FUNDAMENTAL CONCEPTS OF CONVECTION

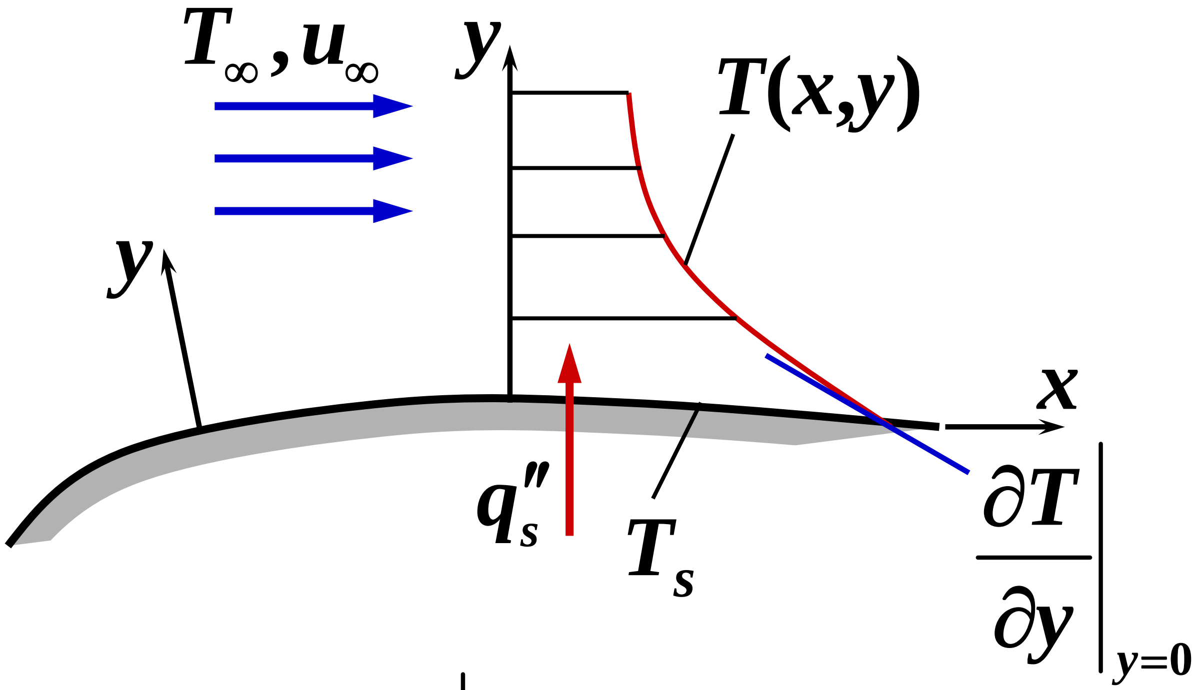
- **Convection Heat Transfer Coefficient**
- **Conservation Equations**
- **Boundary Layer Approximation**
- **Reynolds Analogy**
- **Turbulent Flow**

- Forced Convection
- Free Convection
- Boiling and Condensation

  - External flows
  - Internal flows

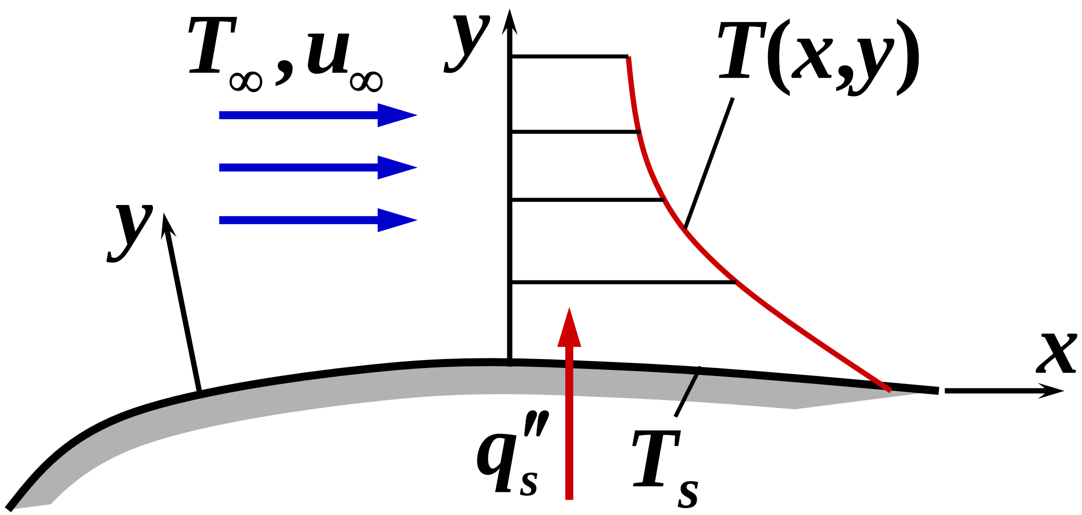
  - Laminar flows
  - Turbulent flows

Convection Heat Transfer Coefficient



$$q_s''(x) = -k_f \left.\frac{\partial T}{\partial y}\right|_{y=0} \equiv h(x)\left(T_s - T_\infty\right)$$

$$h(x) = \frac{-k_f}{T_s - T_\infty} \left.\frac{\partial T}{\partial y}\right|_{y=0}$$



total heat transfer rate over A_s

$$q_s = \int_{A_s} q''_s(x)\,dA_s = \int_{A_s} h(x)\left(T_s - T_\infty\right)dA_s$$

If T_s = constant,

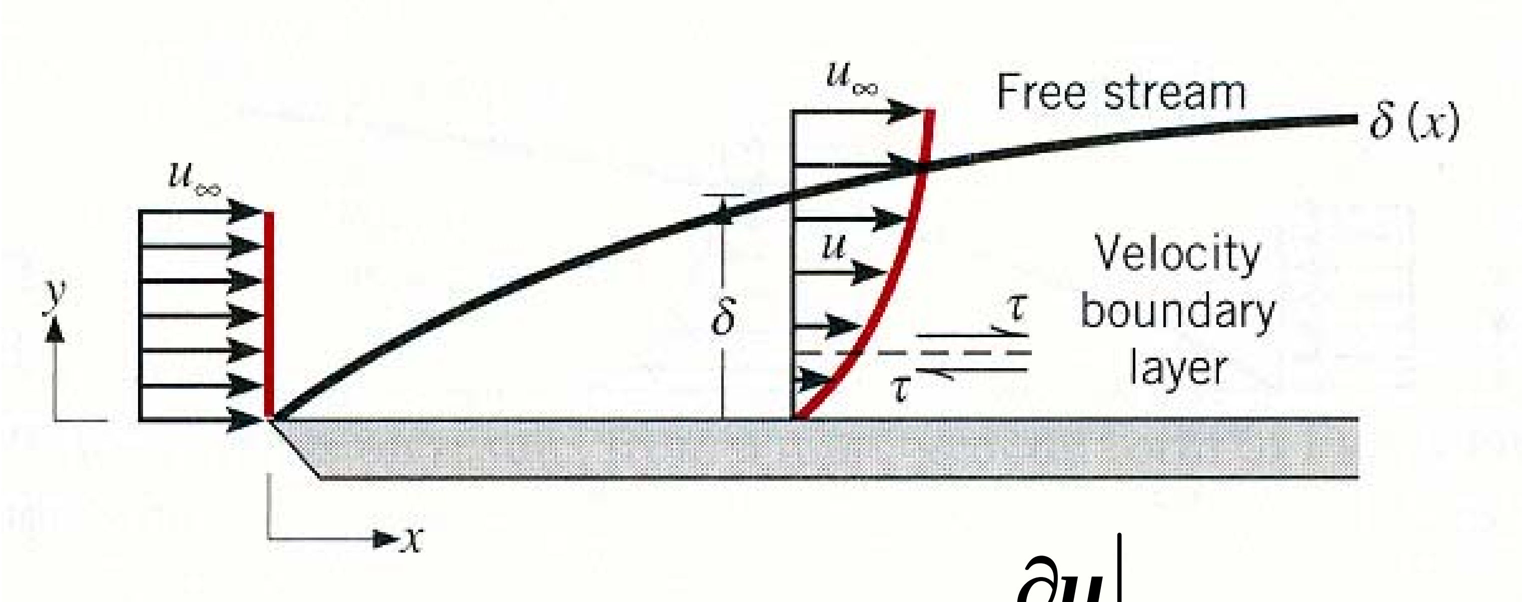
$$q_s = \left(T_s - T_\infty\right)\int_{A_s} h(x)\,dA_s \equiv \bar{h}A_s\left(T_s - T_\infty\right)$$

