Tokama Plasma Physics

Plasma properties

- Quasi-neutrality
- Ability of carrying currents

Plasma dynamics

- Particle orbits
- Kinetic, fluid-like behaviors

Tokamak Plasma confinements

- β =(plasma pressure)/(magnetic pressure)< 0.1
- Collisions : ion collision time 1-100 ms
- Anomalous transport : instabilities, disruptions

Table 2.1.1 Typical tokamak plasmas

Plasma volume	1–100 m^3	
Total plasma mass	10^{-4}–10^{-2}	g
Ion concentration	10^{19}–10^{20}	m^{-3}
Temperature	1–40 keV	
Pressure	0.1–5 atmo	
Ion thermal velocity	100–1000	km/s
Electron thermal velocity	0.01*c*–0.1*c*	
Magnetic field	1–10 T	
Total plasma current	0.1–7 MA	

10^{-5} X atmosphere

10^{5} X atmosphere

Tokama Plasma Properties

- Debye shielding : Debye length

$$\lambda_D = (\varepsilon_o T / ne^2)^{1/2}$$

$$= 2.35 \times 10^5 \left(\frac{T}{n} \right)^{1/2} m, T \, in \, keV$$

λ_D lies in the Range of 10^{-2} to 10^{-1} mm

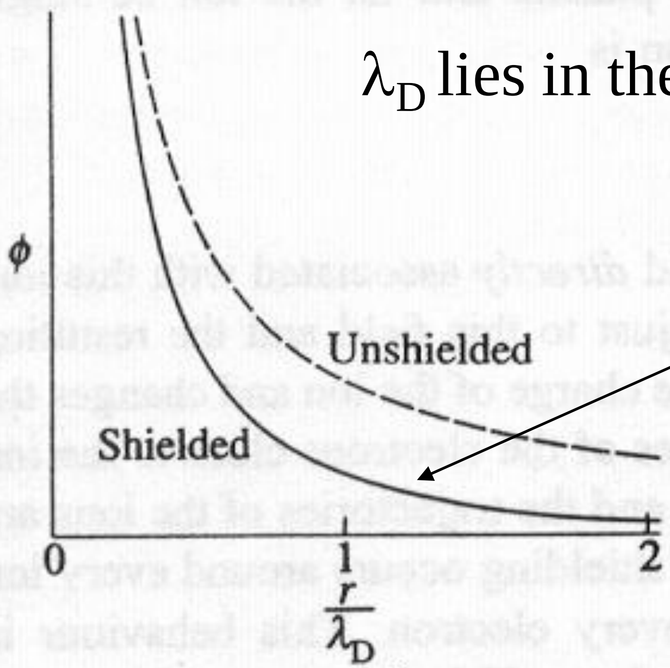


Fig. 2.2.2 Graphs of the unshielded and shielded potential around an ion.

