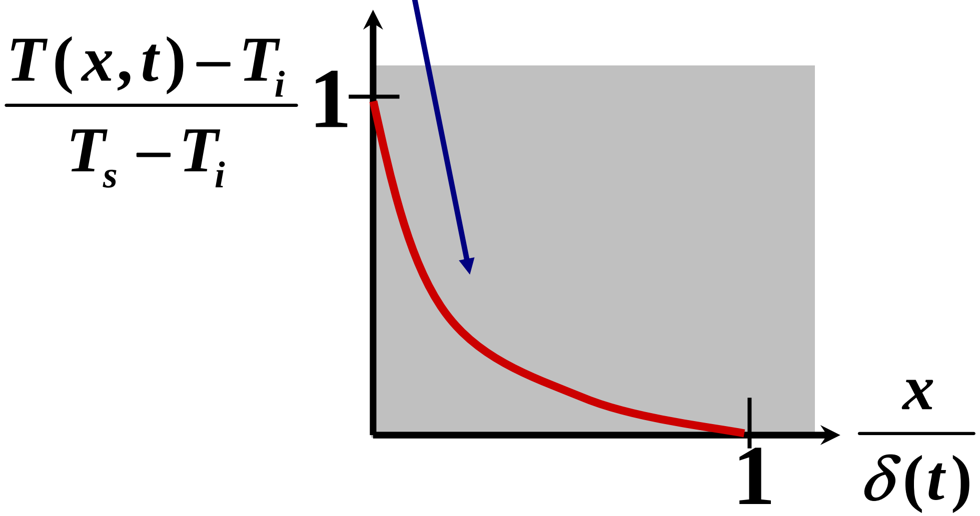
i.c. $T(x,0) = T_i$

b.c. $T(0,t) = T_s$

$T(\infty,t) = T_i$

similarity solution $\theta(x,t) = \dfrac{T(x,t) - T_i}{T_s - T_i}$

$$= \theta\left(\frac{x}{\delta(t)}\right) = \theta(\eta)$$



similarity variable

$$\eta = \frac{x}{\delta(t)}$$

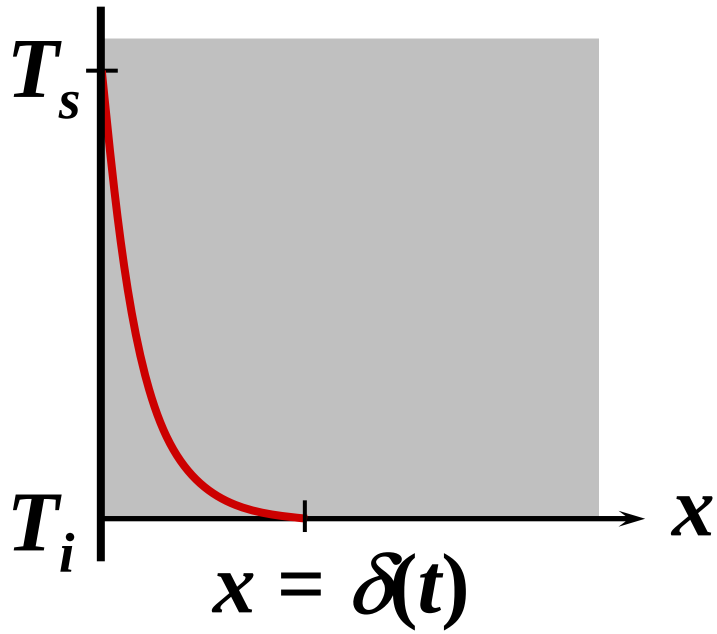
Scaling analysis

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial T}{\partial t} \sim \frac{\Delta T}{t}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \sim \frac{\Delta T}{\delta^2}$$

$$\frac{\Delta T}{t} \sim \alpha \frac{\Delta T}{\delta^2} \rightarrow \delta^2 \sim \alpha t \rightarrow \delta \sim \sqrt{\alpha t}$$



Let $\eta = \dfrac{x}{\delta(t)} = \dfrac{x}{2\sqrt{\alpha t}}$

$$\theta(x,t)=\frac{T(x,t)-T_i}{T_s-T_i}=\theta(\eta),\quad \eta=\frac{x}{2\sqrt{\alpha t}}$$

$$\frac{\partial T}{\partial t}=\left(T_s-T_i\right)\frac{\partial\theta}{\partial t}=\left(T_s-T_i\right)\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial t}$$

$$=\left(T_s-T_i\right)\frac{x}{2\sqrt{\alpha}}\left(-\frac{1}{2t\sqrt{t}}\right)\frac{d\theta}{d\eta}=-\frac{\left(T_s-T_i\right)}{2t}\eta\frac{d\theta}{d\eta}$$

$$\frac{\partial T}{\partial x}=\left(T_s-T_i\right)\frac{\partial\theta}{\partial x}=\left(T_s-T_i\right)\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial x}$$

$$=\left(T_s-T_i\right)\frac{1}{2\sqrt{\alpha t}}\frac{d\theta}{d\eta}$$

$$\eta = \frac{x}{2\sqrt{\alpha t}}, \qquad \frac{\partial T}{\partial x} = \left(T_s - T_i\right)\frac{1}{2\sqrt{\alpha t}}\frac{d\theta}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\left(T_s - T_i\right)\frac{1}{2\sqrt{\alpha t}}\frac{d\theta}{d\eta}\right)$$

$$= \left(T_s - T_i\right)\frac{1}{2\sqrt{\alpha t}}\frac{d}{d\eta}\left(\frac{d\theta}{d\eta}\right)\frac{\partial \eta}{\partial x} = \left(T_s - T_i\right)\frac{1}{4\alpha t}\frac{d^2\theta}{d\eta^2}$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\rightarrow -\frac{\left(T_s - T_i\right)}{2t}\eta\frac{d\theta}{d\eta} = \alpha\left(T_s - T_i\right)\frac{1}{4\alpha t}\frac{d^2\theta}{d\eta^2}$$

$$\theta'' + 2\eta\theta' = 0$$

i.c. $T(x,0) = T_i$:

$$\eta \to \infty, \quad \theta(\infty) = 0$$

b.c. $T(0,t) = T_s$

$$\eta = 0, \quad \theta(0) = 1$$

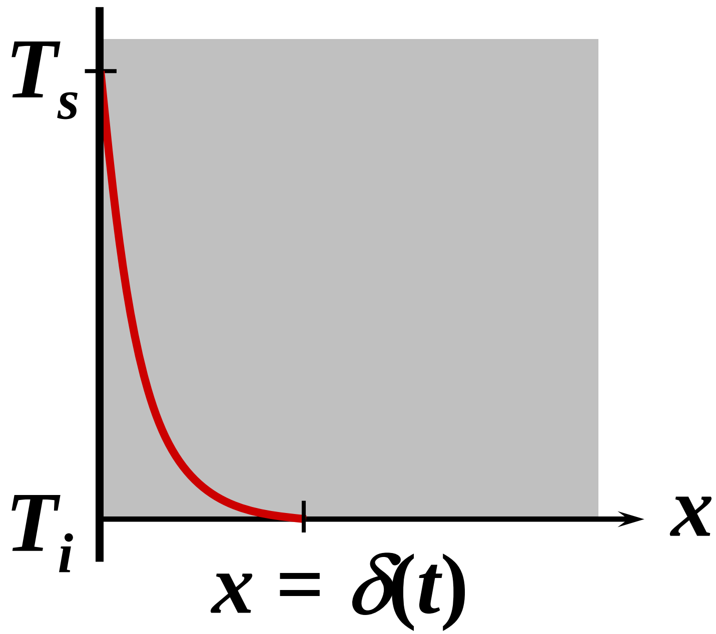
$$T(\infty,t) = T_i$$

$$\eta \to \infty, \quad \theta(\infty) = 0$$

merge into one

$$\theta(x,t) = \frac{T(x,t) - T_i}{T_s - T_i}$$

$$\eta = \frac{x}{2\sqrt{\alpha t}}$$



Similarity solution

$$\theta'' + 2\eta\theta' = 0: \ \theta(0) = 1, \quad \theta(\infty) = 0$$

integrating factor e^{η^2}

$$\frac{d}{d\eta}\left(e^{\eta^2}\theta'\right) = 0 \rightarrow \theta' = \frac{d\theta}{d\eta} = C_1 e^{-\eta^2} \rightarrow \ d\theta = C_1 e^{-\eta^2} d\eta$$

$$\int_0^\eta d\theta = \int_0^\eta C_1 e^{-u^2} du \ \rightarrow \theta(\eta) - \theta(0) = C_1 \int_0^\eta e^{-u^2} du$$

or $\quad \theta(\eta) = 1 + C_1 \int_0^\eta e^{-u^2} du$

$$\theta(\infty) = 0 = 1 + C_1 \int_0^\infty e^{-u^2} du$$

error function: $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du, \ \mathrm{erf}(\infty) = 1$

$$\theta(\infty) = 0 = 1 + C_1 \int_0^\infty e^{-u^2} du$$

$$\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2} \rightarrow C_1 = -\frac{2}{\sqrt{\pi}}$$

$$\theta(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du = 1 - \mathrm{erf}(\eta)$$

$$\theta(x,t) = \frac{T(x,t) - T_i}{T_s - T_i} = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

erfc: complimentary error function

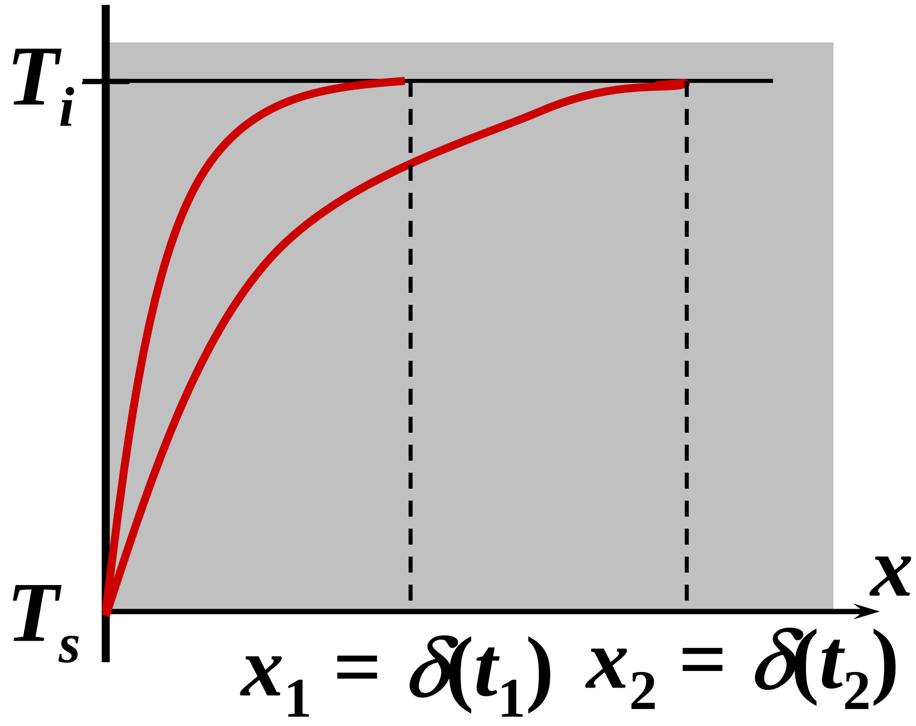
$$\theta(x,t)=\frac{T(x,t)-T_i}{T_s-T_i}=\frac{T(x,t)-T_s+T_s-T_i}{T_s-T_i}$$

$$=1-\frac{T(x,t)-T_s}{T_i-T_s}$$

$$\frac{T(x,t)-T_s}{T_i-T_s}=1-\theta(x,t)$$

$$=1-\left[1-\mathrm{erf}(\eta)\right]$$

$$=\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$



Heat flux through the wall

$$q_s'' = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = -k\left(T_s - T_i\right)\frac{d\theta}{d\eta}\frac{\partial \eta}{\partial x}\bigg|_{x=0}$$

$$= -\frac{k\left(T_s - T_i\right)}{2\sqrt{\alpha t}}\theta'(0)$$

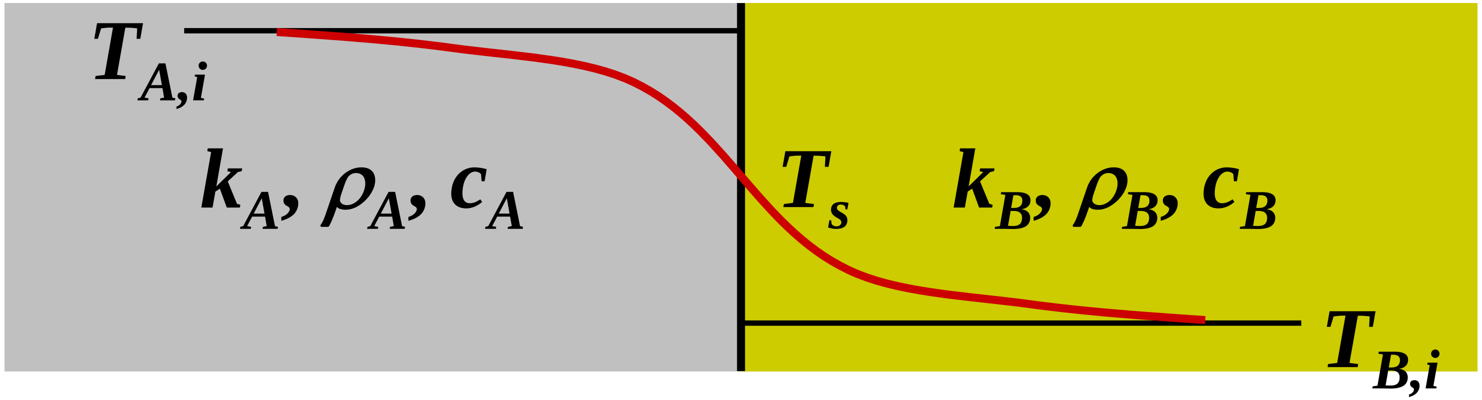
$$\boxed{\theta = \frac{T - T_i}{T_s - T_i} \qquad \eta = \frac{x}{2\sqrt{\alpha t}}}$$

$$\theta(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}du$$

$$\rightarrow \frac{d\theta}{d\eta} = -\frac{2}{\sqrt{\pi}}e^{-\eta^2} \rightarrow \theta'(0) = -\frac{2}{\sqrt{\pi}}$$

$$q_s'' = -\frac{k\left(T_s - T_i\right)}{2\sqrt{\alpha t}}\left(-\frac{2}{\sqrt{\pi}}\right) = \frac{k\left(T_s - T_i\right)}{\sqrt{\pi \alpha t}}$$

Interfacial contact between two semi-infinite solids



$$T_A: \quad T_A(x,t) = A_1 + A_2 \mathrm{erf}(\eta)$$

$$T_B: \quad T_B(x,t) = B_1 + B_2 \mathrm{erf}(\eta)$$

Boundary and interfacial conditions

$$T_A(-\infty,t) = T_{A,i}, T_B(\infty,t) = T_{B,i}, \quad T_s = T_A(0,t) = T_B(0,t)$$

$$q_s'' = -k_A \left.\frac{\partial T_A}{\partial x}\right|_{x=0} = -k_B \left.\frac{\partial T_B}{\partial x}\right|_{x=0}$$

$$T_s = \frac{(k\rho c)_A^{1/2} T_{A,i} + (k\rho c)_B^{1/2} T_{B,i}}{(k\rho c)_A^{1/2} + (k\rho c)_B^{1/2}}$$

The interface temperature is not function of time.