average heat transfer coefficient:

$$\bar{h} = \frac{1}{A_s}\int_{A_s} h\,dA_s$$

Convection Boundary Layer

Velocity (or momentum) boundary layer

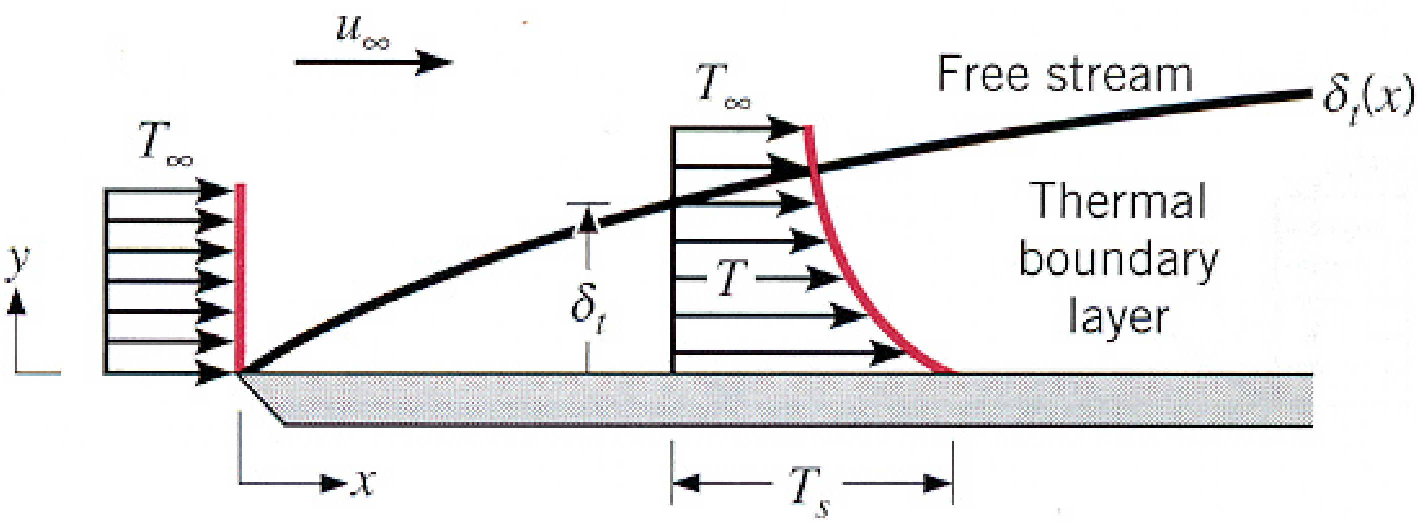


wall shear stress: $\tau_s = \mu \left.\dfrac{\partial u}{\partial y}\right|_{y=0}$

μ**: dynamic viscosity**

friction coefficient: $C_f = \dfrac{\tau_s}{\rho u_\infty^2 / 2}$

Temperature (or thermal) boundary layer

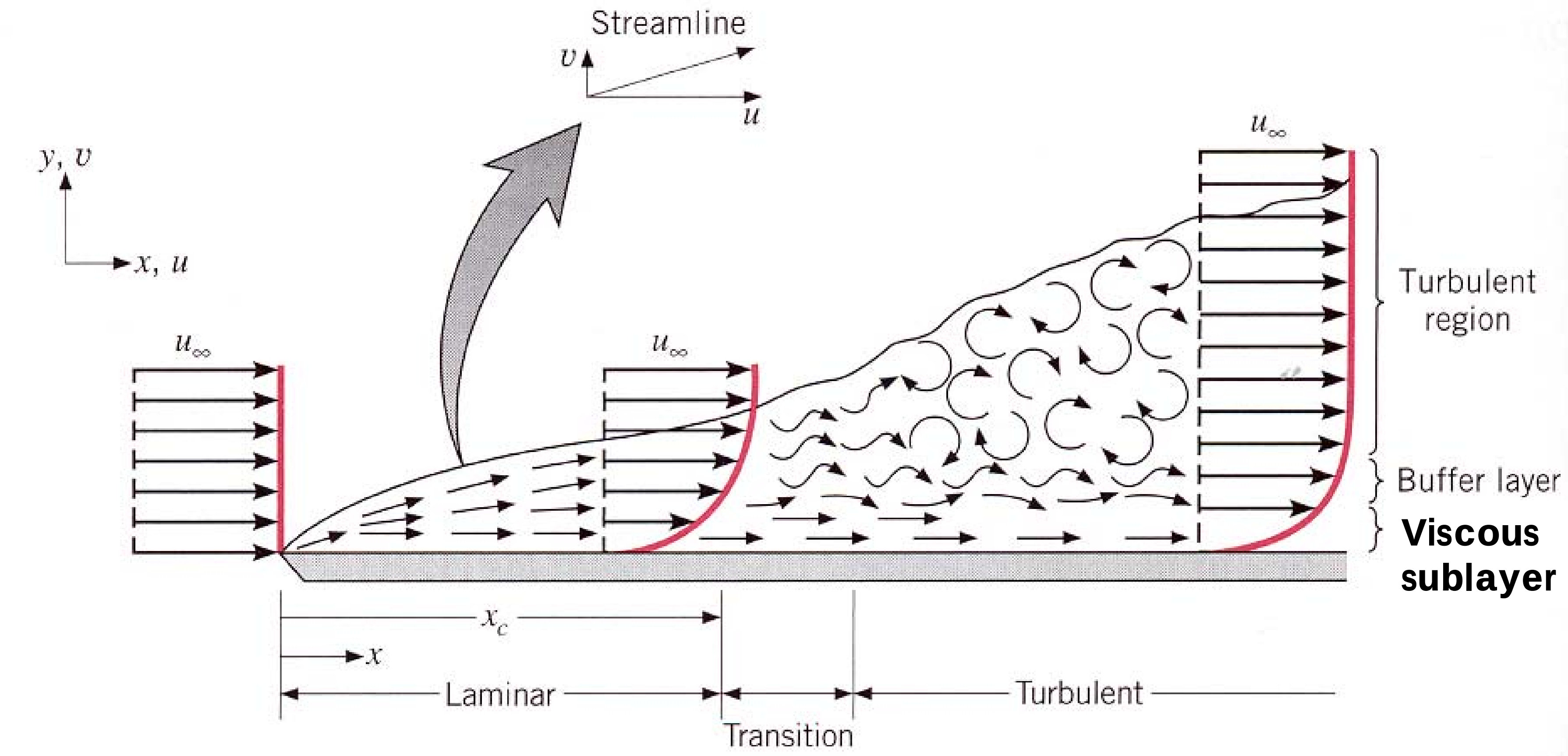


local heat flux: $q_s''(x) = -k_f \left.\dfrac{\partial T}{\partial y}\right|_{y=0}$

local heat transfer coefficient:

$$h(x) = \frac{-k_f}{T_s - T_\infty} \left.\frac{\partial T}{\partial y}\right|_{y=0}$$