$$\phi = \frac{e}{4\pi\varepsilon_o} e^{-\sqrt{2} r / \lambda_D}$$

Plasma parameter $n\lambda_D^3 >> 1$

Collective behavior

$$\lambda_D << L$$

Quasi-neutrality

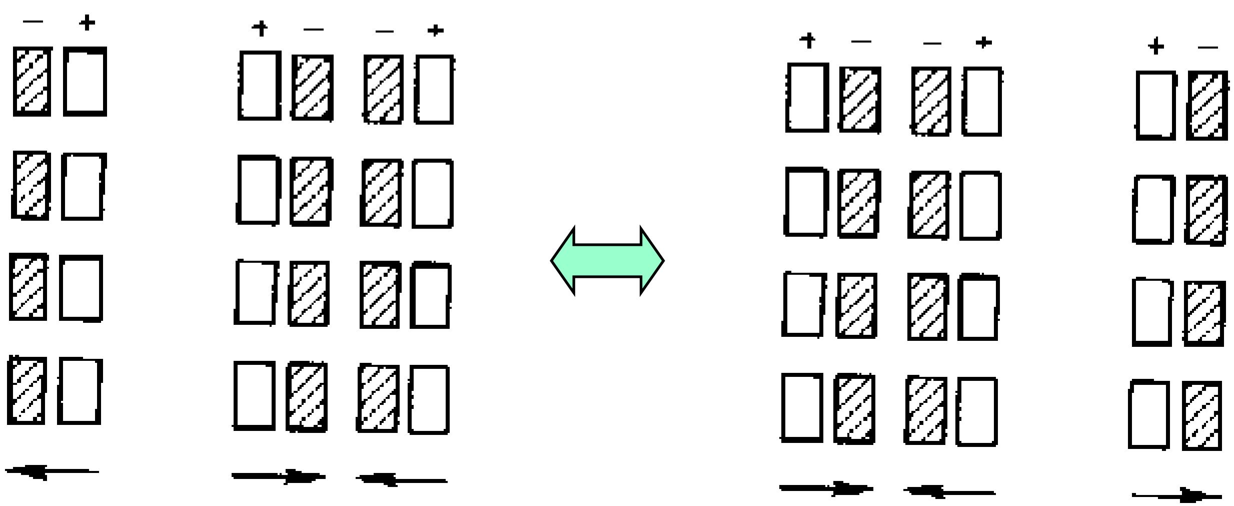
Tokama Plasma Properties

- Plasma oscillation : plasma frequency

$$\omega_p = (ne^2 / m_e \varepsilon_o)^{1/2} = 56.4(n)^{1/2} s^{-1}$$

$$\omega_{pi} = (ne_i^2 / m_i \varepsilon_o)^{1/2}$$

$\omega_p = 5.6\text{x}10^{11}\ s^{-1}$ for $n = 10^{20}\ m^{-3}$



Collective behavior

$\omega_{pe}\ \tau > 1$ (plasma frequency > collision frequency)

Single Particle Motion under Constant Field

- Equation of motion : Lorentz equation $$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$
- Motion in electric fields

$$\left.\begin{aligned} \vec{F} = m\frac{d\vec{v}}{dt} = q\vec{E} = -q\nabla V \\ dW = \vec{F}\cdot d\vec{x} = m\frac{d\vec{v}}{dt}\cdot d\vec{x} = m\vec{v}\cdot d\vec{v} = d(\frac{1}{2}mv^2) \end{aligned}\right\} \nabla(\frac{1}{2}mv^2 + qV) = 0$$

- Cyclotron motion in magnetic fields

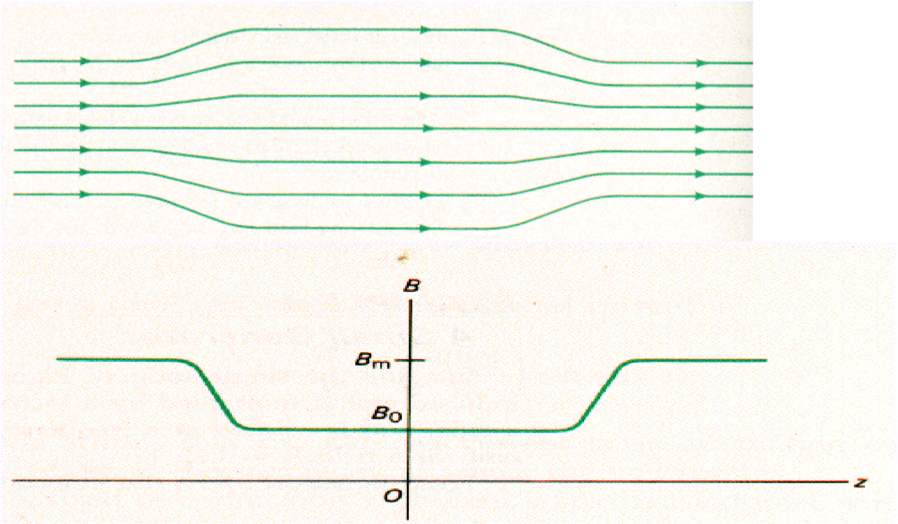
$$\frac{d^2\vec{v}}{dt^2} = \frac{q}{m}\frac{d\vec{v}}{dt}\times\vec{B} = \frac{q^2}{m^2}(\vec{v}\times\vec{B})\times\vec{B} = (\frac{qB}{m})^2\vec{v}$$

Cyclotron frequency : $\omega_c = \dfrac{qB}{m}$, Larmor radius : $r_L = \dfrac{mv}{qB}$

- Magnetization criteria $\omega_c\,\tau > 1 \quad r_L < a$

Particle Motion along the Magnetic Field

$$\vec{F} = m\frac{d\vec{v}}{dt} = -\mu\nabla B \quad \longleftrightarrow \quad \vec{F} = q\vec{E} = -q\nabla V$$



- Magnetic mirrors and nozzles :

magnetic moment conserved!

$$\mu = IA = \left(\frac{e\,\omega_c}{2\pi}\right)\left(\frac{\pi\, v_\perp^2}{\omega_c^2}\right) = \frac{mv_\perp^2/2}{B}$$