Ex) A: man, B: wood (pine) or steel (AISI 1302)

Assume $T_A = 36^\circ\mathrm{C},\ T_B = 10^\circ\mathrm{C}$

$$k_A = 628\ \mathrm{W/m\cdot K},\ \rho_A = 993\ \mathrm{kg/m^3},\ c_A = 4718\ \mathrm{J/kg\cdot K}$$

wood: $k_B = 0.12\ \mathrm{W/m\cdot K},\ \rho_B = 510\ \mathrm{kg/m^3},\ c_B = 1380\ \mathrm{J/kg\cdot K}$

$$T_s = 35.9^\circ\mathrm{C}$$

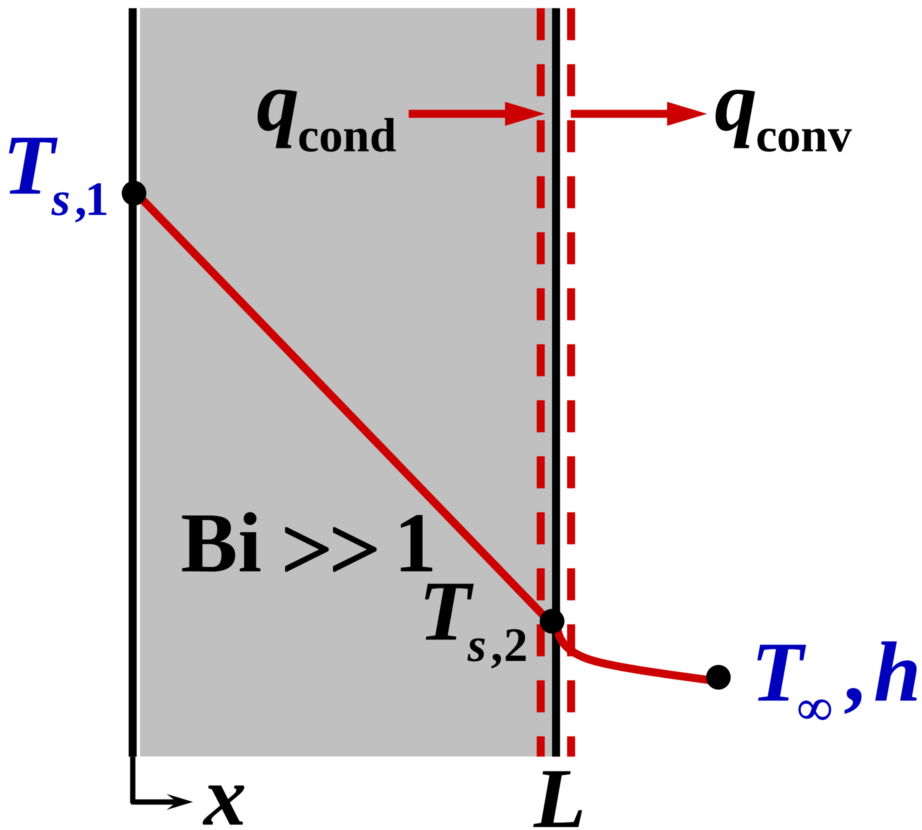
steel: $k_B = 15.1\ \mathrm{W/m\cdot K},\ \rho_B = 8055\ \mathrm{kg/m^3},\ c_B = 480\ \mathrm{J/kg\cdot K}$

$$T_s = 33.9^\circ\mathrm{C}$$

Objects with Constant Surface Temperatures

Utilization of solution to convection boundary condition

$$\frac{kA}{L}\left(T_{s,1}-T_{s,2}\right)=hA\left(T_{s,2}-T_{\infty}\right)$$



$$\frac{T_{s,1}-T_{s,2}}{T_{s,2}-T_{\infty}}=\frac{L/kA}{1/hA}$$

$$=\frac{hL}{k}\equiv \mathrm{Bi}$$

As $\mathrm{Bi}\rightarrow\infty,\ T_{s,2}\rightarrow T_{\infty}$

- **Semi-Infinite Solid**

$$q_s'' = \frac{k\left(T_s - T_i\right)}{\sqrt{\pi \alpha t}}$$

In dimensionless form

$$q^* \equiv \frac{q_s'' L_c}{k\left(T_s - T_i\right)} = \frac{k\left(T_s - T_i\right)}{\sqrt{\pi \alpha t}} \frac{L_c}{k\left(T_s - T_i\right)} = \frac{L_c}{\sqrt{\pi \alpha t}}$$

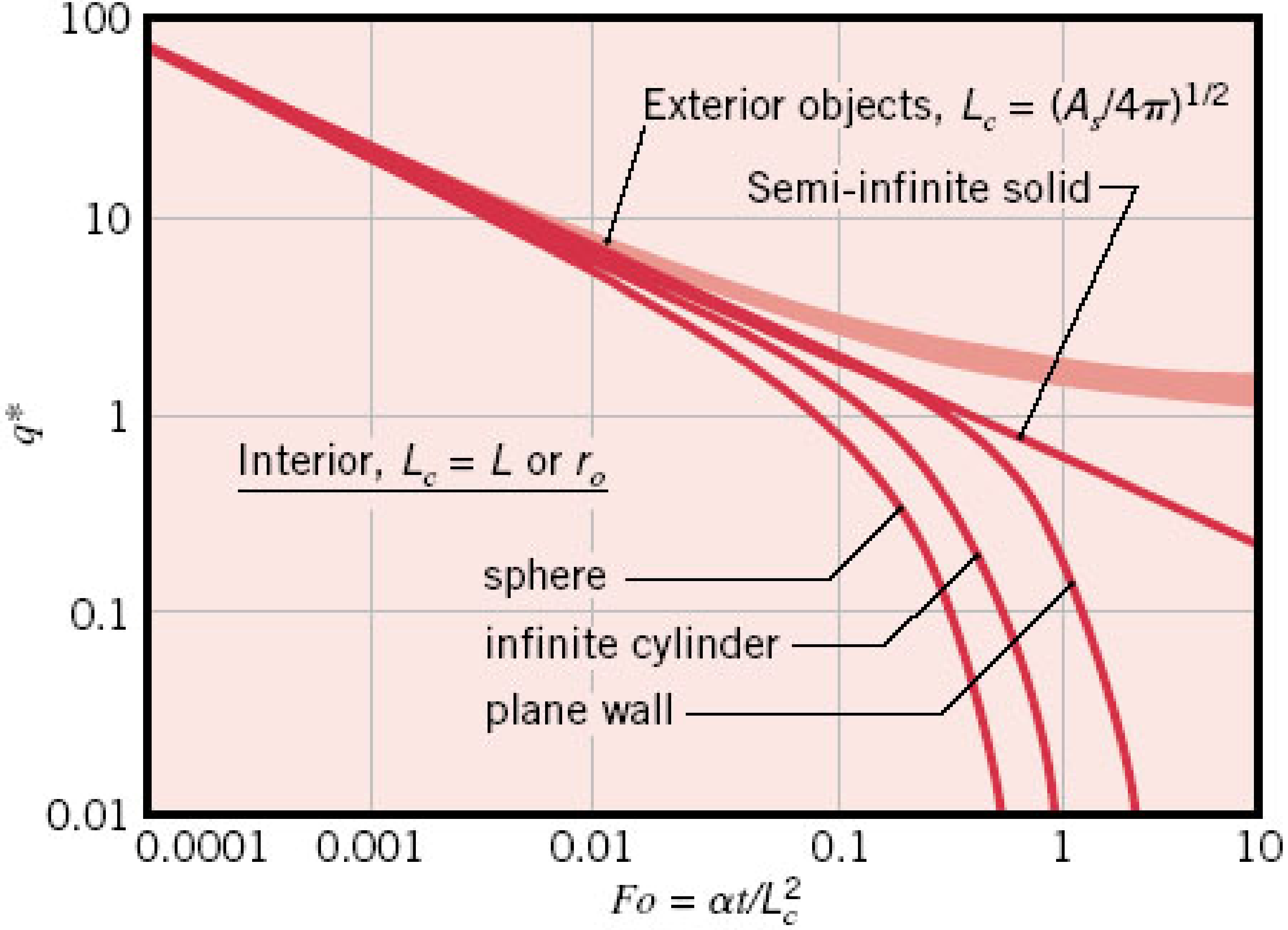
$$\mathrm{Fo} = \frac{\alpha t}{L_c^2} \rightarrow \alpha t = L_c^2 \mathrm{Fo} \qquad q^* = \frac{1}{\sqrt{\pi \mathrm{Fo}}}$$

- **Plane wall, Cylinder, and Sphere for** $\mathrm{Bi} \rightarrow \infty$

Summary of transient heat transfer results for constant surface temperature cases

Geometry	Length Scale, L_c	$q^*(Fo)$ Exact Solutions	$q^*(Fo)$ Approximate Solutions $Fo < 0.2$	$q^*(Fo)$ Approximate Solutions $Fo \geq 0.2$	Maximum Error (%)
Semi-infinite	L (arbitrary)	$\frac{1}{\sqrt{\pi Fo}}$	Use exact solution.		none
Interior Cases					
Plane wall of thickness $2L$	L	$2\sum_{n=1}^{\infty} \exp(-\zeta_n^2 Fo) \quad \zeta_n = (n - \frac{1}{2})\pi$	$\frac{1}{\sqrt{\pi Fo}}$	$2 \exp(-\zeta_1^2 Fo) \quad \zeta_1 = \pi/2$	1.7
Infinite cylinder	r_o	$2\sum_{n=1}^{\infty} \exp(-\zeta_n^2 Fo) \quad J_0(\zeta_n) = 0$	$\frac{1}{\sqrt{\pi Fo}} - 0.50 - 0.65\, Fo$	$2 \exp(-\zeta_1^2 Fo) \quad \zeta_1 = 2.4050$	0.8
Sphere	r_o	$2\sum_{n=1}^{\infty} \exp(-\zeta_n^2 Fo) \quad \zeta_n = n\pi$	$\frac{1}{\sqrt{\pi Fo}} - 1$	$2 \exp(-\zeta_1^2 Fo) \quad \zeta_1 = \pi$	6.3
Exterior Cases					
Sphere	r_o	$\frac{1}{\sqrt{\pi Fo}} + 1$	Use exact solution.		none
Various shapes (Table 4.1, cases 12–15)	$(A_s/4\pi)^{1/2}$	none	$\frac{1}{\sqrt{\pi Fo}} + q_s^*, \quad q_s^*$ from Table 4.1		7.1

[a] $q^* \equiv q_s'' L_c / k(T_s - T_i)$ and $Fo \equiv \alpha t / L_c^2$ where L_c is the length scale given in the table, T_s is the object surface temperature, and T_i is (a) the initial object temperature for the interior cases and (b) the temperature of the infinite medium for the exterior cases.



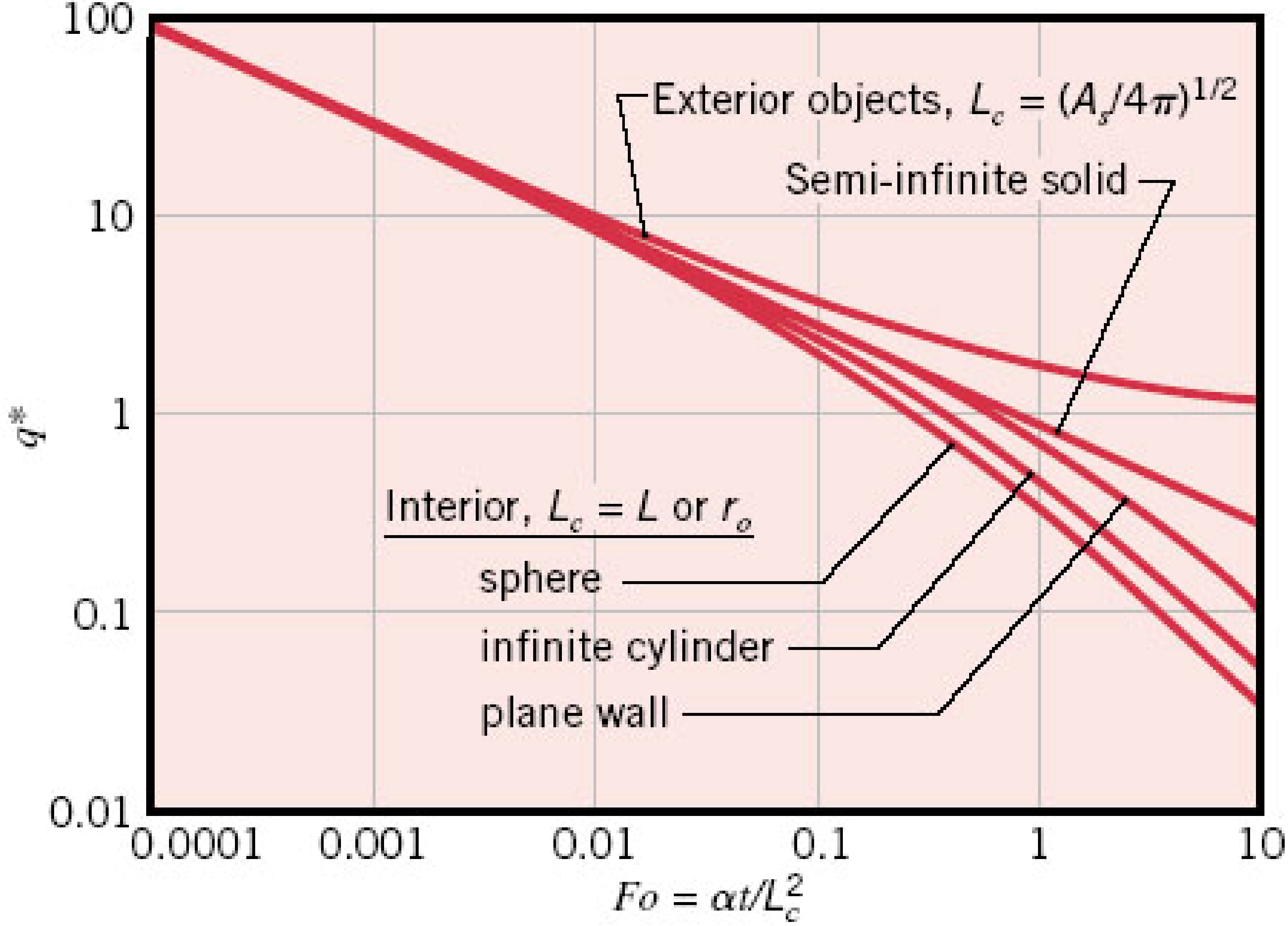
100
10
1
0.1
0.01
q^*
Exterior objects, $L_c = (A_s/4\pi)^{1/2}$
Semi-infinite solid
Interior, $L_c = L$ or r_o
sphere
infinite cylinder
plane wall
0.0001
0.001
0.01
0.1
1
10
$Fo = \alpha t/L_c^2$