Laminar and Turbulent Flows



laminar: molecular diffusion

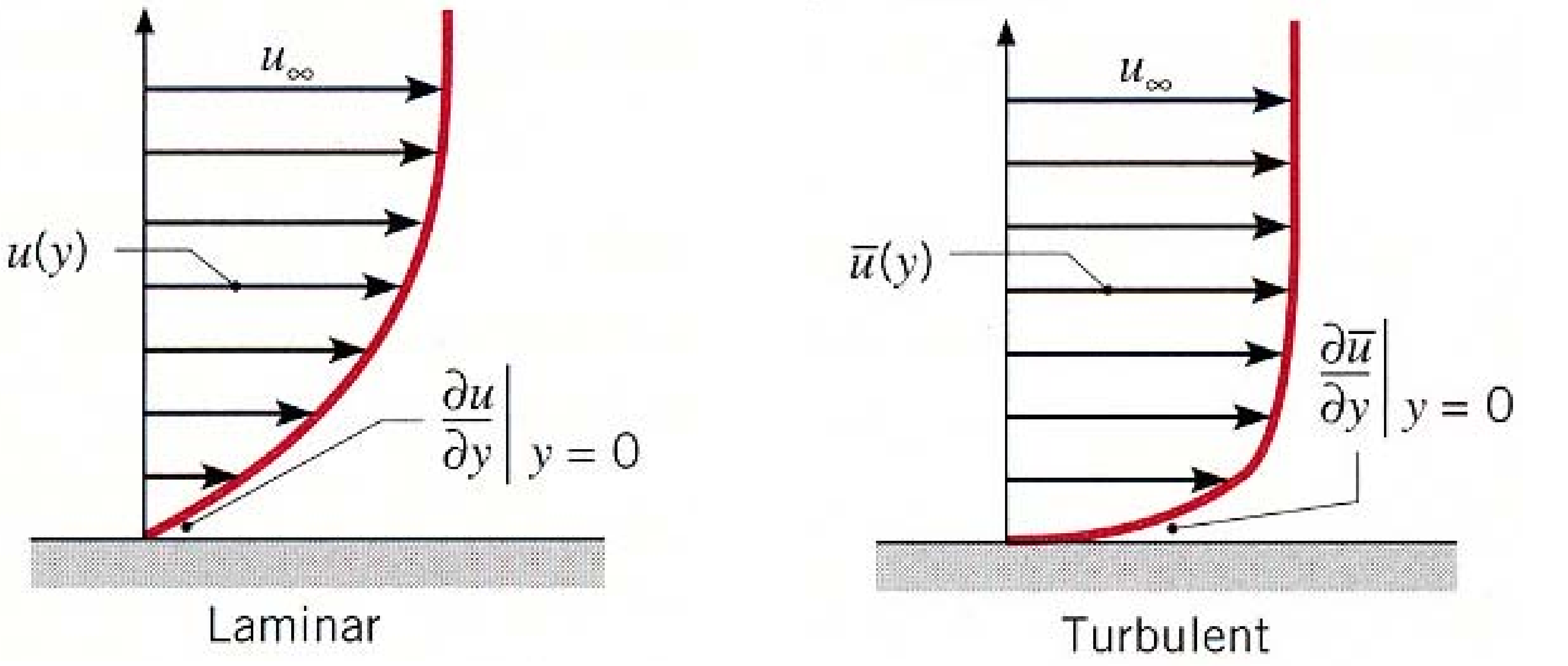
turbulent: eddy motion (fluctuation)

Critical Reynolds number

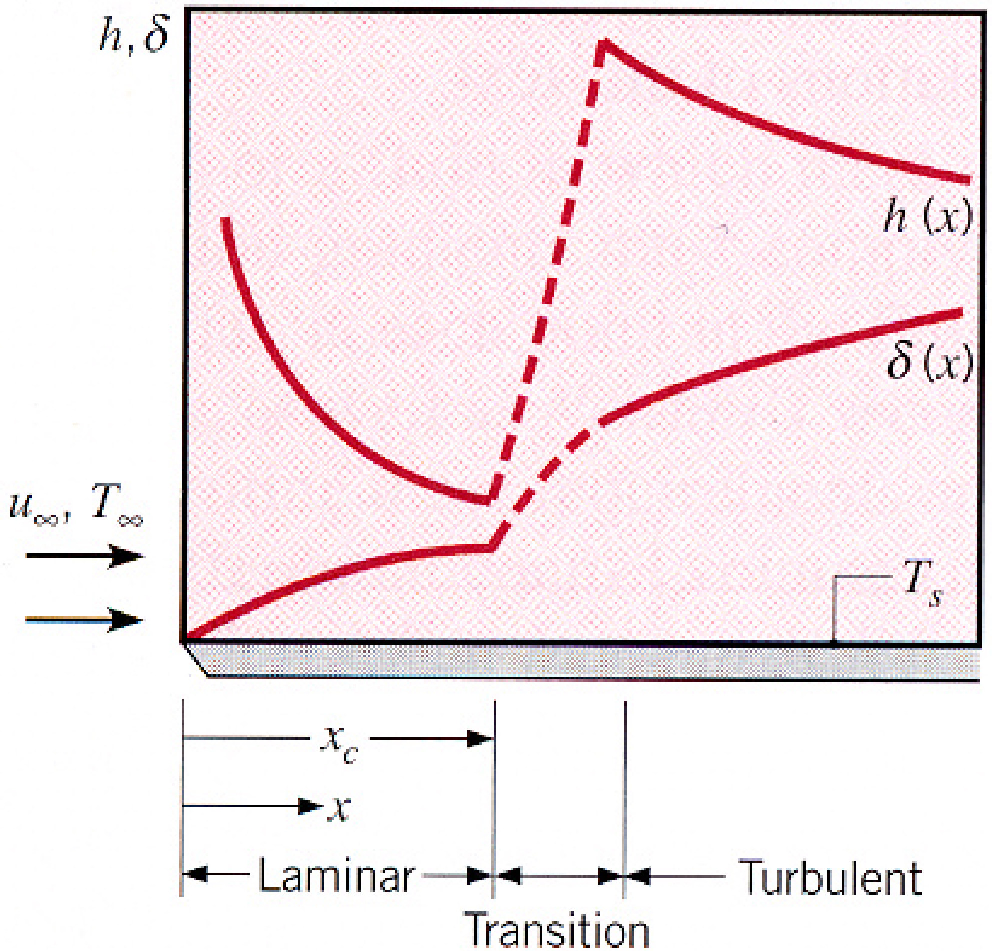
external flow: $$\mathrm{Re}_x = \frac{\rho u_\infty x}{\mu} = \frac{u_\infty x}{\nu} \approx 5 \times 10^5$$

internal flow: $$\mathrm{Re}_D = \frac{\rho u_m D}{\mu} = \frac{u_m D}{\nu} \approx 2,300$$

Comparison of laminar and turbulent velocity profile in the boundary layer



$$\left.\frac{\partial u}{\partial y}\right|_{y=0,\,\mathrm{lam}} < \left.\frac{\partial \overline{u}}{\partial y}\right|_{y=0,\,\mathrm{turb}}$$



Variation of heat transfer coefficient

$$h = \frac{-k_f}{T_s - T_\infty} \left. \frac{\partial T}{\partial y} \right|_{y=0} \sim \frac{-k_f}{T_s - T_\infty} \frac{T_\infty - T_s}{\delta(x)} \sim \frac{k_f}{\delta(x)}$$