Single Particle Motion : Cross-Field Drift

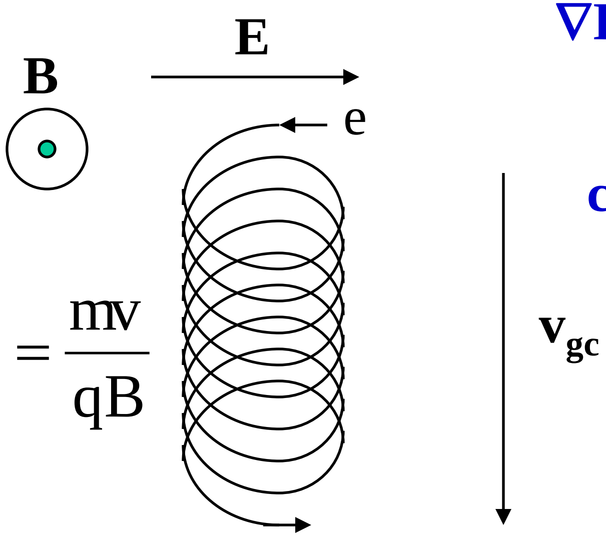
- Cross-field guiding center motion : ExB drift motions

$$\vec{v}_{EXB} = \frac{\vec{E}\times\vec{B}}{B^2}$$

$$0 = (\vec{E} + \vec{v}\times\vec{B})\times\vec{B} = \vec{E}\times\vec{B} + \vec{B}(\vec{v}\cdot\vec{B}) - \vec{v}B^2 = \vec{E}\times\vec{B} - v_\perp B^2$$

For general forces perpendicular to the magnetic field

$$\vec{v}_F = \frac{\vec{F}\times\vec{B}}{qB^2}$$



$$r_L = \frac{mv}{qB}$$

Electron ExB drift motion

∇B drift

$$\vec{F} = m\frac{d\vec{v}}{dt} = -\mu\nabla B \qquad \vec{v}_{\nabla B} = \frac{-\mu\nabla B\times\vec{B}}{qB^2}$$

curvature drift

$$\vec{F} = m\frac{d\vec{v}}{dt} = \frac{mv_{//}^2}{R^2}\nabla R \qquad \vec{v}_C = \frac{-mv_{//}^2\nabla B\times\vec{B}}{qB^3}$$

polarization drift

$$\vec{v}_C = \frac{1}{\omega_c B}\frac{d\vec{E}}{dt} = \frac{m}{qB^2}\frac{d\vec{E}}{dt}$$

$$\vec{j}_p = \frac{\rho_m}{B^2}\frac{d\vec{E}}{dt}$$ **polarization current density**

Results of Collisions

- **Plasma diffusion and other transport processes**
- **Transfer energy between particles**
- **Responsible for the electrical resistivity of the plasma**

Complexity of Collision Processes

- **The concept of collision in a plasma is subtle**
- **Relative velocity dependency of collision processes**
- **Integrated effects of interactions between particles of all velocities**
- **Effective collision frequency depends on the specific process**

Coulomb Collisions :
Large-angle scattering and small-angle collisions

Effective collision cross-section

$$4\sigma_o \ln \Lambda$$

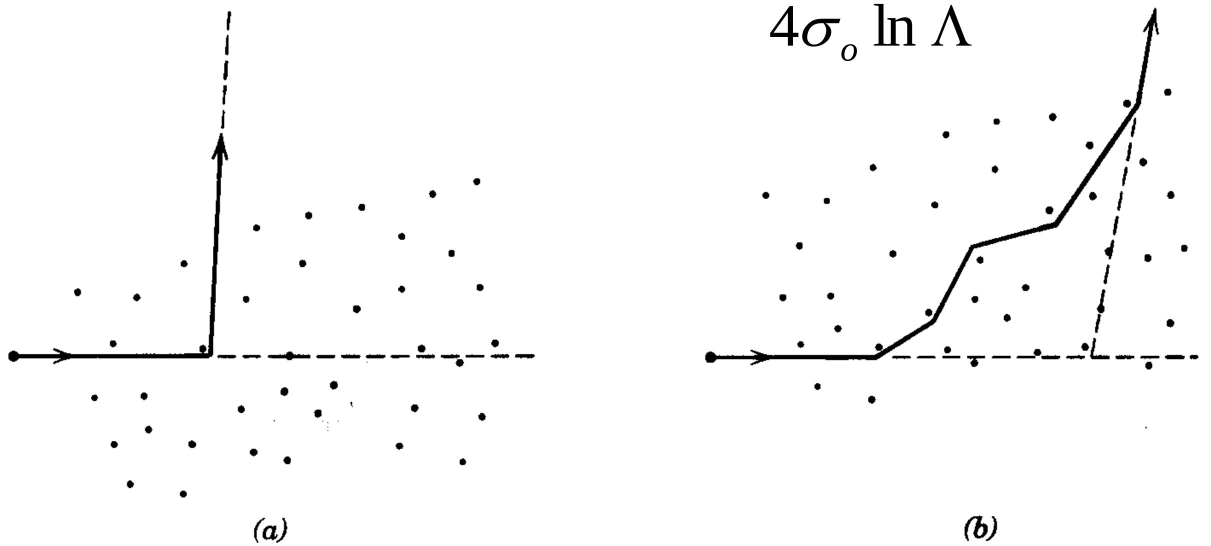


FIGURE 3.6. The processes that lead to large-angle Coulomb scattering: (a) single large-angle event; (b) cumulative effect of many small-angle events.