Objects with Constant Surface Heat Fluxes

Summary of transient heat transfer results for constant surface heat flux cases

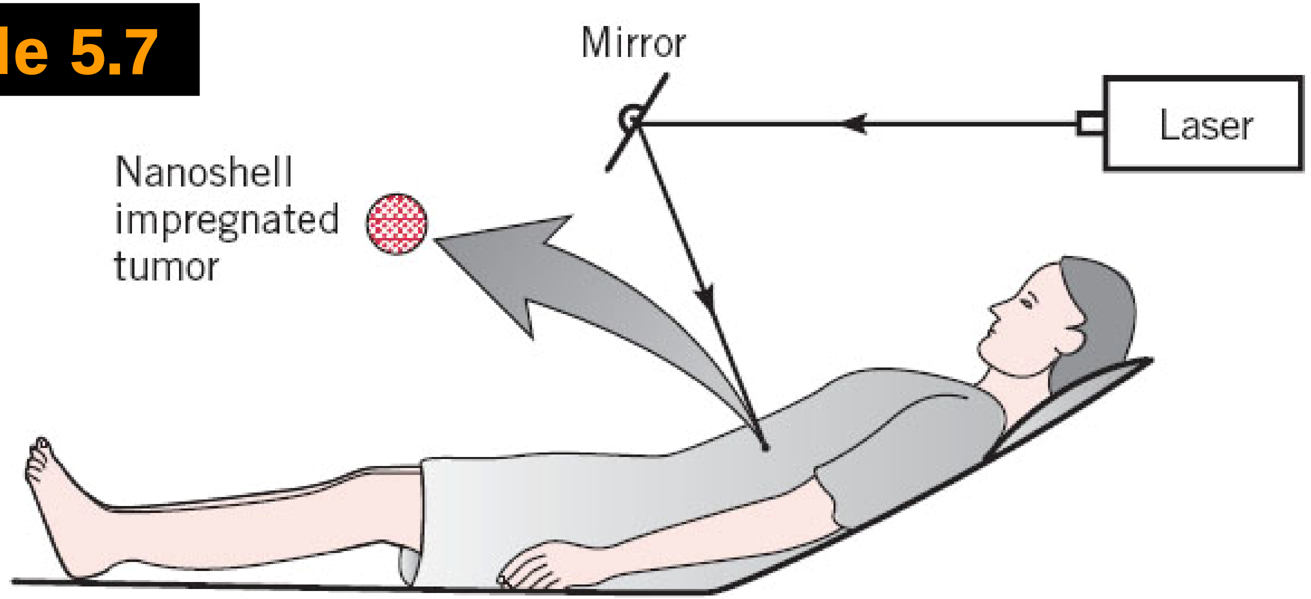
Geometry	Length Scale, L_c	$q^*(Fo)$ Exact Solutions		Approximate Solutions $Fo < 0.2$	Approximate Solutions $Fo \geq 0.2$	Maximum Error (%)
Semi-infinite	L (arbitrary)	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}}$		Use exact solution.		none
Interior Cases						
Plane wall of thickness $2L$	L	$\left[Fo + \frac{1}{3} - 2\sum_{n=1}^{\infty}\frac{\exp(-\zeta_n^2 Fo)}{\zeta_n^2}\right]^{-1}$	$\zeta_n = n\pi$	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}}$	$\left[Fo + \frac{1}{3}\right]^{-1}$	5.3
Infinite cylinder	r_o	$\left[2Fo + \frac{1}{4} - 2\sum_{n=1}^{\infty}\frac{\exp(-\zeta_n^2 Fo)}{\zeta_n^2}\right]^{-1}$	$J_1(\zeta_n) = 0$	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} - \frac{\pi}{8}$	$\left[2Fo + \frac{1}{4}\right]^{-1}$	2.1
Sphere	r_o	$\left[3Fo + \frac{1}{5} - 2\sum_{n=1}^{\infty}\frac{\exp(-\zeta_n^2 Fo)}{\zeta_n^2}\right]^{-1}$	$\tan(\zeta_n) = \zeta_n$	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} - \frac{\pi}{4}$	$\left[3Fo + \frac{1}{5}\right]^{-1}$	4.5
Exterior Cases						
Sphere	r_o	$[1 - \exp(Fo)\mathrm{erfc}(Fo^{1/2})]^{-1}$		$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} + \frac{\pi}{4}$	$\frac{0.77}{\sqrt{Fo}} + 1$	3.2
Various shapes (Table 4.1, cases 12–15)	$(A_s/4\pi)^{1/2}$	none		$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} + \frac{\pi}{4}$	$\frac{0.77}{\sqrt{Fo}} + q_{ss}^*$	unknown

[a] $q^* \equiv q_s'' L_c / k(T_s - T_i)$ and $Fo \equiv \alpha t / L_c^2$, where L_c is the length scale given in the table, T_s is the object surface temperature, and T_i is (a) the initial object temperature for the interior cases and (b) the temperature of the infinite medium for the exterior cases.



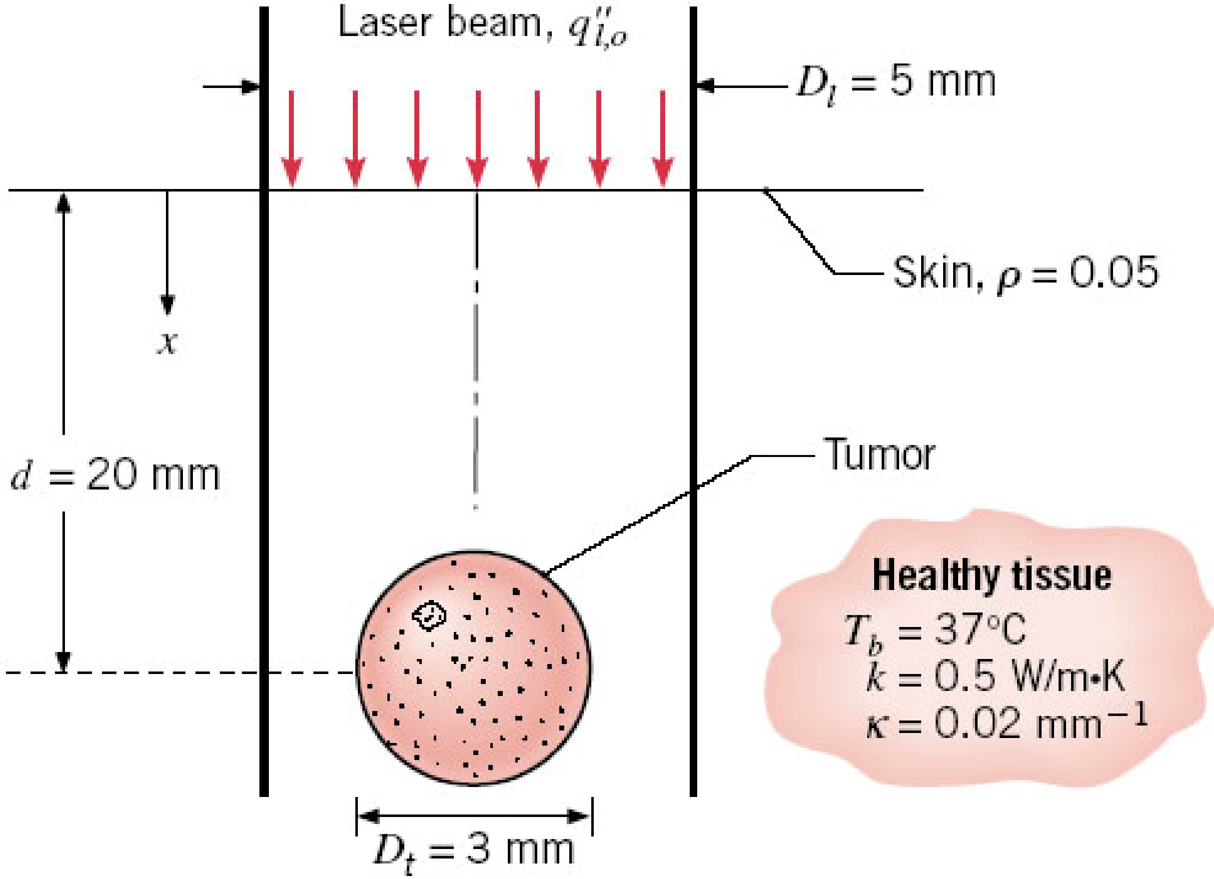
100
10
1
0.1
0.01
q^*
0.0001
0.001
0.01
0.1
1
10
$Fo = \alpha t/L_c^2$
Exterior objects, $L_c = (A_s/4\pi)^{1/2}$
Semi-infinite solid
Interior, $L_c = L$ or r_o
sphere
infinite cylinder
plane wall

Example 5.7



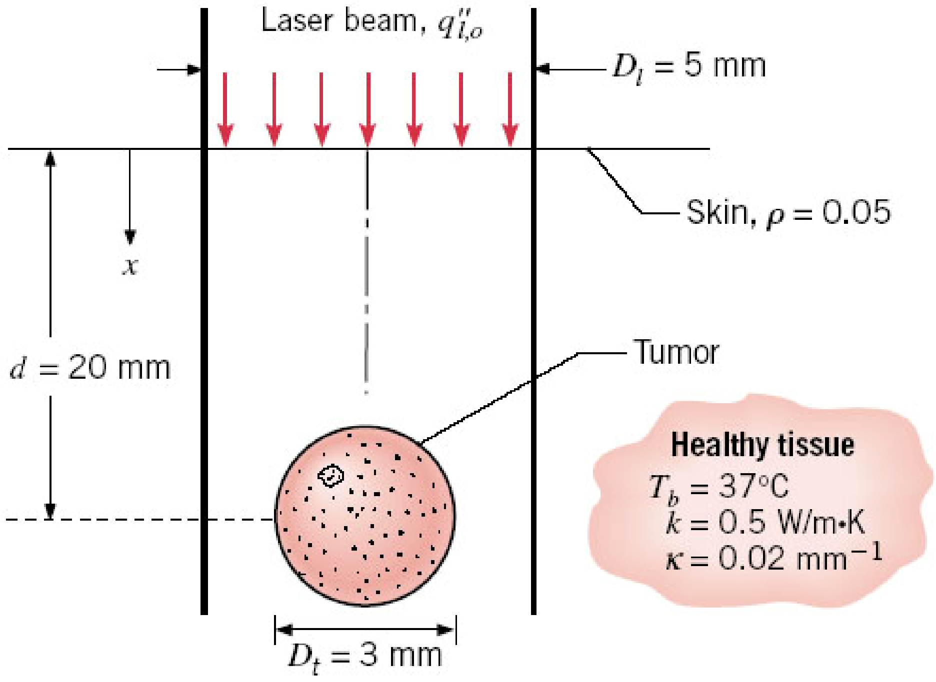
Cancer treatment by laser heating using nanoshells

1) Prior to treatment, antibodies are attached to the nanoscale particles.
2) Doped particles are then injected into the patient's bloodstream and distributed throughout the body.
3) The antibodies are attracted to malignant sites, and therefore carry and adhere the nanoshells only to cancerous tissue.
4) A laser beam penetrates through the tissue between the skin and the cancer, is absorbed by the nanoshells, and, in turn, heats and destroys the cancerous tissues.