Conservation Equations

Continuity Equation: Mass Conservation

$$\frac{dm}{dt} = 0 \;\rightarrow\; \frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{u} \right) = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} \;=\; \frac{d\rho}{dt} + \rho \nabla \cdot \underline{u} = 0$$

incompressible flow

$$\frac{d\rho}{dt} = 0 \;\rightarrow\; \nabla \cdot \vec{u} = 0$$

2-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum Equations

Newton's 2nd law of motion

$$m\frac{d\vec{u}}{dt} = \sum \vec{F}, \qquad \rho\frac{d\vec{u}}{dt} = \sum \vec{f}$$

f : force per unit mass

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = \sum \vec{f}$$

force: body force, surface force

body force: gravitational force, centrifugal force, electromagnetic force

surface force: viscous force, pressure force

incompressible flow with constant viscosity

$$\rho \frac{d\vec{u}}{dt} = \rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \mu\nabla^2\vec{u} + \rho\vec{f}$$

2-dimensional steady flow

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f_x$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + f_y$$

Energy Equation

1st law of thermodynamics

rate change of internal energy
= heat transferred in the system
+ work done on the system by forces
+ internal heat generation

internal energy: thermal energy,
kinetic energy

heat transfer: conduction, radiation

work: work done by body force
and surface force

energy per unit mass of fluid

thermal energy: e

kinetic energy: $\frac{1}{2}V^2 \quad \left(V^2 = \vec{u}\cdot\vec{u}\right)$

total energy: $\phi \equiv e + \frac{1}{2}V^2$

Total energy equation $\underset{\sim}{\tau} = \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$\rho\frac{d}{dt}\left(e + \frac{1}{2}V^2\right) = -\nabla\cdot\vec{q}'' - p\nabla\cdot\vec{u} - \vec{u}\cdot\nabla p$$

$$+\left(\nabla\vec{u}\right):\underset{\sim}{\tau} + \vec{u}\cdot\left(\nabla:\underset{\sim}{\tau}\right) + \rho\vec{f}\cdot\vec{u} + \dot{q}$$

$$\vec{q}'' = \vec{q}''_{\text{cond}} + \vec{q}''_{\text{rad}} = -k\nabla T + \vec{q}''_{\text{rad}}$$

Mechanical energy equation

$$\rho \frac{d}{dt}\left(\frac{1}{2}V^2\right) = -\vec{u}\cdot\nabla p + \vec{u}\cdot\left(\nabla : \underset{\sim}{\tau}\right) + \rho \vec{f}\cdot\vec{u}$$

Thermal energy equation

$$\rho \frac{de}{dt} = -\nabla\cdot\vec{q}'' - p\nabla\cdot\vec{u} + \left(\nabla\vec{u}\right) : \underset{\sim}{\tau} + \dot{q}$$

$\left(\nabla\vec{u}\right) : \underset{\sim}{\tau} \equiv \mu\Phi$ **: viscous dissipation**

Thermal Energy Equation

$$\rho\frac{de}{dt} = -\nabla\cdot\vec{q}'' - p\nabla\cdot\vec{u} + \dot{q} + \mu\Phi$$

thermal energy equation for enthalpy

$$i = e + \frac{p}{\rho} \quad \text{or} \quad e = i - \frac{p}{\rho}$$

$$\frac{de}{dt} = \frac{di}{dt} - \frac{d}{dt}\left(\frac{p}{\rho}\right) = \frac{di}{dt} - \frac{1}{\rho}\frac{dp}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt}$$

$$\rho\left(\frac{di}{dt} - \frac{1}{\rho}\frac{dp}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt}\right) = -\nabla\cdot\vec{q}'' - p\nabla\cdot\vec{u} + \dot{q} + \mu\Phi$$

$$\rho\left(\frac{di}{dt}-\frac{1}{\rho}\frac{dp}{dt}+\frac{p}{\rho^2}\frac{d\rho}{dt}\right)=-\nabla\cdot\vec{q}''-p\nabla\cdot\vec{u}+\dot{q}+\mu\Phi$$

$$\rho\frac{di}{dt}=-\nabla\cdot\vec{q}''+\frac{dp}{dt}-\frac{p}{\rho}\frac{d\rho}{dt}-p\nabla\cdot\vec{u}+\dot{q}+\mu\Phi$$

$$=-\nabla\cdot\vec{q}''+\frac{dp}{dt}-\frac{p}{\rho}\left(\frac{d\rho}{dt}+\rho\nabla\cdot\vec{u}\right)+\dot{q}+\mu\Phi$$

$$\rho\frac{di}{dt}=-\nabla\cdot\vec{q}''+\frac{dp}{dt}+\dot{q}+\mu\Phi$$

When it is assumed that (ideal gas)

$$de = c_v dT \quad \textbf{and} \quad di = c_p dT$$

$$\rho \frac{de}{dT} = \rho c_v \frac{dT}{dt} = \rho c_v \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T \right)$$

$$= -\nabla \cdot \vec{q}'' - p \nabla \cdot \vec{u} + \dot{q} + \mu \Phi$$

$$\rho \frac{di}{dt} = \rho c_p \frac{dT}{dt} = \rho c_p \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T \right)$$

$$= -\nabla \cdot \vec{q}'' + \frac{dp}{dt} + \dot{q} + \mu \Phi$$

When pressure work and viscous dissipation is negligible, and internal heat generation is not present,

$$\rho c_p\left(\frac{\partial T}{\partial t}+\vec{u}\cdot\nabla T\right)=-\nabla\cdot\left(\vec{q}''_{\mathrm{cond}}+\vec{q}''_{\mathrm{rad}}\right)$$

$$=\nabla\cdot\left(k\nabla T\right)-\nabla\cdot\vec{q}''_{\mathrm{rad}}$$

When the fluid is transparent to radiation and its thermal conductivity is constant,

$$\frac{\partial T}{\partial t}+\vec{u}\cdot\nabla T=\alpha\nabla^2 T$$

steady-state: $\vec{u}\cdot\nabla T=\alpha\nabla^2 T$

For a 2-dimensional flow

$$\vec{u} \cdot \nabla T = \alpha \nabla^2 T$$

$$\rightarrow u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Summary

2-dimensional, steady, incompressible, constant property, transparent to radiation, no internal heat generation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