- Impact parameters for small angle Coulomb scattering
  - minimum from Coulomb force balance
  - maximum from Debye length
  - ratio gives importance of small angle scattering

$$\frac{e^2}{4\pi\varepsilon_o r_o^2} = \frac{mv^2}{r_o} \qquad \lambda_D = \frac{\varepsilon_o kT}{ne^2}$$

$$\Lambda \sim \frac{r_{max}}{r_{min}} \sim \frac{\lambda_D}{r_o} \sim \frac{Z}{12\pi} n \lambda_D^3$$

Binary Coulomb Interaction inside the Debye Sphere

Rutherford scattering

$$\cot(\chi/2) = \frac{4\pi\varepsilon_o m_e v^2 r}{e^2}$$

$$r_o = \frac{e^2}{4\pi\varepsilon_o m_e v^2}$$

$$\sigma_o = \pi r_o^2 = \frac{e^4}{64\pi\varepsilon_o^2 E_e^2}$$

Parallel momentum change

$$\delta p = -\frac{2m_e v}{1+(r/r_o)^2}$$

$$\frac{dp}{dt} = -2nm_e v^2 \int_0^{\lambda_D} \frac{2\pi r dr}{1+(r/r_o)^2}$$

$$= -2nm_e v^2 \sigma_o \ln(1+(r/r_o)^2)\Big|_0^{\lambda_D} = -4\ln\Lambda n m_e v^2 \sigma_o$$

$$\lambda_D >> r_o$$

$$\ln\Lambda = \ln(\lambda_D / r_o)$$

Coulomb logarithm

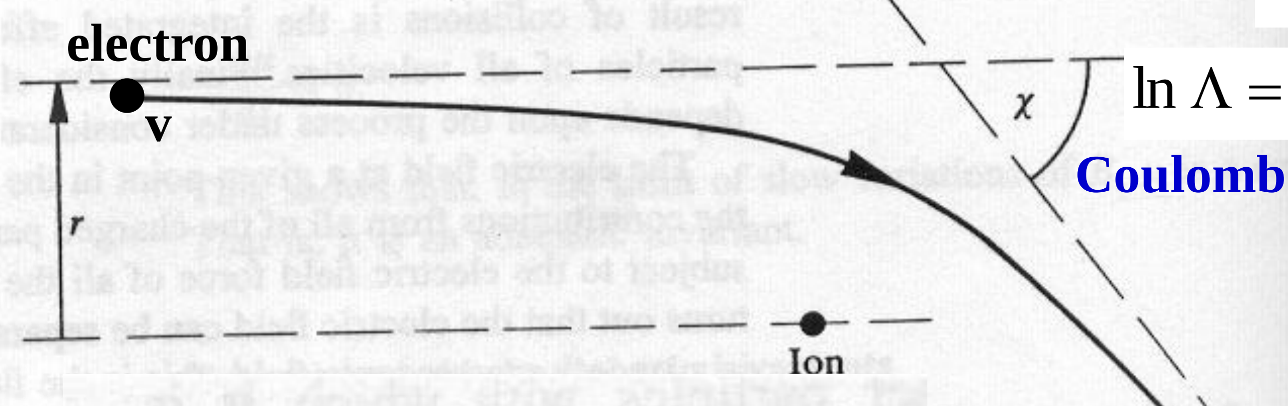
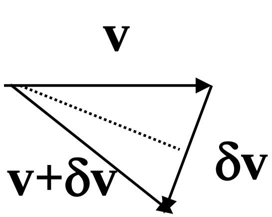


Fig. 2.8.1 Geometry of electron–ion collision with impact parameter r and scattering angle χ.