Known:

1) **Size of a small sphere**
2) **Thermal conductivity, reflectivity, and extinction coefficient of tissue**
3) **Depth of sphere below the surface of the skin**



laser heat flux

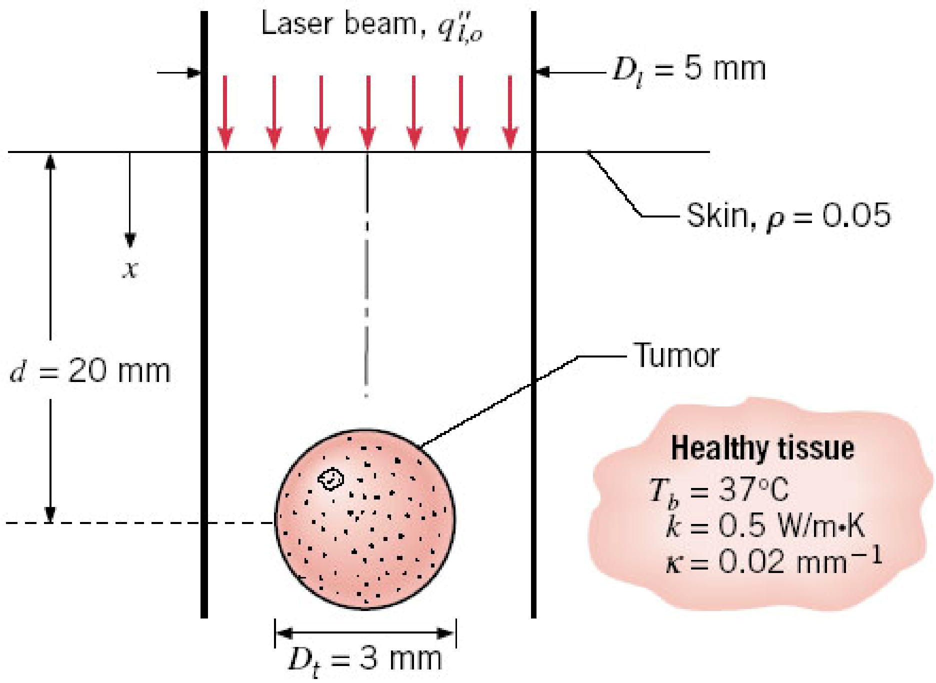
$$q''_l(x) = q''_{l,o}(1-\rho)e^{-\kappa x}$$

$$\rho = v_f^{-1} = 989.1\ \mathrm{kg/m^3},$$

$$c_p = 4180\ \mathrm{J/kg\cdot K}$$

Find:

1) **Heat transfer rate from the tumor to the surrounding healthy tissue for a steady-state treatment temperature of $T_{t,ss}$ = 55ºC at the surface of the tumor.**
2) **Laser power needed to sustain the tumor surface temperature at $T_{t,ss}$ = 55ºC.**
3) **Time for the tumor to reach T_t = 52ºC when heat transfer to the surrounding tissue is neglected. Water property can be used.**
4) **Time for the tumor to reach T_t = 52ºC when heat transfer to the surrounding is considered and the thermal mass of the tumor is neglected.**



laser heat flux

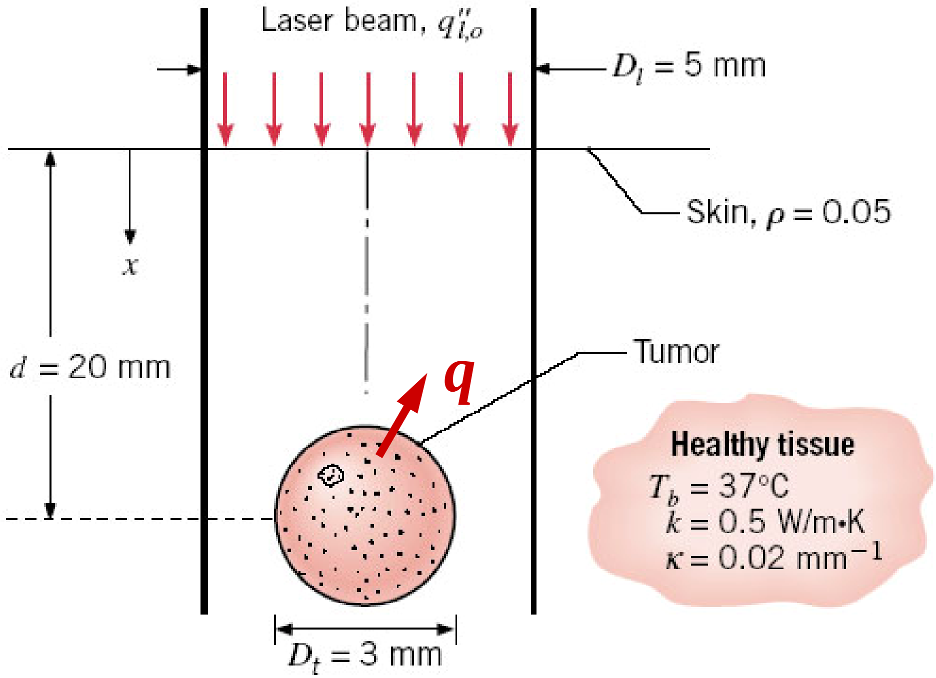
$$q''_l(x) = q''_{l,o}(1-\rho)e^{-\kappa x}$$

$$\rho = v_f^{-1} = 989.1 \text{ kg/m}^3,$$

$$c_p = 4180 \text{ J/kg} \cdot \text{K}$$

Assumptions:

1) **1D conduction in the radial direction.**
2) **Constant properties.**
3) **Healthy tissue can be treated as an infinite medium.**
4) **The treated tumor absorbs all irradiation incident from the laser.**
5) **Lumped capacitance behavior for the tumor.**
6) **Neglect potential nanoscale heat transfer effects.**
7) **Neglect the effect of perfusion.**



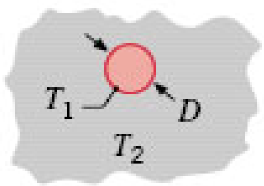
$$q = q_{ss}^{*}\frac{kA_s}{L_c}(T_1 - T_2)$$

$$L_c = \left(\frac{A_s}{4\pi}\right)^{1/2} = \left(\frac{\pi D_t^2}{4\pi}\right)^{1/2} = \frac{D_t}{2}$$

$$q = 1\cdot\frac{k\pi D_t^2}{D_t/2}\left(T_{t,ss} - T_b\right)$$

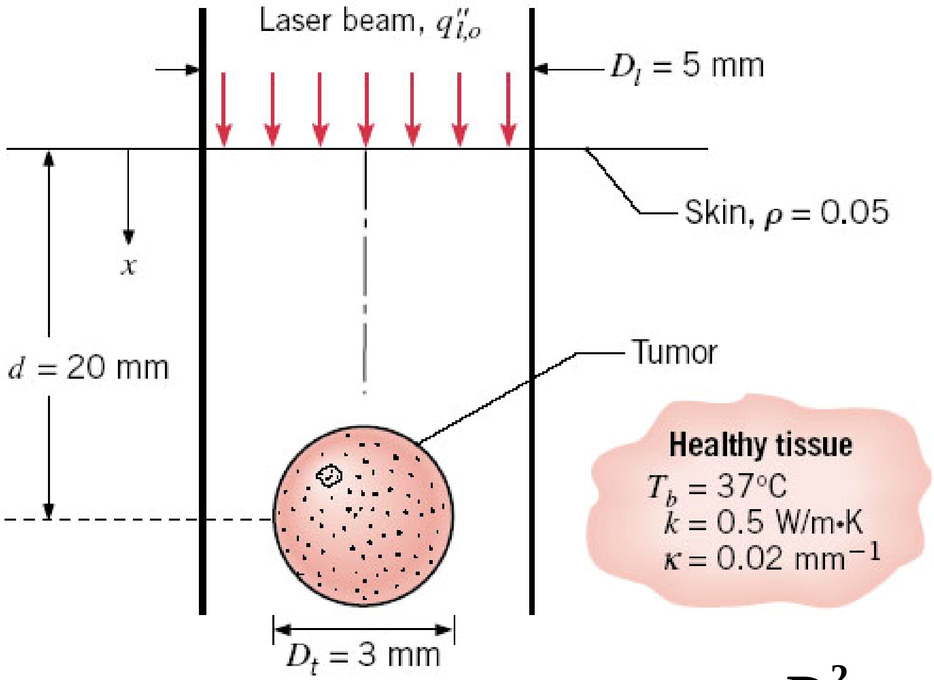
1. Steady-state heat loss q from the tumor (Case 12 of Table 4.1)

(*b*) **Dimensionless conduction heat rates** [$q = q_{ss}^{*}\,kA_s(T_1 - T_2)/L_c$; $L_c \equiv (A_s/4\pi)^{1/2}$]

System	Schematic	Active Area, A_s	q_{ss}^{*}
Case 12 Isothermal sphere of diameter D and temperature T_1 in an infinite medium of temperature T_2		πD^2	1

$$q = 2\pi k D_t\left(T_{t,ss} - T_b\right) = 2\pi \times 0.5\,\mathrm{W/m\cdot K} \times 3\times10^{-3}\,\mathrm{m}\times(55-37)\,^{\circ}\mathrm{C}$$

$$= 0.170\,\mathrm{W}$$



laser heat flux

$$q''_l(x) = q''_{l,o}\left(1-\rho\right)e^{-\kappa x}$$

projected area of the tumor:

$$A_p = \frac{\pi D_t^2}{4}$$

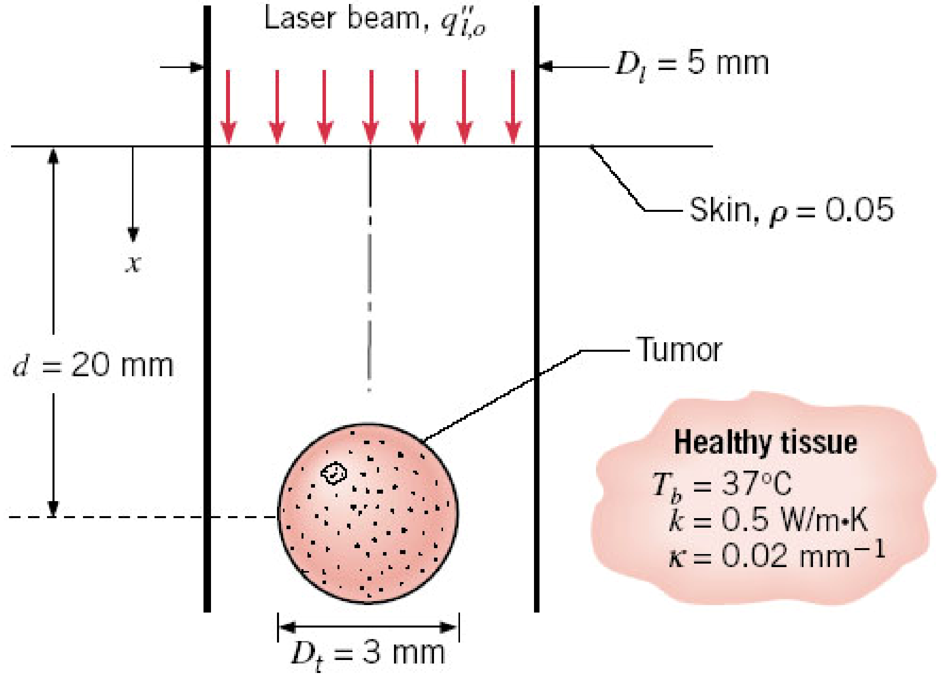
2. Laser power P_l, $\quad P_l = q''_{l,o}\dfrac{\pi D_l^2}{4}$

Energy balance : heat transfer rate from tumor = absorbed laser energy

$$q = 0.170\,\mathrm{W} \approx q''_l(x=d)\frac{\pi D_t^2}{4} = q''_{l,o}\left(1-\rho\right)e^{-\kappa d}\frac{\pi D_t^2}{4}$$

$$P_l = q''_{l,o}\frac{\pi D_l^2}{4} = \frac{q}{\left(1-\rho\right)e^{-\kappa d}\pi D_t^2/4}\frac{\pi D_l^2}{4} = \frac{q D_l^2 e^{\kappa d}}{\left(1-\rho\right)D_t^2}$$

$$= \frac{0.170\mathrm{W}\times(5\times10^{-3}\mathrm{m})^2\times e^{(0.02\mathrm{mm}^{-1}\times 20\mathrm{mm})}}{(1-0.05)\times(3\times10^{-3}\mathrm{m})^2} = 0.74\mathrm{W}$$



laser heat flux

$$q''_l(x) = q''_{l,o}\left(1-\rho\right)e^{-\kappa x}$$

$$\rho = v_f^{-1} = 989.1\ \mathrm{kg/m^3},$$

$$c = 4180\ \mathrm{J/kg\cdot K}$$

3. Time for the tumor to reach T_t = 52°C when heat transfer to the surrounding tissue is neglected.

$$\frac{q''_l(x=d)\pi D_t^2}{4} = q = \rho V c\frac{dT}{dt}, \qquad \frac{q}{\rho V c}\int_{t=0}^{t} dt = \int_{T_b}^{T_t} dT$$

$$t = \frac{\rho V c}{q}(T_t - T_b)$$

$$= \frac{989.1\ \mathrm{kg/m^3}\times(\pi/6)\times(3\times10^{-3}\ \mathrm{m})^3\times4180\ \mathrm{J/kg\cdot K}}{0.170\mathrm{W}}\times\left(55°\mathrm{C}-37°\mathrm{C}\right)$$

$$= 5.16\,\mathrm{s}$$

4. Time for the tumor to reach $T_t = 52^{\circ}\mathrm{C}$ when heat transfer to the surrounding is considered and thermal mass of the tumor is neglected.

Heat transfer between a sphere and an exterior infinite medium subjected to constant heat flux

$$q^* = \frac{1}{1-\exp(\mathrm{Fo})\mathrm{erfc}(\mathrm{Fo}^{1/2})}, \qquad q^* \equiv \frac{q''_s L_c}{k\left(T_s - T_i\right)}$$

$$q^* \equiv \frac{q''_s L_c}{k\left(T_t - T_b\right)} = \frac{q''_s A_s L_c}{A_s k\left(T_t - T_b\right)} = \frac{q}{\pi D_t^2 k\left(T_t - T_b\right)}\frac{D_t}{2} = \frac{q}{2\pi k D_t\left(T_t - T_b\right)}$$

$$\frac{q}{2\pi k D_t\left(T_t - T_b\right)} = \frac{1}{1-\exp(\mathrm{Fo})\mathrm{erfc}(\mathrm{Fo}^{1/2})}$$

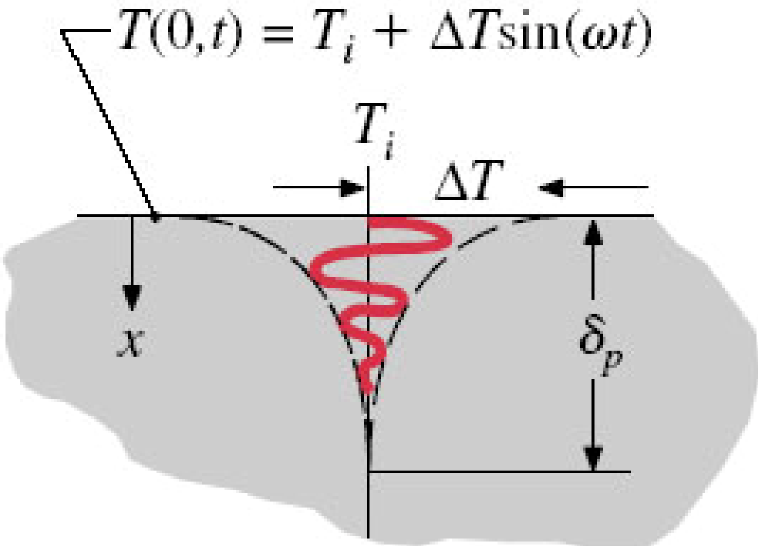
By trial and error, $\mathrm{Fo} = 10.3 = \dfrac{\alpha t}{L_c^2} = \dfrac{\alpha t}{\left(D_t/2\right)^2} = \dfrac{4\alpha t}{D_t^2} \rightarrow t = \dfrac{D_t^2}{4\alpha}\mathrm{Fo}$

$$t = \frac{D_t^2}{4\alpha}\mathrm{Fo} = \frac{\rho c_p D_t^2}{4k}\mathrm{Fo}$$

$$= \frac{989.1\ \mathrm{kg/m^3} \times 4180\ \mathrm{J/kg\cdot K} \times \left(0.003\ \mathrm{m}\right)^2}{4 \times 0.50\ \mathrm{W/m\cdot K}} \times 10.3 = 192\ \mathrm{s}$$