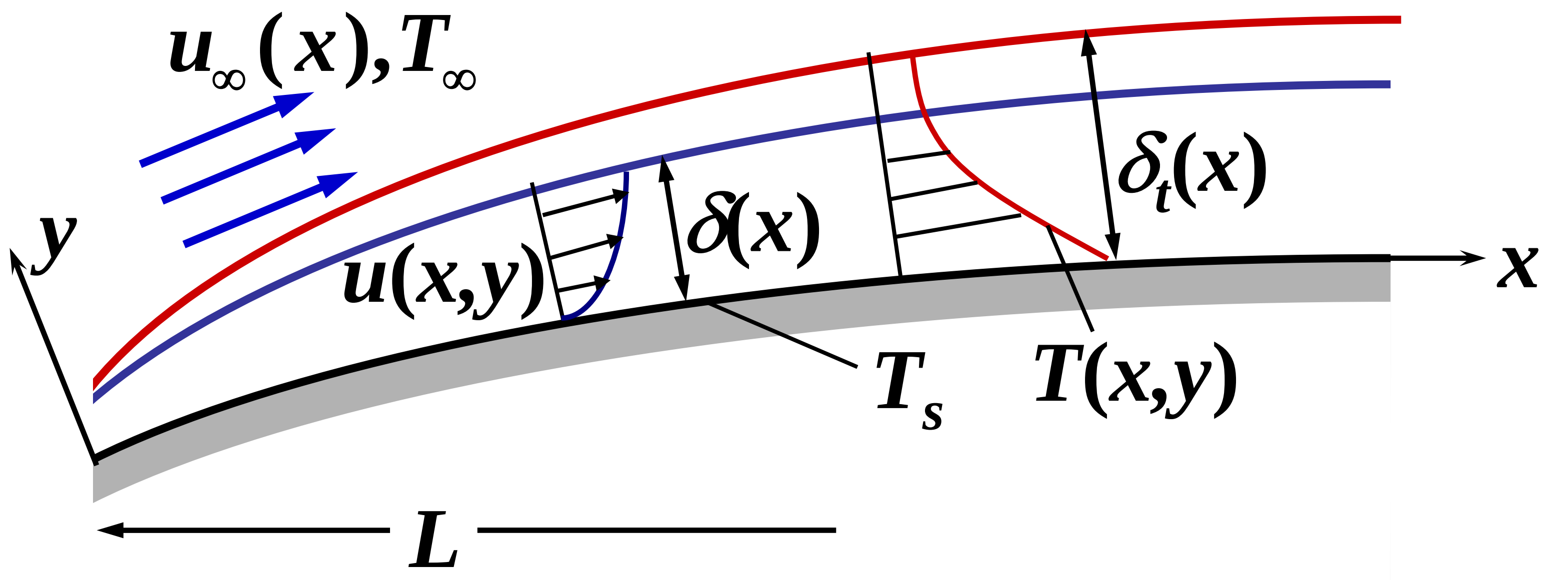
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f_x$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + f_y$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

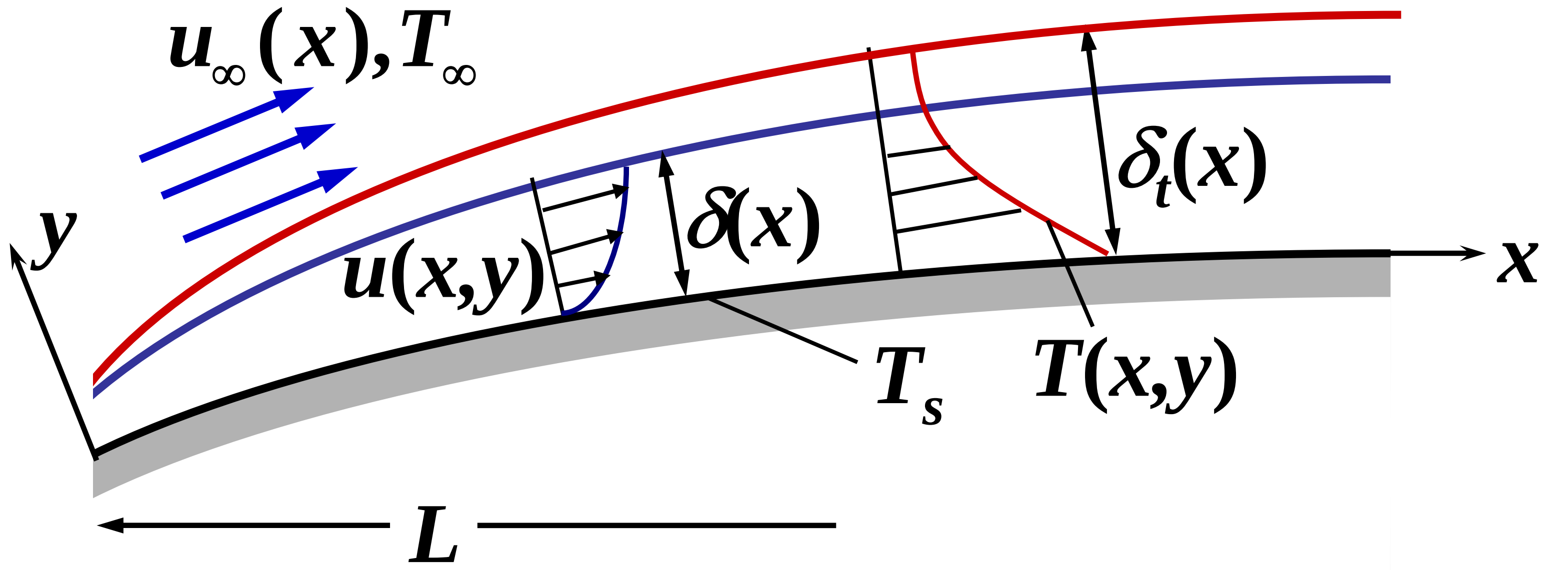
Boundary Layer Approximation

2-D Boundary Layer Flow



$\delta(x)$: thickness of velocity boundary layer

$\delta_t(x)$: thickness of temperature boundary layer



scaling $x \sim L, \;\; y \sim \delta$ **or** $\delta_t, \;\; u \sim u_\infty$

from continuity $\dfrac{\partial u}{\partial x} \sim \dfrac{\partial v}{\partial y} \rightarrow \dfrac{u_\infty}{L} \sim \dfrac{v}{\delta} \rightarrow v \sim \dfrac{u_\infty \delta}{L}$

pressure $p_s = p + \dfrac{1}{2}\rho u_\infty^2 \rightarrow p \sim \rho u_\infty^2$

temperature $\Delta T = T_s - T_\infty$

dimensionless variables

$$x \sim L,\ y \sim \delta \text{ or } \delta_t,\ u \sim u_\infty,\ v \sim \frac{u_\infty \delta}{L},$$

$$p \sim \rho u_\infty^2,\ \Delta T = T_s - T_\infty$$

$$x^* = \frac{x}{L},\quad y^* = \frac{y}{\delta}\left(\text{or} = \frac{y}{\delta_t}\right),\quad u^* = \frac{u}{u_\infty},$$

$$v^* = \frac{vL}{u_\infty \delta},\quad p^* = \frac{p}{\rho u_\infty^2},\quad T^* = \frac{T - T_\infty}{T_s - T_\infty}$$

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ \rightarrow \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

momentum equation in the streamwise direction

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\cancel{\frac{\nu}{u_\infty L}\frac{\partial^2 u^*}{\partial x^{*2}}}+\frac{\nu}{u_\infty L}\left(\frac{L}{\delta}\right)^2\frac{\partial^2 u^*}{\partial y^{*2}}$$

for a high Reynolds number flow:

$$\mathrm{Re}_L=\frac{u_\infty L}{\nu}>>1,\ \ \mathrm{Re}_L\sim\left(\frac{L}{\delta}\right)^2$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

momentum equation in the wall-normal direction

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\left(\frac{\delta}{L}\right)^2\left(u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}\right)$$

$$=-\frac{\partial p^*}{\partial y^*}+\frac{\nu}{u_\infty L}\left(\frac{\delta}{L}\right)^2\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\nu}{u_\infty L}\frac{\partial^2 v^*}{\partial y^{*2}}$$

$$\mathrm{Re}_L=\frac{u_\infty L}{\nu}>>1,\ \ \mathrm{Re}_L\sim\left(\frac{L}{\delta}\right)^2$$

$\frac{\partial p}{\partial y}=0\ \ \rightarrow p=p(x)$ **in the boundary layer**