Coulomb Logarithm

electron–electron collisions

$$\ln\Lambda = 14.9 - \tfrac{1}{2}\ln(n_e/10^{20}) + \ln T_e, \qquad T_e \text{ in keV}$$

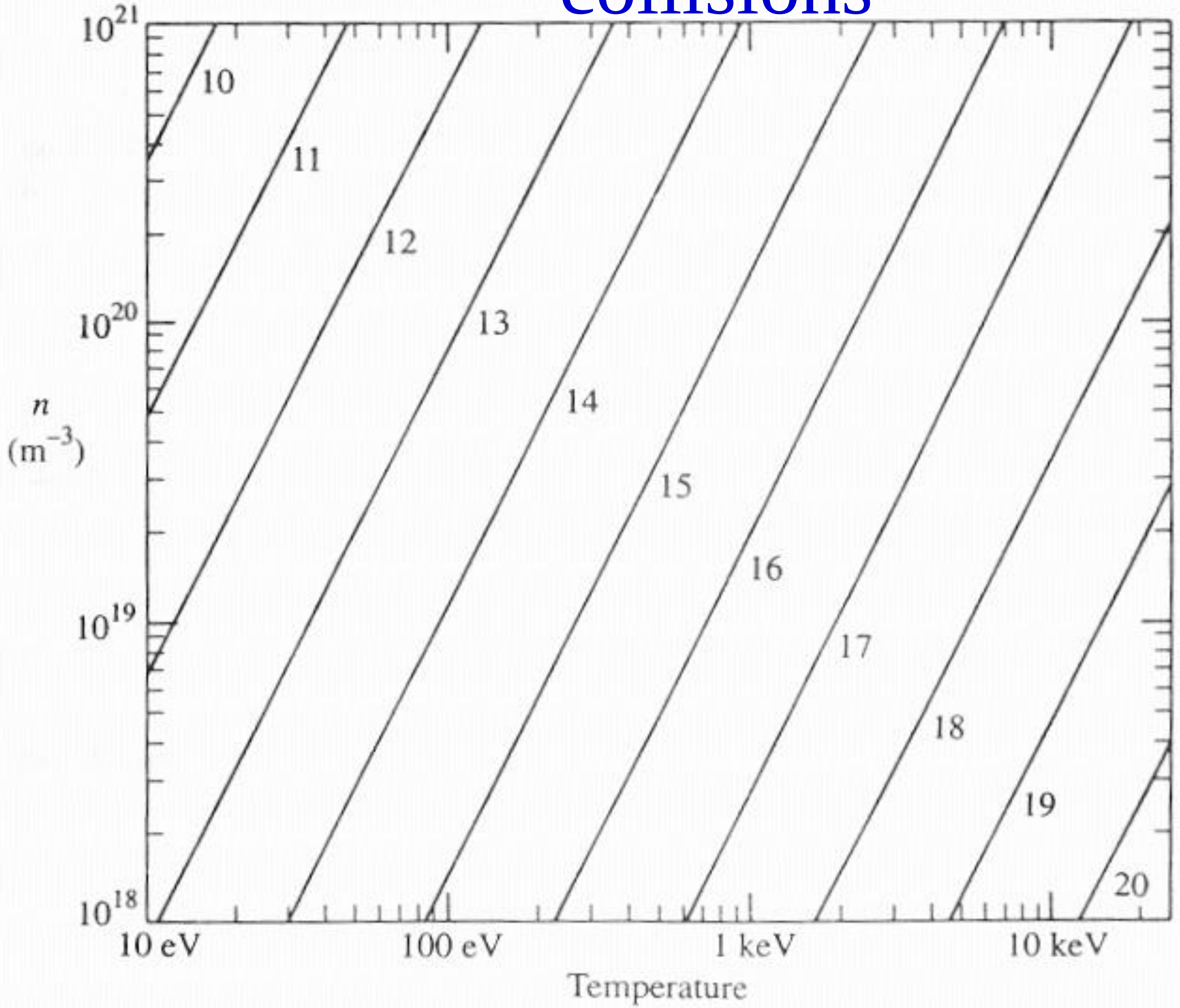
electron–ion collisions $(T \gtrsim 10\,\text{eV})$

$$\ln\Lambda = 15.2 - \tfrac{1}{2}\ln(n_e/10^{20}) + \ln T_e, \qquad T_e \text{ in keV}$$

ion–ion collisions (singly charged ions, $T \lesssim 10(m_i/m_p)$ keV

$$\ln\Lambda = 17.3 - \tfrac{1}{2}\ln(n_e/10^{20}) + \tfrac{3}{2}\ln T, \qquad T \text{ in keV.}$$