Periodic Heating

- **Oscillating surface temperature**



thermal penetration depth

(reduction of temperature amplitude by 90% relative to that of surface)

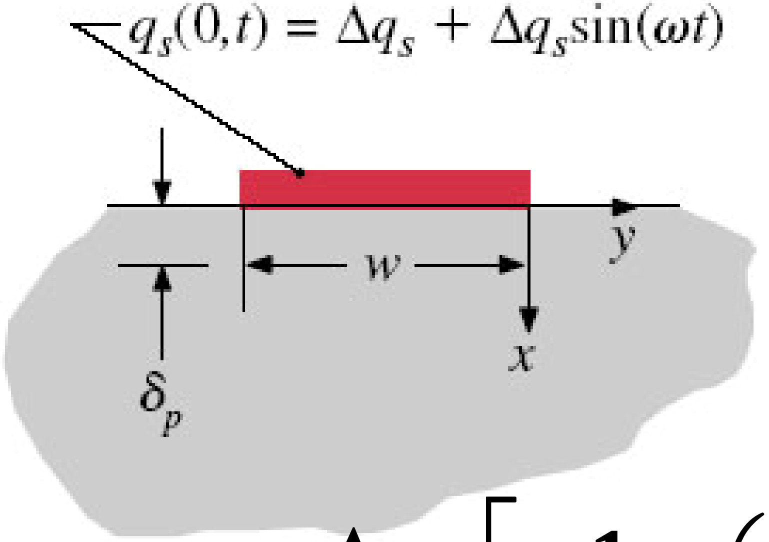
$$\delta_p \equiv 4\sqrt{\frac{\alpha}{\omega}}$$

Quasi-steady state temperature distribution

$$\frac{T(x,y)-T_i}{\Delta T} = \exp\left[-x\sqrt{\frac{\omega}{2\alpha}}\right]\sin\left[\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right]$$

Surface heat flux $\quad q_s''(t) = k\Delta T\sqrt{\frac{\omega}{\alpha}}\sin\left(\omega t + \frac{\pi}{4}\right)$

- **Sinusoidal heating by a strip**



$$L \gg w$$

$$\delta_p \approx \sqrt{\frac{\alpha}{\omega}}$$

$$\Delta T \approx \frac{\Delta q_s}{L\pi k}\left[-\frac{1}{2}\ln\left(\frac{\omega}{2}\right) - \ln\left(\frac{w^2}{4\alpha}\right) + C_1\right]$$

$$= \frac{\Delta q_s}{L\pi k}\left[-\frac{1}{2}\ln\left(\frac{\omega}{2}\right) + C_2\right]$$

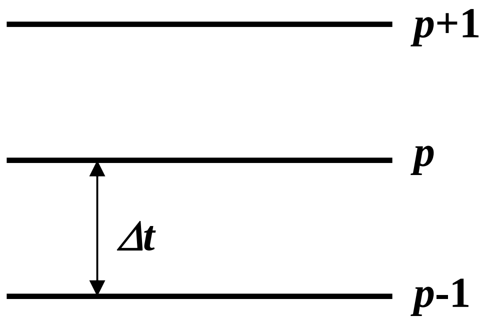
C_1: **depends on thermal contact resistance at interface between heater strip and underlying material**

Numerical Method

Finite Difference Method

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,y,t+\Delta t)=T(x,y,t)+\left.\frac{\partial T}{\partial t}\right|_{x,y,t}(\Delta t)+\frac{1}{2}\frac{\partial^2 T}{\partial t^2}(\Delta t)^2+\cdots$$

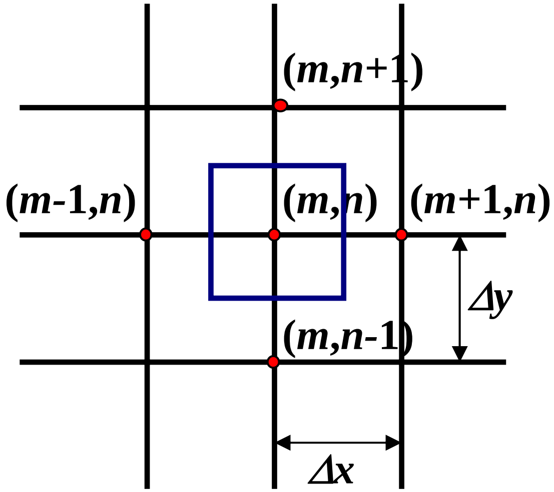


in time

$$\frac{\partial T}{\partial t}=\frac{T(x,y,t+\Delta t)-T(x,y,t)}{\Delta t}+O(\Delta t)\approx\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\Delta t}$$

truncation error: $O(\Delta t)$

first order accuracy in time



in space

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

$$=\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\alpha\Delta t}$$

Explicit Method (Euler Method) : forward difference

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}=\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\alpha\Delta t}$$

$$\frac{T_{m+1,n}^{p}+T_{m-1,n}^{p}-2T_{m,n}^{p}}{(\Delta x)^2}+\frac{T_{m,n+1}^{p}+T_{m,n-1}^{p}-2T_{m,n}^{p}}{(\Delta y)^2}=\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\alpha\Delta t}$$

If $\Delta x=\Delta y,\ \ T_{m,n}^{p+1}=\mathrm{Fo}\left(T_{m+1,n}^{p}+T_{m-1,n}^{p}+T_{m,n+1}^{p}+T_{m,n-1}^{p}\right)+\left(1-4\mathrm{Fo}\right)T_{m,n}^{p}$

stability criterion: $\left(1-4\mathrm{Fo}\right)\geq 0$ or $\mathrm{Fo}=\dfrac{\alpha\Delta t}{(\Delta x)^2}\leq\dfrac{1}{4}$ or $\Delta t\leq\dfrac{(\Delta x)^2}{4\alpha}$

If the system is one-dimensional in x,

$$T_{m}^{p+1}=\mathrm{Fo}\left(T_{m+1}^{p}+T_{m-1}^{p}\right)+\left(1-2\mathrm{Fo}\right)T_{m}^{p}$$

stability criterion: $\left(1-2\mathrm{Fo}\right)\geq 0$ or $\mathrm{Fo}=\dfrac{\alpha\Delta t}{(\Delta x)^2}\leq\dfrac{1}{2}$ or $\Delta t\leq\dfrac{(\Delta x)^2}{2\alpha}$

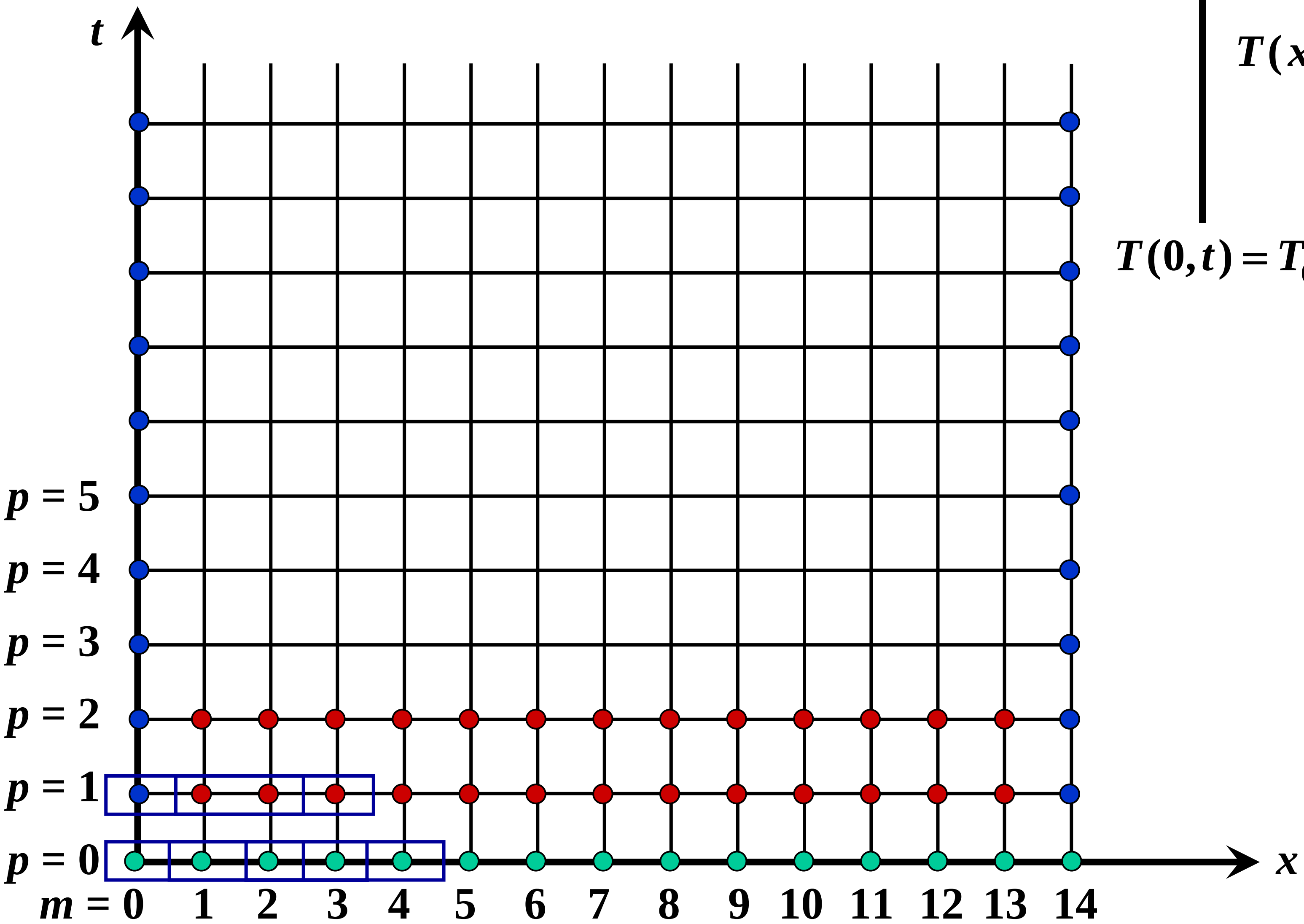
one-dimensional in x,

$$T_m^{p+1} = \mathrm{Fo}\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + \left(1 - 2\mathrm{Fo}\right)T_m^{p}$$

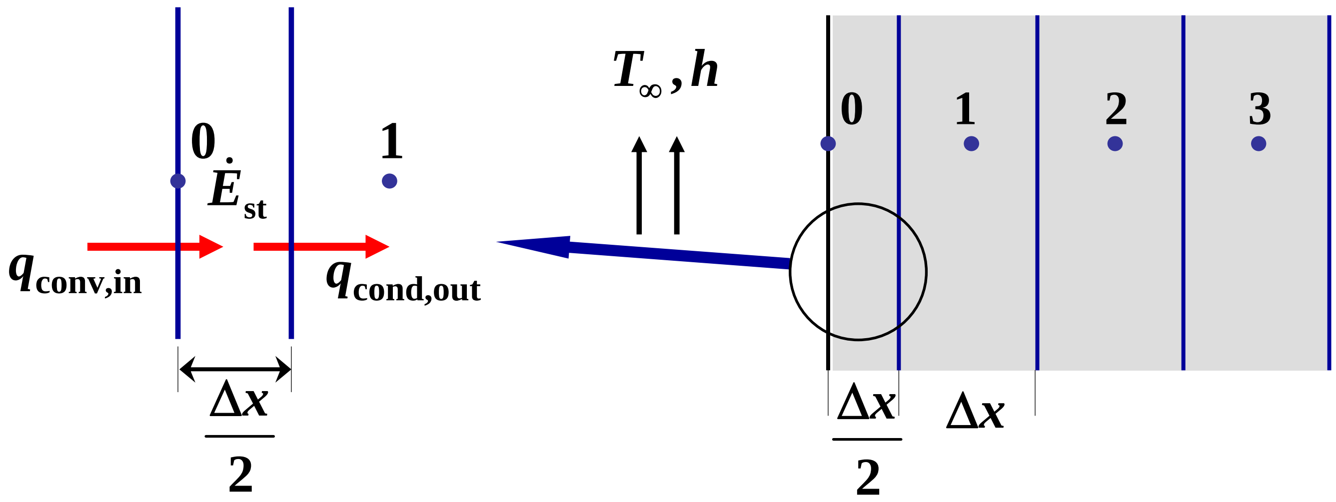
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = T_i$$

$$T(0,t) = T_0 \qquad T(L,t) = T_L$$



Boundary node subjected to convection



$$hA(T_\infty - T_0^p) - kA\frac{T_0^p - T_1^p}{\Delta x} = \rho c A \frac{\Delta x}{2}\frac{T_0^{p+1} - T_0^p}{\Delta t}$$

$$T_0^{p+1} = \frac{2h\Delta t}{\rho c \Delta x}(T_\infty - T_0^p) + \frac{2\alpha\Delta t}{(\Delta x)^2}(T_1^p - T_0^p) + T_0^p$$

See Table 5.3 (p. 306)

$$\frac{2h\Delta t}{\rho c \Delta x} = \frac{2h\Delta x}{k}\cdot\frac{\alpha\Delta t}{(\Delta x)^2} = 2\mathrm{BiFo}$$

$$T_0^{p+1} = 2\mathrm{Fo}\left(T_1^p + \mathrm{Bi}T_\infty\right) + \left(1 - 2\mathrm{Fo} - 2\mathrm{BiFo}\right)T_0^p$$

stability criterion: $\left(1 - 2\mathrm{Fo} - 2\mathrm{BiFo}\right) \geq 0$ or $\mathrm{Fo}\left(1 + \mathrm{Bi}\right) \leq 1/2$