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$p_s = p+\frac{1}{2}\rho u_\infty^2 \rightarrow 0=\frac{dp}{dx}+\rho u_\infty\frac{du_\infty}{dx}\rightarrow\frac{dp}{dx}=-\rho u_\infty\frac{du_\infty}{dx}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_\infty\frac{du_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

flow over a flat plate: $u_\infty =$ **constant** $\rightarrow \frac{du_\infty}{dx}=0$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

$$u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{\nu}{u_\infty L}\frac{\alpha}{\nu}\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\nu}{u_\infty L}\frac{\alpha}{\nu}\left(\frac{L}{\delta_t}\right)^2\frac{\partial^2 T^*}{\partial y^{*2}}$$

$$=\cancel{\frac{1}{\mathrm{Re}_L\cdot\mathrm{Pr}}\frac{\partial^2 T^*}{\partial x^{*2}}}+\frac{1}{\mathrm{Re}_L\cdot\mathrm{Pr}}\left(\frac{L}{\delta_t}\right)^2\frac{\partial^2 T^*}{\partial y^{*2}}$$

$$\mathrm{Re}_L=\frac{u_\infty L}{\nu},\ \ \mathrm{Pr}=\frac{\nu}{\alpha}$$

Pr: **Prandtl number**

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

Summary: 2-D Boundary layer equations

continuity: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

momentum: $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

energy: $$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

Boundary layer approximation

- **Parabolic in the streamwise direction**
- **Pressure is constant across the boundary layer.**

Fundamental Form of Solutions

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{dp^*}{dx^*}+\frac{1}{\mathrm{Re}_L}\left(\frac{L}{\delta}\right)^2\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$=-\frac{dp^*}{dx^*}+f(\mathrm{Re}_L)\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{1}{\mathrm{Re}_L}\frac{\alpha}{\nu}\left(\frac{L}{\delta_t}\right)^2\frac{\partial^2 T^*}{\partial y^{*2}}$$

$$=g(\mathrm{Re}_L,\mathrm{Pr})\frac{\partial^2 T^*}{\partial y^{*2}}$$

$$u^*=u^*\left(x^*,y^*,\frac{dp^*}{dx^*},\mathrm{Re}_L\right),$$

$$T^*=T^*\left(x^*,y^*,\frac{dp^*}{dx^*},\mathrm{Re}_L,\mathrm{Pr}\right)$$

Friction coefficient

$$C_f = \frac{\tau_s}{\rho u_\infty^2 / 2}$$

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{u_\infty}{\delta} \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0} \qquad y^* = \frac{y}{\delta}, \;\; u^* = \frac{u}{u_\infty}$$

$$C_f = \frac{\tau_s}{\rho u_\infty^2 / 2} = 2 \frac{\mu}{\rho u_\infty} \frac{1}{\delta} \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0}$$

$$= 2 \frac{\nu}{u_\infty L} \frac{L}{\delta} \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0} = \frac{2}{\mathrm{Re}_L} \frac{L}{\delta} \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0}$$

$$C_f = C_f(x^*, \mathrm{Re}_L)$$

Convection heat transfer coefficient

$$h = \frac{-k_f}{T_s - T_\infty}\left.\frac{\partial T}{\partial y}\right|_{y=0} \qquad y^* = \frac{y}{\delta_t},\ T^* = \frac{T - T_\infty}{T_s - T_\infty}$$

$$= \frac{-k_f\left(T_s - T_\infty\right)}{T_s - T_\infty}\frac{1}{\delta_t}\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} \quad = \frac{-k_f}{\delta_t}\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

Nusselt number

local Nusselt number

$$\mathrm{Nu}_x = \frac{hx}{k_f} = -\frac{x}{\delta_t}\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} \quad = -x^*\frac{L}{\delta}\frac{\delta}{\delta_t}\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$\mathrm{Nu}_x = \mathrm{Nu}(x^*, \mathrm{Re}_L, \mathrm{Pr})$$

average Nusselt number $\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(\mathrm{Re}_L, \mathrm{Pr})$

Reynolds Analogy

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

When $\frac{dp}{dx}=0$ **and** $\Pr = 1$**, that is,** $\nu = \alpha$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2},\quad u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

$$\delta = \delta_t \sim \frac{L}{\sqrt{\mathrm{Re}_L}}$$

Let $x^* = \dfrac{x}{L},\ y^* = \dfrac{y}{\delta} \equiv \dfrac{y}{L}\sqrt{\mathrm{Re}_L},\ u^* = \dfrac{u}{u_\infty},$

$$v^* = \frac{vL}{u_\infty \delta} \equiv \frac{v}{u_\infty}\sqrt{\mathrm{Re}_L},\ T^* = \frac{T - T_s}{T_\infty - T_s}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial {y^*}^2},\quad u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{\partial^2 T^*}{\partial {y^*}^2}$$

boundary conditions

$$u^*(x^*,0) = 0,\ u^*(x^*,\infty) = 1$$

$$T^*(x^*,0) = 0,\ T^*(x^*,\infty) = 1$$

Thus, $u^*(x^*, y^*) = T^*(x^*, y^*)$

$$x^* = \frac{x}{L},\ y^* = \frac{y}{\delta} \equiv \frac{y}{L}\sqrt{\mathrm{Re}_L},\ u^* = \frac{u}{u_\infty},$$

$$v^* = \frac{v}{u_\infty}\sqrt{\mathrm{Re}_L},\ T^* = \frac{T - T_s}{T_\infty - T_s}$$

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \frac{u_\infty \sqrt{\mathrm{Re}_L}}{L} \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0}$$

$$C_f = \frac{\tau_s}{\rho u_\infty^2 / 2} = \frac{2\mu u_\infty \sqrt{\mathrm{Re}_L}}{\rho u_\infty^2 L} \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0}$$

$$= 2\frac{\nu}{u_\infty L}\sqrt{\mathrm{Re}_L} \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \frac{2}{\sqrt{\mathrm{Re}_L}} \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0}$$

$$h = \frac{-k_f}{T_s - T_\infty} \left.\frac{\partial T}{\partial y}\right|_{y=0} = \frac{-k_f\left(T_\infty - T_s\right)}{T_s - T_\infty} \frac{\sqrt{\mathrm{Re}_L}}{L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$= \frac{k_f\sqrt{\mathrm{Re}_L}}{L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$\mathrm{Nu}_L = \frac{hL}{k_f} = \frac{L}{k_f} \frac{k_f\sqrt{\mathrm{Re}_L}}{L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = \sqrt{\mathrm{Re}_L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$C_f = \frac{2}{\sqrt{\mathrm{Re}_L}} \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \frac{2}{\sqrt{\mathrm{Re}_L}} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$= \frac{2}{\sqrt{\mathrm{Re}_L}} \frac{\mathrm{Nu}_L}{\sqrt{\mathrm{Re}_L}} = \frac{2\mathrm{Nu}_L}{\mathrm{Re}_L}$$

Stanton number

$$\mathrm{St} = \frac{h}{\rho u_\infty c_p} = \frac{hL}{k_f}\frac{k}{\rho c_p}\frac{\nu}{u_\infty L}\frac{1}{\nu} = \mathrm{Nu}_L\,\mathrm{Re}_L^{-1}\,\mathrm{Pr}^{-1}$$