Coulomb Logarithm for electron-ion collisions



Relative importance of Coulomb Collisions

- Coulomb collision cross section $\sigma_i \sim \pi b^2 \sim \dfrac{\pi e^4}{(4\pi 4_o)^2 T_e^2} \sim 10^{-17} \dfrac{1}{T_e^2(eV)} m^2$
- Effect of small angle scattering : $\ln \Lambda$ $\sigma_i \sim \pi b^2 \ln \Lambda \sim 10^{-17} \dfrac{\ln \Lambda}{T_e^2(eV)} m^2$
- In order to define an average electron collision frequency, evaluate the frictional force on a distribution of electrons drifting through essentially stationary ions $\vec{F} = -n_e m \langle \nu_{ei} \vec{v} \rangle$
- Collision frequency $\nu_{ei} = n_e < \sigma_{ei} v_e >$
- Various Coulomb collision frequencies from the frictional forces

$$F_z = -n_e m < \nu_{ei} > u_z$$

$$< \nu_{ei} > = \frac{2^{1/2} n_i Z^2 e^4 \ln \Lambda}{12 \pi^{3/2} \varepsilon_o^2 m^{1/2} T_e^{3/2}}$$

$$< \nu_{ee} > \approx \frac{n_e e^4 \ln \Lambda}{\varepsilon_o^2 m^{1/2} T_e^{3/2}} \approx \frac{< \nu_{ei} >}{n_i Z^2 / n_e}$$

$$< \nu_{ii} > = \frac{n_i Z^4 e^4 \ln \Lambda}{12 \pi^{3/2} \varepsilon_o^2 M^{1/2} T_i^{3/2}} \approx \left(\frac{m T_e^3}{M T_i^3}\right)^{1/2} < \nu_{ei} >$$

Kinetic Equations

- Liouville equation and kinetic equations
- Fokker-Planck equation
- Gyro-averaged kinetic equations
- Fokker-Planck equation for a plasma
- Fokker-Planck coefficients for Maxwellian distributions

Liouville Equation and Kinetic Equations

- **Distribution function** $F(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_N, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N, t)$

$$f = f_j^{(1)}(\vec{x}_1, \vec{v}_1, t) = \int F(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_N, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N, t) d\vec{x}_2 \cdots d\vec{x}_N d\vec{v}_2 \cdots d\vec{v}_N$$

- **Liouville equation**

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_i \left(\frac{\partial F}{\partial \vec{x}_i} \cdot \vec{v}_i + \frac{\partial F}{\partial \vec{v}_i} \cdot \vec{a}_i \right) = 0$$

- **Vlasov equation**

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} = \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{e_j(\vec{E} + \vec{v} \times \vec{B})}{m_j} \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

- **Fokker-Planck equation**

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{e_j(\vec{E} + \vec{v} \times \vec{B})}{m_j} \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t} \right)_c$$

- **Drift kinetic equation** : slowly varying compared to gyro motion

$$f(\vec{x}, v_{ll}, v_{\perp})$$

- **Gyro-kinetic equation :** electromagnetic fields vary significantly across gyro radius --> averaging their effect over gyro radius

Fokker-Planck Equation

- **Fokker-Planck equation contains the cumulative contribution of many small angle scattering.**

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{f(\vec{x},\vec{v},t+\Delta t) - f(\vec{x},\vec{v},t)}{\Delta t}$$

$$f(\vec{x},\vec{v},t+\Delta t) = \int f(\vec{x},\vec{v}-\Delta\vec{v},t)\psi(\vec{v}-\Delta\vec{v},\Delta\vec{v})d(\Delta\vec{v})$$

$$f(\vec{x},\vec{v}-\Delta\vec{v},t)\psi(\vec{v}-\Delta\vec{v},\Delta\vec{v}) = f(\vec{x},\vec{v},t)\psi(\vec{v},\Delta\vec{v})$$

$$-\sum_\alpha \frac{\partial}{\partial v_\alpha}(f\psi)\Delta v_\alpha + \frac{1}{2}\sum_{\alpha,\beta}\frac{\partial^2}{\partial v_\alpha \partial v_\beta}(f\psi)\Delta v_\alpha \Delta v_\beta$$

$$\int \psi(\vec{v},\Delta\vec{v})d(\Delta\vec{v}) = 1$$

$$f(\vec{x},\vec{v},t+\Delta t) - f(\vec{x},\vec{v},t)$$

$$= -\sum_\alpha \frac{\partial}{\partial v_\alpha}\left(f(\vec{x},\vec{v},t)\int \psi(\vec{v},\Delta\vec{v})\Delta v_\alpha d(\Delta\vec{v})\right)$$

$$+\frac{1}{2}\sum_{\alpha,\beta}\frac{\partial^2}{\partial v_\alpha \partial v_\beta}\left(f(\vec{x},\vec{v},t)\int \psi(\vec{v},\Delta\vec{v})\Delta v_\alpha \Delta v_\beta d(\Delta\vec{v})\right)$$

$< \Delta v_\alpha > \Delta t$ — **coefficient of dynamic friction**

$< \Delta v_\alpha \Delta v_\beta > \Delta t$ — **diffusion tensor**

Then,

$$\left(\frac{\partial f}{\partial t}\right)_c = -\sum_\alpha \frac{\partial}{\partial v_\alpha}(< \Delta v_\alpha > f) + \frac{1}{2}\sum_{\alpha,\beta}\frac{\partial^2}{\partial v_\alpha \partial v_\beta}(< \Delta v_\alpha \Delta v_\beta > f)$$

Gyro-Averaged Kinetic Equation

- Simpler kinetic equation averaged over the fast Larmor motion is appropriate for slowly varying plasma phenomena.