Example 5.9

Steady operation

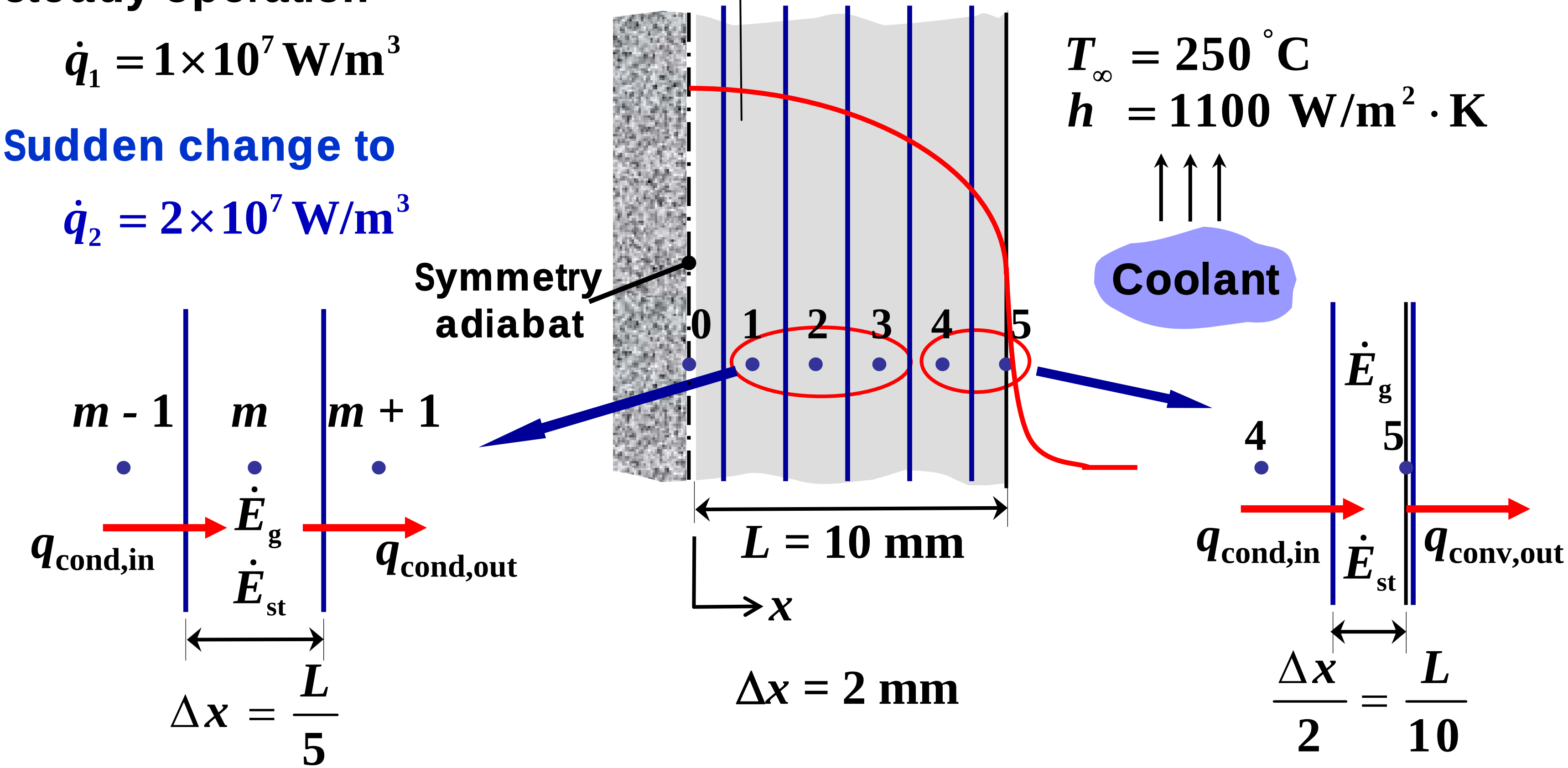
$\dot{q}_1 = 1\times10^7\ \mathrm{W/m^3}$

Sudden change to

$\dot{q}_2 = 2\times10^7\ \mathrm{W/m^3}$

Fuel element: $\alpha = 5\times10^{-6}\ \mathrm{W/m^3}$, $k = 30\mathrm{W/m\cdot K}$

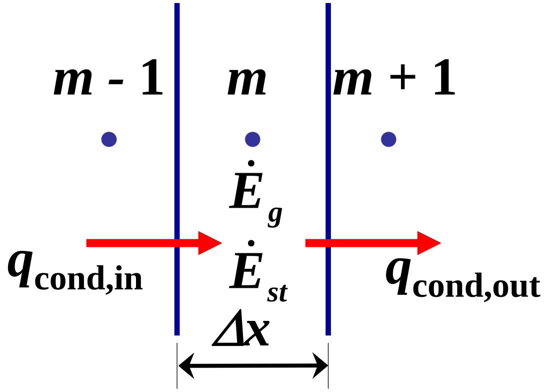
$T_\infty = 250\ ^\circ\mathrm{C}$
$h = 1100\ \mathrm{W/m^2\cdot K}$



Find:

Temperature distribution at 1.5 s after a change in operating power by using the explicit finite difference method

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_{\text{g}} = \dot{E}_{\text{st}}$$



$$kA\frac{T_{m-1}^{p} - T_{m}^{p}}{\Delta x} - kA\frac{T_{m}^{p} - T_{m+1}^{p}}{\Delta x} + \dot{q}A\Delta x$$

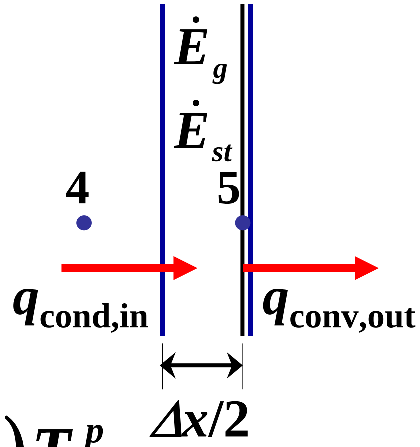
$$= \rho A\Delta x c\frac{T_{m}^{p+1} - T_{m}^{p}}{\Delta t}$$

Thus, $T_{m}^{p+1} = \text{Fo}\left[T_{m-1}^{p} - T_{m+1}^{p} + \frac{\dot{q}(\Delta x)^{2}}{k}\right] + (1 - 2\text{Fo})T_{m}^{p}, \quad m = 1, 2, 3, 4$

For node 0, set $T_{m-1}^{p} = T_{m+1}^{p}, \quad T_{0}^{p+1} = \text{Fo}\left[\frac{\dot{q}(\Delta x)^{2}}{k}\right] + (1 - 2\text{Fo})T_{0}^{p}$

For node 5, $kA\frac{T_{4}^{p} - T_{5}^{p}}{\Delta x} - hA\left(T_{5}^{p} - T_{\infty}\right) + \dot{q}A\frac{\Delta x}{2}$

$$= \rho A\frac{\Delta x}{2}c\frac{T_{5}^{p+1} - T_{5}^{p}}{\Delta t}$$



or $T_{5}^{p+1} = 2\text{Fo}\left[T_{4}^{p} + \text{Bi}T_{\infty} + \frac{\dot{q}(\Delta x)^{2}}{2k}\right] + (1 - 2\text{Fo} - 2\text{BiFo})T_{5}^{p}$

Δt: stability criterion

$$1 - 2\text{Fo} \geq 0,\ \ 1 - 2\text{Fo} - 2\text{BiFo} \geq 0$$

or $\text{Fo} \leq 0.5,\ \ \text{Fo}\left(1 + \text{Bi}\right) \leq 0.5$

$$\text{Bi} = \frac{h\Delta x}{k} = \frac{1100\ \text{W/m}^2 \cdot \text{K}(0.002\ \text{m})}{30\ \text{W/m} \cdot \text{K}} = 0.0733$$

Thus, $\text{Fo} \leq 0.466$

$$\Delta t = \frac{\text{Fo}(\Delta x)^2}{\alpha} \leq \frac{0.466(2 \times 10^{-3}\ \text{m})^2}{5 \times 10^{-6}\ \text{m}^2/\text{s}} \leq 0.373\ \text{s}$$

choose $\Delta t = 0.3\ \text{s}$

Then, $\text{Fo} = \dfrac{5 \times 10^{-6}\ \text{m}^2/\text{s}(0.3\ \text{s})}{(2 \times 10^{-3}\ \text{m})^2} = 0.375$

nodal equations

$$T_0^{p+1} = 0.375(2T_1^p + 2.67) + 0.250T_0^p$$
$$T_1^{p+1} = 0.375(T_0^p + T_2^p + 2.67) + 0.250T_1^p$$
$$T_2^{p+1} = 0.375(T_1^p + T_3^p + 2.67) + 0.250T_2^p$$
$$T_3^{p+1} = 0.375(T_2^p + T_4^p + 2.67) + 0.250T_3^p$$
$$T_4^{p+1} = 0.375(T_3^p + T_5^p + 2.67) + 0.250T_4^p$$
$$T_5^{p+1} = 0.750(T_4^p + 19.67) + 0.250T_5^p$$

Initial distribution: steady-state solution with $\dot{q} = \dot{q}_1 = 1\times 10^7\ \mathrm{W/m^3}$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s,\ \ T_s = T_\infty + \frac{\dot{q}L}{h}$$

$$T_s = T_5 = T_\infty + \frac{\dot{q}L}{h} = 250 + \frac{10^7 \times 0.01}{1100} = 340.91^\circ\mathrm{C}$$

$$T(x) = \frac{10^7 \times 0.01}{2\times 30}\left(1 - \frac{x^2}{L^2}\right) + 340.91 = 16.67\left(1 - \frac{x^2}{L^2}\right) + 340.91^\circ\mathrm{C}$$

$$T(x) = 16.67\left(1 - \frac{x^2}{L^2}\right) + 340.91^\circ\mathrm{C}$$

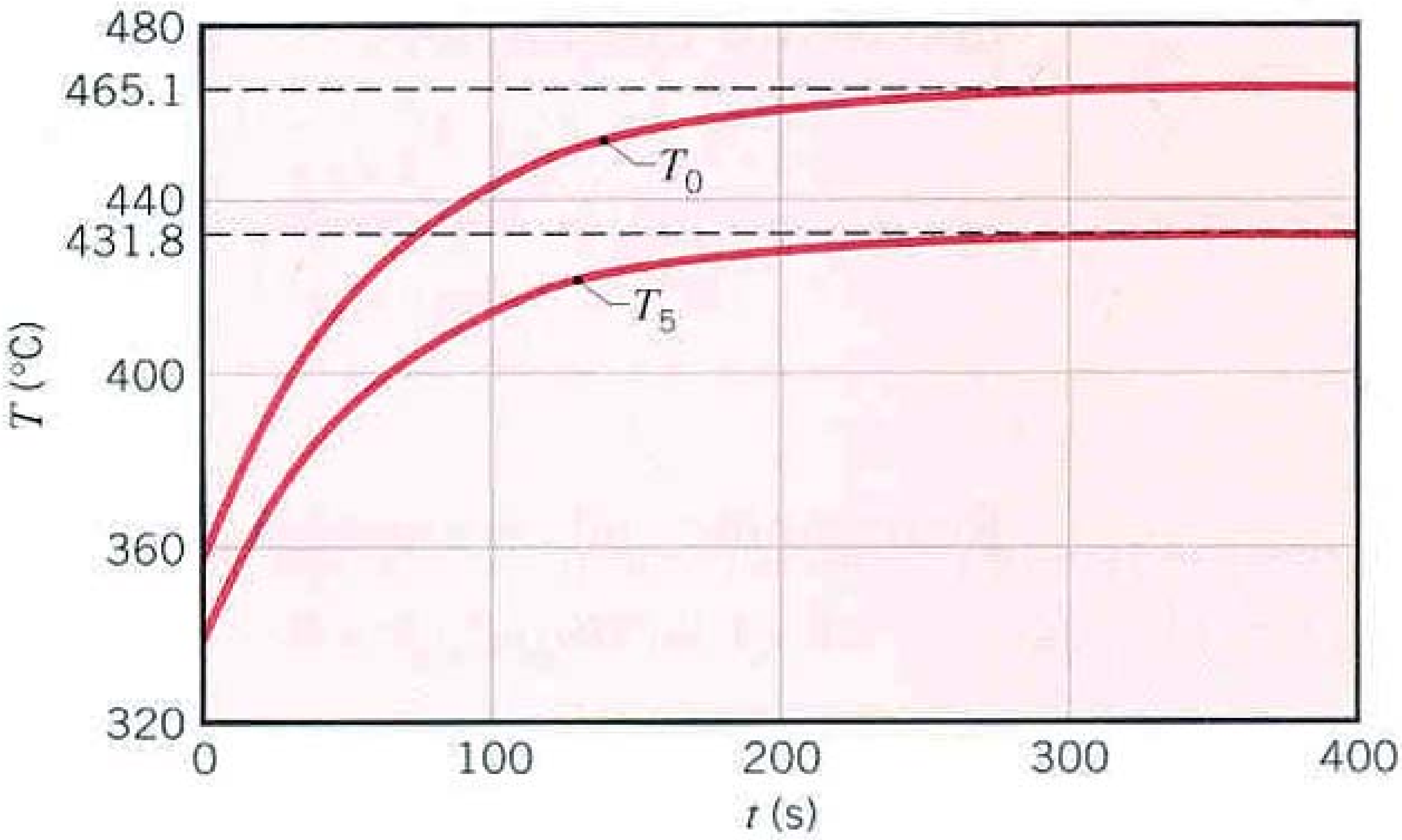
Calculated nodal temperatures

Tabulated Nodal Temperatures

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5
0	0	357.58	356.91	354.91	351.58	346.91	340.91
1	0.3	358.08	357.41	355.41	352.08	347.41	341.41
2	0.6	358.58	357.91	355.91	352.58	347.91	341.88
3	0.9	359.08	358.41	356.41	353.08	348.41	342.35
4	1.2	359.58	358.91	356.91	353.58	348.89	342.82
5	1.5	360.08	359.41	357.41	354.07	349.37	343.27
∞	∞	465.15	463.82	459.82	453.15	443.82	431.82

Comments:

Expanding the finite difference solution, the new steady-state condition may be determined.