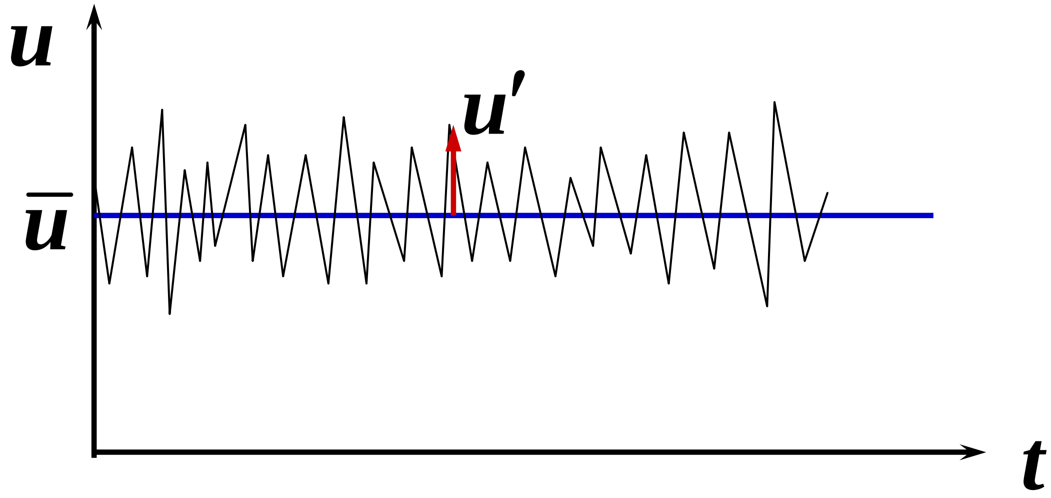
When $\mathrm{Pr} = 1,\ \ \mathrm{St} = \dfrac{\mathrm{Nu}_L}{\mathrm{Re}_L}$

$$C_f = \frac{2\mathrm{Nu}_L}{\mathrm{Re}_L} \rightarrow \frac{C_f}{2} = \mathrm{St}$$

Modified Reynolds analogy or Chilton-Colburn analogy

$$\frac{C_f}{2} = \mathrm{St}\,\mathrm{Pr}^{2/3} = j_H \quad \left(0.6 < \mathrm{Pr} < 60\right)$$

Turbulent Flow



time-average: $\bar{u} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_0}^{t_0+\tau} u\,dt$

instantaneous velocity: $u = \bar{u} + u'$

$\bar{u}$ **: time-averaged velocity**

u' **: fluctuating component**

Turbulent Boundary Layer Equations

continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

momentum:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$

energy:

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_p\overline{v'T'}\right)$$

Turbulence Modeling

Eddy diffusivity for momentum: ε_M

$$\tau_{\text{tot}} = \tau_l + \tau_t = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}$$

Reynolds stress: $\tau_t = -\rho \overline{u'v'}$

Boussinesque type model:

$$\tau_t = -\rho \overline{u'v'} = \rho \varepsilon_M \frac{\partial \bar{u}}{\partial y}$$

$$\tau_{\text{tot}} = \rho \left(\nu + \varepsilon_M \right) \frac{\partial \bar{u}}{\partial y}$$

Eddy diffusivity for heat transfer: ε_H

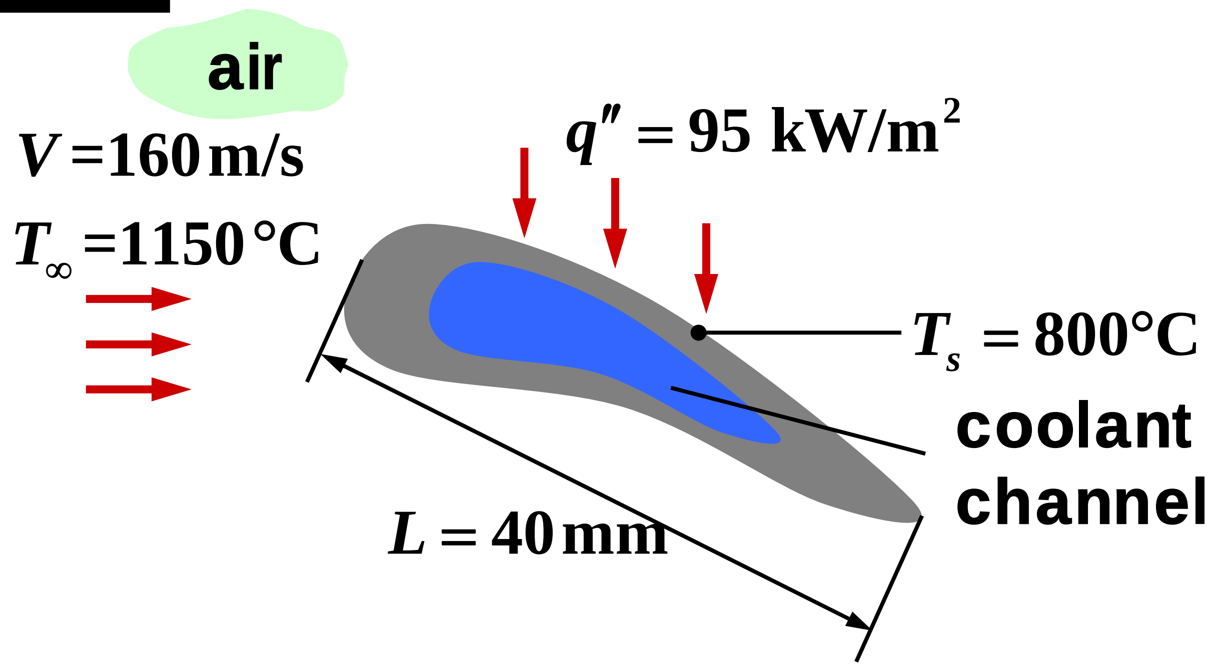
$$q''_{\mathrm{tot}} = q''_l + q''_t = -k\frac{\partial \bar{T}}{\partial y} + \rho c_p \overline{v'T'}$$

Boussinesque type model:

$$q''_t = \rho c_p \overline{v'T'} = -\rho c_p \varepsilon_H \frac{\partial \bar{T}}{\partial y}$$

$$q''_{\mathrm{tot}} = -\rho c_p \left(\alpha + \varepsilon_H\right)\frac{\partial \bar{T}}{\partial y}$$

Example 6.5



original conditions

Find:

1) **heat flux to the blade if the surface temperature is reduced to** $700°\text{C}$

2) **heat flux at the same dimensionless location for a similar blade having a chord length of** $L = 80\ \text{mm}$ **when** $T_\infty = 1150°\text{C}$, $V = 80\ \text{m/s}$, **and** $T_s = 800°\text{C}$

1) $q_1'' = h_1 \left(T_\infty - T_{s,1} \right)$

$$\mathrm{Nu} = \frac{hL}{k} = f\left(x^*, \mathrm{Re}_L, \mathrm{Pr}\right)$$

x^*, Re_L, Pr are independent on T_s.

L and k are also unchanged.