  Independent variables: **r**, **v** --> **r**, $v_\perp$, $v_{//}$

- For the motion slower than the bounce frequency along the field line, averaging over this motion reduces the phase space to four dimensions.

- **Drift Kinetic Equation** is the equation for the gyro-averaged distribution function. $\bar{f} = \frac{1}{2\pi}\int f d\phi$

By expanding the Fokker-Planck equation with $\frac{1}{\omega_c}\frac{\partial}{\partial t} << 1$ $\quad \frac{\rho}{L} << 1$

$$\frac{\partial \bar{f}}{\partial t} + \vec{v}_g \cdot \nabla \bar{f} + [e_j \vec{E}\cdot\vec{v}_g + \mu\frac{\partial B}{\partial t})\frac{\partial \bar{f}}{\partial K} = \left(\frac{\partial \bar{f}}{\partial t}\right)_c \qquad \mu = \frac{m_j v_\perp^2/2}{B} \qquad K = \frac{1}{2}m_j v^2$$

convection

$$\vec{v}_g = v_{//}\frac{\vec{B}}{B} + \frac{\vec{E}\times\vec{B}}{B^2} + \frac{(v_{//}^2 + v_\perp^2/2)\vec{B}\times\nabla B}{\omega_{cj}B^2}$$

Gyro-Kinetic Equation

- Procedure of averaging over the fast Larmor motion can be applied to some components of the electromagnetic fields vary significantly across a particle's Larmor orbit.
- Collisionless Vlasov equations for $k_\perp v_\perp / \omega_c \sim O(1)$ with the perturbed quantities in the form of $e^{-i\omega t + i\vec{k}_\perp \cdot \vec{x}}$, and writing $f = f_o + \delta f$ and $\boldsymbol{E} = \boldsymbol{E}_o - \nabla\phi - i\omega\boldsymbol{A}$ then for isotropic f_o

$$\delta f = e_j \phi \frac{\partial f_o}{\partial K} + g e^{iL}$$

Gyro-kinetic equation $g(\mu, K, \vec{x})$

$$L = \frac{v_\perp \times \vec{B} \cdot \vec{k}_\perp}{B\omega_{cj}}$$

$$\frac{\partial g}{\partial t} + v_{//} \frac{\vec{B}}{B} \cdot \nabla g + i\vec{k}_\perp \cdot \vec{v}_g g$$

$$= -[\omega \frac{\partial f_o}{\partial K} - \frac{\vec{B} \times \nabla f_o \cdot \vec{k}_\perp}{B\omega_{cj}}][J_o(z) e_j (\phi - v_{//} A_{//}) + \frac{2J_1(z)}{z} \mu B_{//}] \qquad z = k_\perp v_\perp / \omega_{cj}$$

- **In the long wavelength limit (z-> 0 and L-> 0), f and g would be obtained by linearizing the drift kinetic equation.**
- **Bounce averaged drift kinetic equation, or bounce averaged gyro-kinetic equation for low frequency by averaging over longitudinal motion : $x_\perp$, μ , K**

Relaxation Processes

- **Slowing-down time** $$\tau_s = -\frac{v}{< dv/dt >} = -\frac{v}{< \Delta v_{//} >}$$
- **Deflection time** $$\tau_d = \frac{v^2}{< dv_\perp^2 / dt >} = \frac{v^2}{< (\Delta v_\perp)^2 >}$$
- **Energy balance equation**

$$\frac{dE_f}{dt} = -< \Delta E > = -\frac{m}{2}(2v < \Delta v_{//} > + < (\Delta v_{//})^2 > + < (\Delta v_\perp)^2 >)$$

$$= -F_d v - \frac{m}{2}(< (\Delta v_{//})^2 > + < (\Delta v_\perp)^2 >)$$

$F_d = m < \Delta v_{//} >$ **: dynamic friction force**

rate of energy transfer to field particles

work done by test particles against dynamic friction

energy increases of test particles

- **Fast test particles** $v / v_{T1} >> 1$
- **Slow test particles** $v / v_{T1} << 1$