Implicit Method (fully) : backward difference

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}=\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\alpha\Delta t}$$

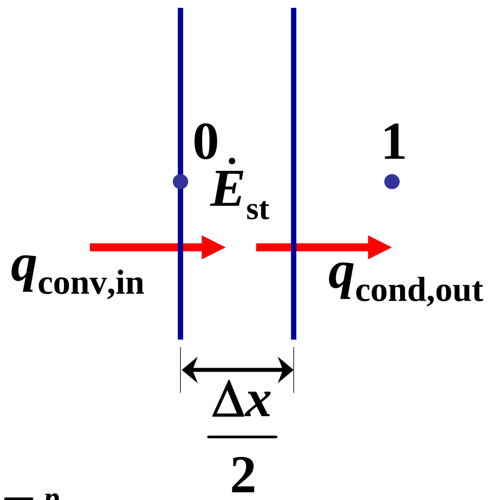
$$\frac{T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}-2T_{m,n}^{p+1}}{(\Delta x)^2}+\frac{T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}-2T_{m,n}^{p+1}}{(\Delta y)^2}=\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\alpha\Delta t}$$

If $\Delta x=\Delta y,\ \ (1+4\mathrm{Fo})T_{m,n}^{p+1}-\mathrm{Fo}\left(T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}+T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)=T_{m,n}^{p}$

stability criterion : no restriction

If the system is one-dimensional in x,

$$(1+2\mathrm{Fo})T_{m}^{p+1}-\mathrm{Fo}\left(T_{m+1}^{p+1}+T_{m-1}^{p+1}\right)=T_{m}^{p}$$



Boundary node subjected to convection

$$(1+2\mathrm{Fo}+2\mathrm{FoBi})T_{0}^{p+1}-2\mathrm{Fo}T_{1}^{p+1}=2\mathrm{FoBi}T_{\infty}+T_{0}^{p}$$

Explicit Method

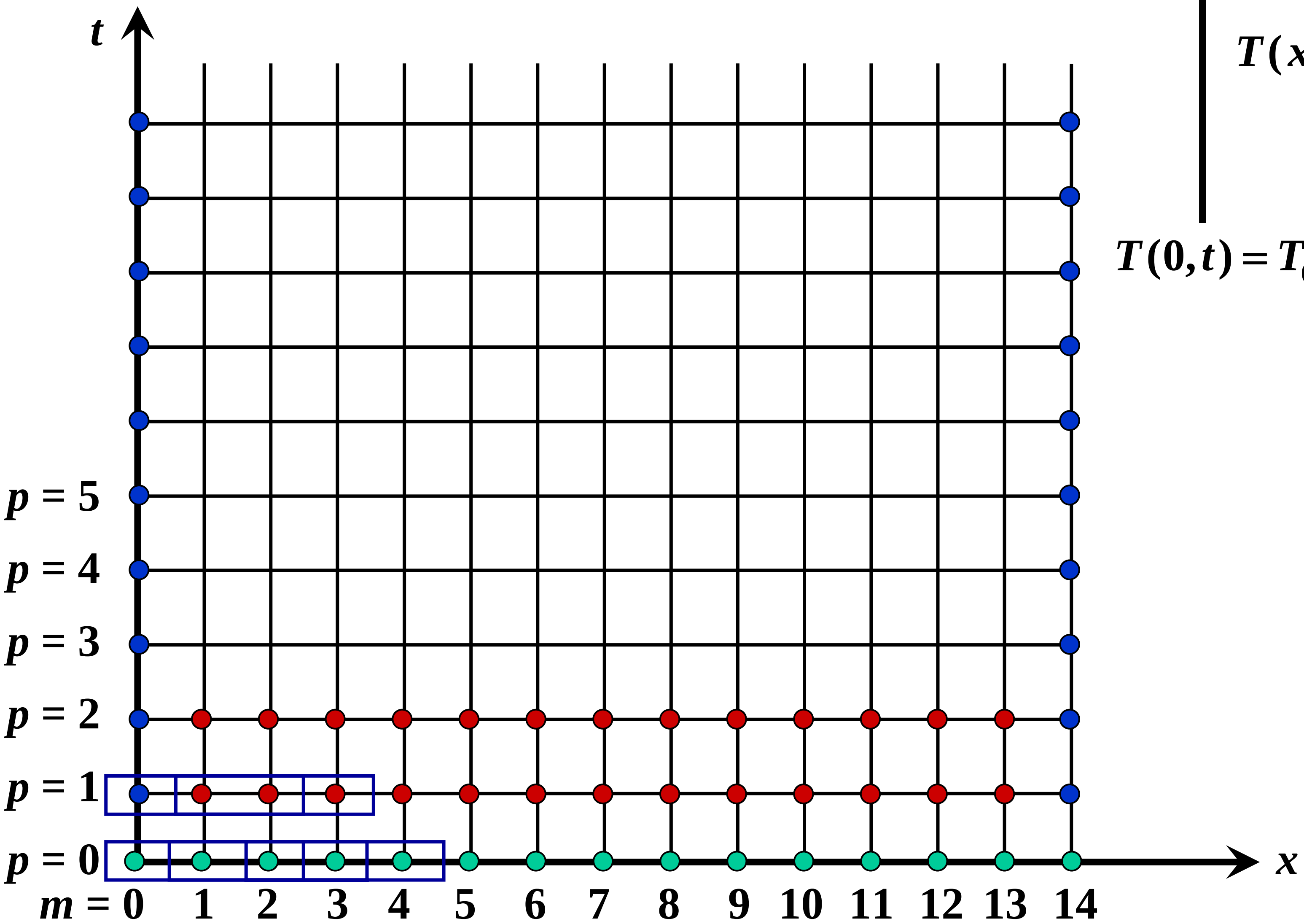
one-dimensional in x,

$$T_m^{p+1} = \mathrm{Fo}\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + \left(1 - 2\mathrm{Fo}\right)T_m^{p}$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = T_i$$

$$T(0,t) = T_0 \qquad T(L,t) = T_L$$



Implicit Method (fully)

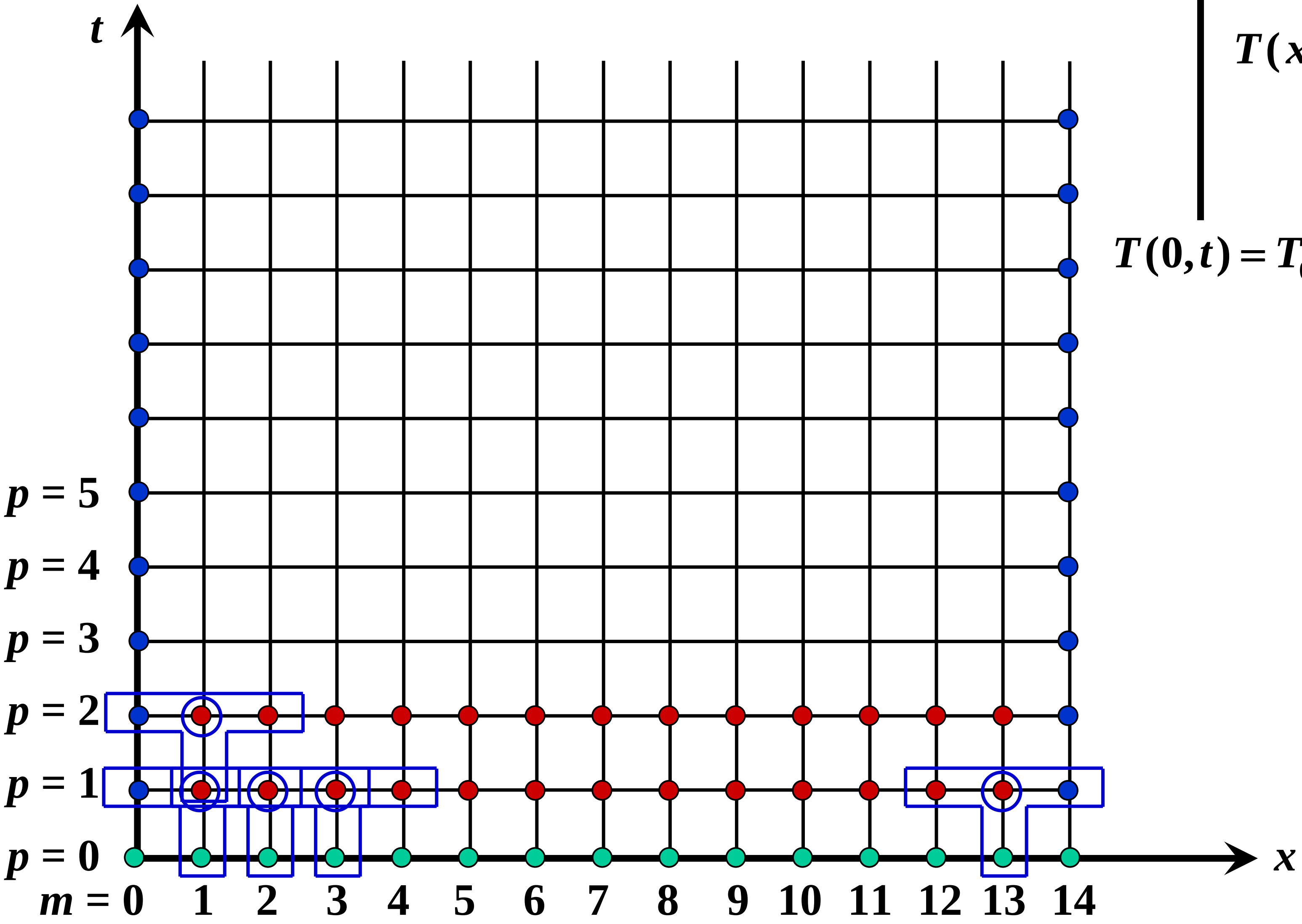
one-dimensional in x,

$$\left(1+2\mathrm{Fo}\right)T_m^{p+1}-\mathrm{Fo}\left(T_{m+1}^{p+1}+T_{m-1}^{p+1}\right)=T_m^p$$

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}$$

$$T(x,0)=T_i$$

$$T(0,t)=T_0 \qquad T(L,t)=T_L$$



Crank – Nicolson Method

$$T(x,t+\Delta t)=T(x,t)+\left.\frac{\partial T}{\partial t}\right|_{x,t}(\Delta t)+\frac{1}{2}\left.\frac{\partial^2 T}{\partial t^2}\right|_{x,t}(\Delta t)^2+O\left[(\Delta t)^3\right]$$

$$T(x,t-\Delta t)=T(x,t)-\left.\frac{\partial T}{\partial t}\right|_{x,t}(\Delta t)+\frac{1}{2}\left.\frac{\partial^2 T}{\partial t^2}\right|_{x,t}(\Delta t)^2+O\left[(\Delta t)^3\right]$$

$$T(x,t+\Delta t)-T(x,t-\Delta t)=2\left.\frac{\partial T}{\partial t}\right|_{x,t}(\Delta t)+O\left[(\Delta t)^3\right]$$

$$\frac{\partial T}{\partial t}=\frac{T(x,t+\Delta t)-T(x,t-\Delta t)}{2\Delta t}+O\left[(\Delta t)^2\right]$$

Second order accuracy in time, but serious stability problem

backward difference : $$\frac{T_m^{p+1}-T_m^p}{\alpha\Delta t}=\frac{T_{m+1}^{p+1}-2T_m^{p+1}+T_{m-1}^{p+1}}{(\Delta x)^2}$$

forward difference: $$\frac{T_m^{p+1}-T_m^p}{\alpha(\Delta t)}=\frac{T_{m+1}^{p}-2T_m^{p}+T_{m-1}^{p}}{(\Delta x)^2}$$

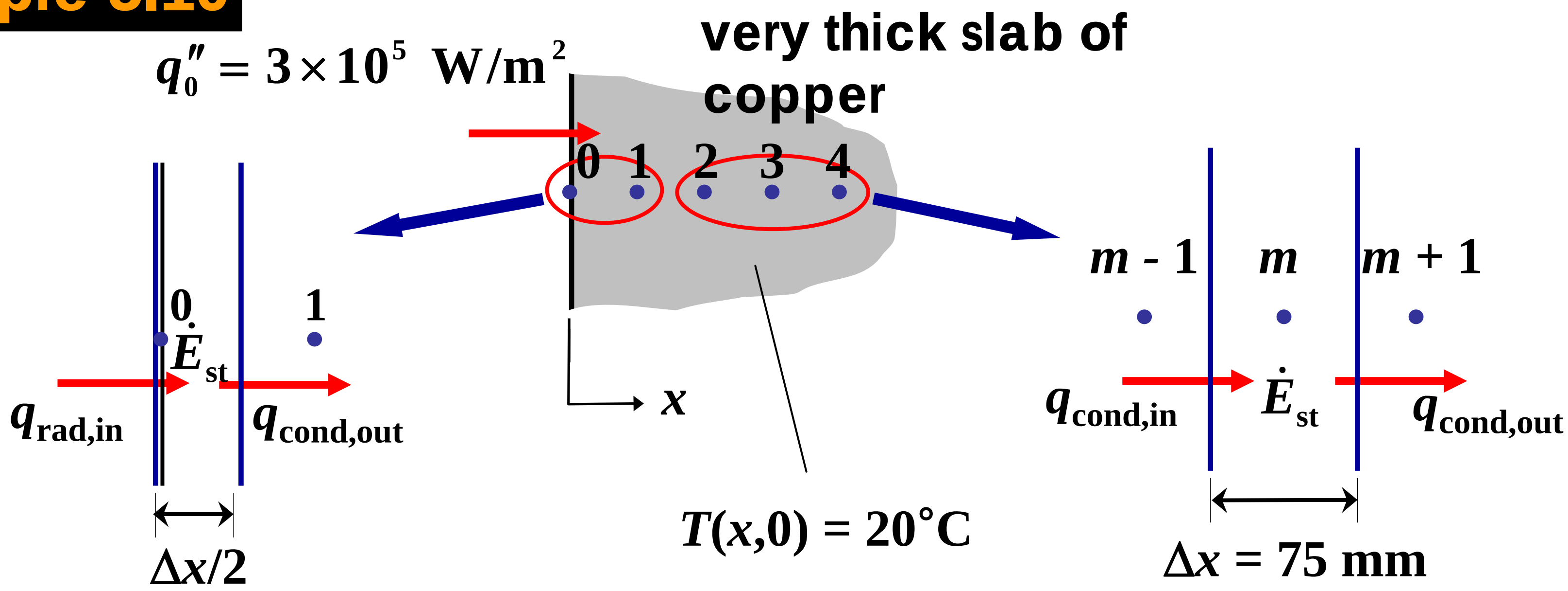
averaging: $$\frac{T_m^{p+1}-T_m^p}{\alpha\Delta t}=\frac{1}{2}\left\{\frac{T_{m+1}^{p+1}-2T_m^{p+1}+T_{m-1}^{p+1}}{(\Delta x)^2}+\frac{T_{m+1}^{p}-2T_m^{p}+T_{m-1}^{p}}{(\Delta x)^2}\right\}$$

$$-\frac{\mathrm{Fo}}{2}T_{m-1}^{p+1}+\left(1+\mathrm{Fo}\right)T_m^{p+1}-\frac{\mathrm{Fo}}{2}T_{m+1}^{p+1}=\frac{\mathrm{Fo}}{2}T_{m-1}^{p}+\left(1-\mathrm{Fo}\right)T_m^{p}+\frac{\mathrm{Fo}}{2}T_{m+1}^{p}$$

stability criterion: $1-\mathrm{Fo}\geq 0$ **or** $\mathrm{Fo}\leq 1$

explicit method: $$\mathrm{Fo}=\frac{\alpha\Delta t}{(\Delta x)^2}\leq\frac{1}{2}$$

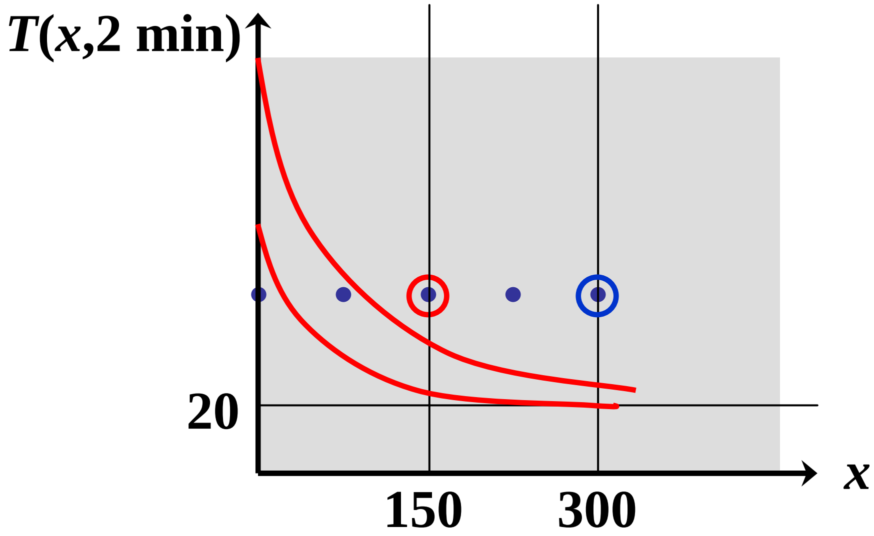
Example 5.10



Find:

1. Using the explicit FDM, determine temperature at the surface and 150 mm from the surface after 2 min, $T(0, 2 \text{ min})$, $T(150 \text{ mm}, 2 \text{ min})$
2. Repeat the calculations using the implicit FDM.
3. Determine the same temperatures analytically.