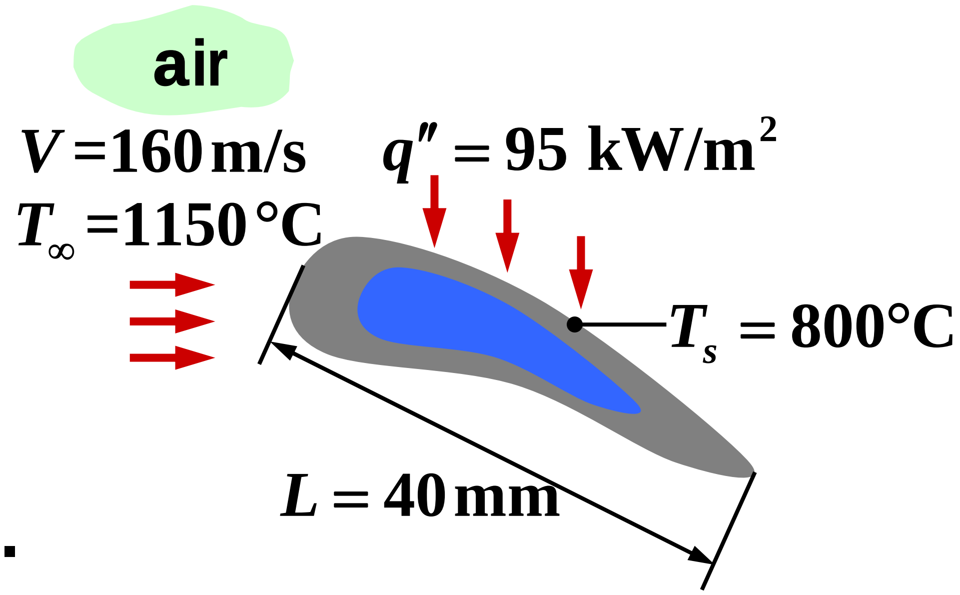
Thus, $h = h_1$

$$q'' = h\left(T_\infty - T_s\right)$$

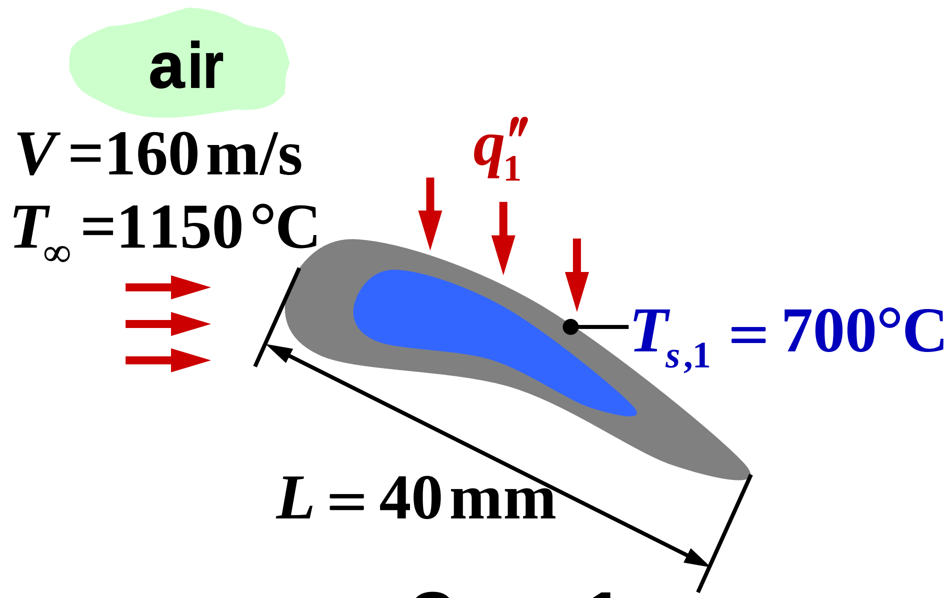
$$h_1 = h = \frac{q''}{T_\infty - T_s}$$

$$q_1'' = \frac{q''\left(T_\infty - T_{s,1}\right)}{T_\infty - T_s}$$

$$= 122\ \mathrm{kW/m^2}$$



original conditions



Case 1

2) $q_2'' = h_2 \left(T_\infty - T_s \right)$

$$\mathrm{Nu} = \frac{hL}{k} = f\left(x^*, \mathrm{Re}_L, \mathrm{Pr}\right)$$

$$\mathrm{Re}_{L,2} = \frac{V_2 L_2}{\nu} = \frac{VL}{\nu} = \mathrm{Re}_L$$

x^*, Pr are also unchanged.

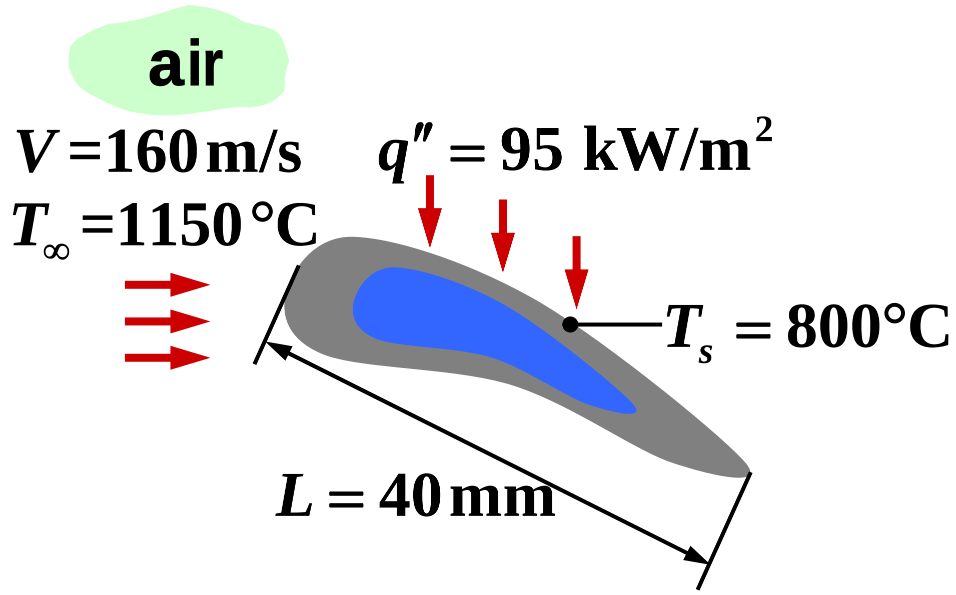
Local Nusselt number remains the same.

$$\mathrm{Nu}_2 = \mathrm{Nu} \rightarrow \frac{h_2 L_2}{k} = \frac{hL}{k}$$

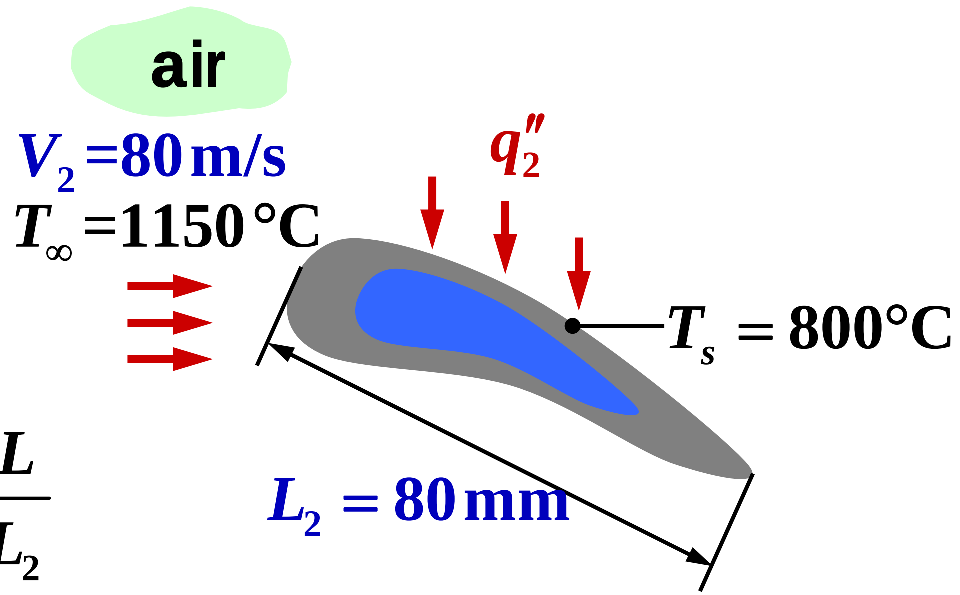
$$h_2 = h\frac{L}{L_2} = \frac{q''}{T_\infty - T_s} \frac{L}{L_2}$$

$$q_2'' = h_2 \left(T_\infty - T_s \right) = \frac{q'' \left(T_\infty - T_s \right)}{T_\infty - T_s} \frac{L}{L_2}$$

$$= 47.5 \, \mathrm{kW/m^2}$$



original conditions



Case 2