Determination of nodal points

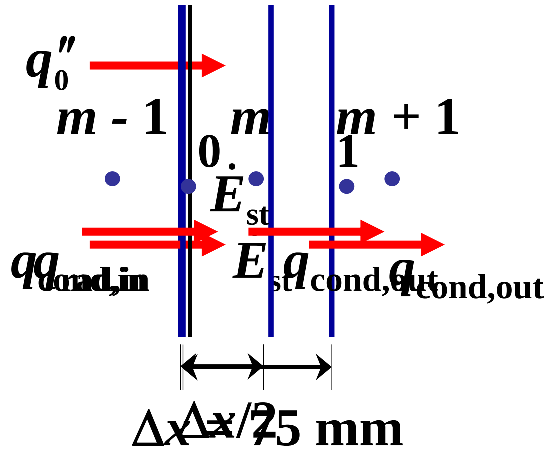


$$\delta(t) \sim \sqrt{\alpha t} \;\; = \sqrt{117 \times 10^{-6} \times 120} = 0.118 = 118 \text{ mm}$$

Table A.1, copper (300 K) : $k = 401 \text{W/m} \cdot \text{K}, \;\; \alpha = 117 \times 10^{-6} \text{m}^2/\text{s}$

$$\delta : 500 \sim 1000 \text{ mm}$$

Explicit FDM



node 0: $\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$

$$q''_0 A - kA\frac{T_0^p - T_1^p}{\Delta x} = \rho A \frac{\Delta x}{2} c \frac{T_0^{p+1} - T_0^p}{\Delta t}$$

or $$T_0^{p+1} = 2\text{Fo}\left(\frac{q''_0 \Delta x}{k} + T_1^p\right) + \left(1 - 2\text{Fo}\right) T_0^p$$

interior nodes: $$T_m^{p+1} = \text{Fo}\left(T_{m+1}^p + T_{m-1}^p\right) + \left(1 - 2\text{Fo}\right) T_m^p$$

time step: stability criterion $$\text{Fo} = \frac{\alpha \Delta t}{\left(\Delta x\right)^2} \le \frac{1}{2}$$

Table A.1, copper (300 K) : $k = 401\,\text{W/m}\cdot\text{K}, \quad \alpha = 117 \times 10^{-6}\,\text{m}^2/\text{s}$

$$\Delta t = \frac{\text{Fo}\left(\Delta x\right)^2}{\alpha} \le \frac{1}{2}\frac{(0.075\ \text{m})^2}{117 \times 10^{-6}\ \text{m}^2/\text{s}} = 24\ \text{s} \rightarrow \text{Fo} = \frac{1}{2}$$

$$2\ \text{min} \rightarrow p = 5$$

$$T_0^{p+1} = 2\mathrm{Fo}\left(\frac{q_0''\Delta x}{k} + T_1^p\right) + \left(1 - 2\mathrm{Fo}\right)T_0^p, \quad T_m^{p+1} = \mathrm{Fo}\left(T_{m+1}^p + T_{m-1}^p\right) + \left(1 - 2\mathrm{Fo}\right)T_m^p$$

$$\mathrm{Fo} = 0.5, \qquad \frac{q_0''\Delta x}{k} = \frac{3\times 10^5 \text{ W/m}^2 (0.075 \text{ m})}{401 \text{ W/m}\cdot\text{K}} = 56.1°\text{C}$$

finite-difference equations

$$T_0^{p+1} = 56.1°\text{C} + T_1^p \quad \text{and} \quad T_m^{p+1} = \frac{T_{m+1}^p + T_{m-1}^p}{2}, \quad m = 1, 2, 3, 4$$

$$T_5 = 20°\text{C}$$

Explicit Finite-Difference Solution for $Fo = \frac{1}{2}$

p	t(s)	T_0	T_1	T_2	T_3	T_4
0	0	20	20	20	20	20
1	24	76.1	20	20	20	20
2	48	76.1	48.1	20	20	20
3	72	104.2	48.1	34.1	20	20
4	96	104.2	69.1	34.1	27.1	20
5	120	125.3	69.1	48.1	27.1	20

After 2 min, $T_0 = 125.3°\text{C}$ **and** $T_2 = 48.1°\text{C}$

Improvement of the accuracy

$$\mathrm{Fo} = \frac{1}{4} \quad (\Delta t = 12\ \mathrm{s}),$$ domain length: 600 mm

$$T_0^{p+1} = \frac{1}{2}(56.1°\mathrm{C} + T_1^p) + \frac{1}{2}T_0^p, \quad T_m^{p+1} = \frac{1}{4}(T_{m+1}^p + T_{m-1}^p) + \frac{1}{2}T_m^p$$

Explicit Finite-Difference Solution for $Fo = \frac{1}{4}$

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20	20	20	20	20	20	20	20	20
1	12	48.1	20	20	20	20	20	20	20	20
2	24	62.1	27.0	20	20	20	20	20	20	20
3	36	72.6	34.0	21.8	20	20	20	20	20	20
4	48	81.4	40.6	24.4	20.4	20	20	20	20	20
5	60	89.0	46.7	27.5	21.3	20.1	20	20	20	20
6	72	95.9	52.5	30.7	22.6	20.4	20.0	20	20	20
7	84	102.3	57.9	34.1	24.1	20.8	20.1	20.0	20	20
8	96	108.1	63.1	37.6	25.8	21.5	20.3	20.0	20.0	20
9	108	113.7	68.0	41.0	27.6	22.2	20.5	20.1	20.0	20.0
10	120	118.9	72.6	44.4	29.6	23.2	20.8	20.2	20.0	20.0

After 2 min, $T_0 = 118.9°\mathrm{C}$ and $T_2 = 44.4°\mathrm{C}$

When Δt = 24 s, $T_0 = 125.3°\mathrm{C}$ and $T_2 = 48.1°\mathrm{C}$

Implicit FDM

node 0: $$q_0'' + k\frac{T_1^{p+1} - T_0^{p+1}}{\Delta x} = \rho \frac{\Delta x}{2} c \frac{T_0^{p+1} - T_0^{p}}{\Delta t}$$

or $$\left(1 + 2\mathrm{Fo}\right) T_0^{p+1} - 2\mathrm{Fo}\, T_1^{p+1} = \frac{2\alpha q_0'' \Delta t}{k \Delta x} + T_0^{p}$$

Arbitrarily choosing, $\mathrm{Fo} = \dfrac{1}{2}$ $(\Delta t = 24\ \mathrm{s})$

$$2T_0^{p+1} - T_1^{p+1} = 56.1^\circ\mathrm{C} + T_0^{p}$$

interior nodes: $-T_{m-1}^{p+1} + 4T_m^{p+1} - T_{m+1}^{p+1} = 2T_m^{p}, \quad m = 1, 2, 3, \cdots, 8$

A set of nine equations must be solved simultaneously for each time increment.

The equations are in the form $[A][T]=[C]$.

$$[A][T] = [C]$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} T_0^{p+1} \\ T_1^{p+1} \\ T_2^{p+1} \\ T_3^{p+1} \\ T_4^{p+1} \\ T_5^{p+1} \\ T_6^{p+1} \\ T_7^{p+1} \\ T_8^{p+1} \end{bmatrix} = \begin{bmatrix} 56.1 + T_0^{p} \\ 2T_1^{p} \\ 2T_2^{p} \\ 2T_3^{p} \\ 2T_4^{p} \\ 2T_5^{p} \\ 2T_6^{p} \\ 2T_7^{p} \\ 2T_8^{p} + T_9^{p+1} \end{bmatrix} \qquad [C]_{p=0} = \begin{bmatrix} 76.1 \\ 40 \\ 40 \\ 40 \\ 40 \\ 40 \\ 40 \\ 40 \\ 60 \end{bmatrix}$$

Implicit Finite-Difference Solution for $Fo = \frac{1}{2}$

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0
1	24	52.4	28.7	22.3	20.6	20.2	20.0	20.0	20.0	20.0
2	48	74.0	39.5	26.6	22.1	20.7	20.2	20.1	20.0	20.0
3	72	90.2	50.3	32.0	24.4	21.6	20.6	20.2	20.1	20.0
4	96	103.4	60.5	38.0	27.4	22.9	21.1	20.4	20.2	20.1
5	120	114.7	70.0	44.2	30.9	24.7	21.9	20.8	20.3	20.1

After 2 min, $T_0 = 114.7°C$ and $T_2 = 44.2°C$

Analytical Solution

$$T(x,t)-T_i=\frac{2q_0''(\alpha t/\pi)^{1/2}}{k}\exp\left(-\frac{x^2}{4\alpha t}\right)-\frac{q_0''x}{k}\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

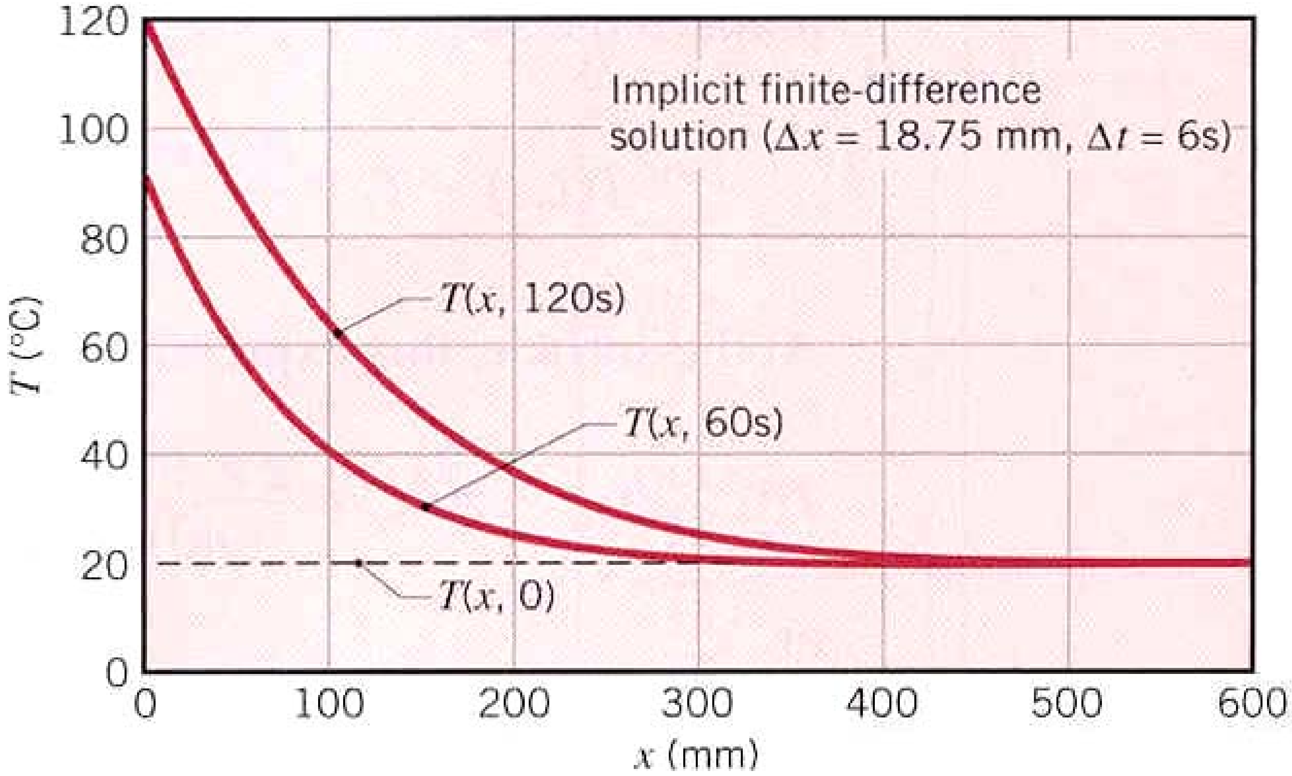
$$T(0,120\text{ s})=120.0^\circ\mathrm{C}$$

$$T(0.15\text{ m},120\text{ s})=45.4^\circ\mathrm{C}$$

Comparison

Method	$T_0 = T(0, 120\text{ s})$	$T_2 = T(0.15\text{ m}, 120\text{ s})$
Explicit ($Fo = \frac{1}{2}$)	125.3	48.1
Explicit ($Fo = \frac{1}{4}$)	118.9	44.4
Implicit ($Fo = \frac{1}{2}$)	114.7	44.2
Exact	120.0	45.4

Implicit method with Δx = 18.75 mm (37 nodalpoints) and Δt = 6 s (Fo = 2.0)



$$T(0, 120\text{ s}) = 119.2^\circ\text{C},\quad T(0.15\text{ m}, 120\text{ s}) = 45.3^\circ\text{C}$$

exact: $$T(0, 120\text{ s}) = 120.0^\circ\text{C},\quad T(0.15\text{ m}, 120\text{ s}) = 45.4^\circ\text{C}$$