Relaxation Processes

- **Heat exchange time** τ_{ij} $$\frac{dT_i}{dt} = \frac{T_j - T_i}{\tau_{ij}}$$

$$\frac{3}{2} n_i \frac{dT_i}{dt} = \int <\Delta E>_{ij} \ f_i 4\pi v_i^2 dv_i$$

for Maxwellian $f_i(v_i) = \frac{n_i}{(2\pi)^{3/2} v_{Ti}^3} e^{-(\frac{v_i^2}{2v_{Ti}^2})}$

$$<\Delta E>_{ij} = -\frac{n_j e^4 Z_i^2 Z_j^2 \ln\Lambda}{4\pi\varepsilon_o^2 m_j}\left(\frac{\Phi(v_i/\sqrt{2}v_{Tj})}{v_i} + (1+\frac{m_j}{m_i})\frac{\Phi'(v_i/\sqrt{2}v_{Tj})}{\sqrt{2}v_{Tj}}\right)$$

$$I = \int \left(v_i\Phi(v_i/\sqrt{2}v_{Tj}) - (1+\frac{m_j}{m_i})\frac{v_i^2\Phi'(v_i/\sqrt{2}v_{Tj})}{\sqrt{2}v_{Tj}}\right) e^{-\frac{v_i^2}{2v_{Ti}^2}} dv_i$$

$$= \frac{T_i - T_j}{(v_{Ti}^3 + v_{Tj}^3)^{3/2}}$$

$$\frac{dT_i}{dt} = -\frac{2n_j e^4 Z_i^2 Z_j^2 \ln\Lambda}{3(2\pi)^{3/2}\varepsilon_o^2 m_i m_j} I$$

for e-i heat exchange $$\tau_{ie} = \tau_{ei} = \frac{m_i}{2m_e}\tau_e$$

$$\tau_{ij} = \frac{3(2\pi)^{3/2}\varepsilon_o^2 m_i m_j}{n_j e^4 Z_i^2 Z_j^2 \ln\Lambda}\left(\frac{T_i}{m_i} + \frac{T_j}{m_j}\right)^{3/2}$$

electron collision frequency $$\tau_e = \frac{3(2\pi)^{3/2}\varepsilon_o^2 m_e^{1/2} T_e^{3/2}}{n e^4 \ln\Lambda}$$

Collision Times

- **Characteristic collision times by taking the test particle to have the average thermal velocity** $m_e v_{Te}^2 = T$

$$\tau_c = \frac{3(2\pi)^{3/2}\varepsilon_o^2 m^{1/2} T^{3/2}}{ne^4 \ln\Lambda}$$

Electron/ion collision times for a plasma with ions of charge Z,

$$\tau_e = \frac{3(2\pi)^{3/2}\varepsilon_o^2 m_e^{1/2} T_e^{3/2}}{n_i Z^2 e^4 \ln\Lambda} \qquad \tau_i = \frac{12\pi^{3/2}\varepsilon_o^2 m_i^{1/2} T_i^{3/2}}{n_i Z^4 e^4 \ln\Lambda} \qquad \tau_e/\tau_i \sim (\frac{m_e}{m_i})^{1/2}$$

$$\tau_e(s) = 1.09\times 10^{16}\frac{T_e(keV)^{3/2}}{n_i Z^2 \ln\Lambda} \qquad \tau_i(s) = 6.60\times 10^{17}\frac{(m_i/m_p)^{1/2} T_i(keV)^{3/2}}{n_i Z^4 \ln\Lambda}$$

Resistivity

- **Force due to electric field is balanced by the frictional force due to collisions,**

Time for electron momentum loss

$$Ee = \frac{m_e v_d}{\tau_c} \qquad E = \eta J = \eta n_e e v_d \longrightarrow \eta = \frac{m_e}{n_e e^2 \tau_c}$$

roughly, $\tau_c \approx \tau_e$

more accurately by taking account of electron-electron collision, Spitzer resistivity

$$\eta_S = 0.51 \frac{m_e}{n_e e^2 \tau_e} = 0.51 \frac{m_e^{1/2} e^2 \ln \Lambda}{3\varepsilon_o^2 (2\pi T_e)^{3/2}}$$

$$= 1.65 \times 10^{-9} \ln \Lambda / T_e(keV)^{3/2}$$

In the presence of magnetic field,

$$\eta_\perp = \frac{m_e}{n_e e^2 \tau_e} = 1.96 \eta_{//}$$

For multi-species,

$$Ee = m_e v_d \sum_j \frac{1}{\tau_{cj}}$$

$$E = \eta \sum_j n_j Z_j e v_d$$

$$\longrightarrow$$

$$\eta = \frac{m_e}{e^2} \frac{\sum_j 1/\tau_{cj}}{\sum_j n_j Z_j}$$

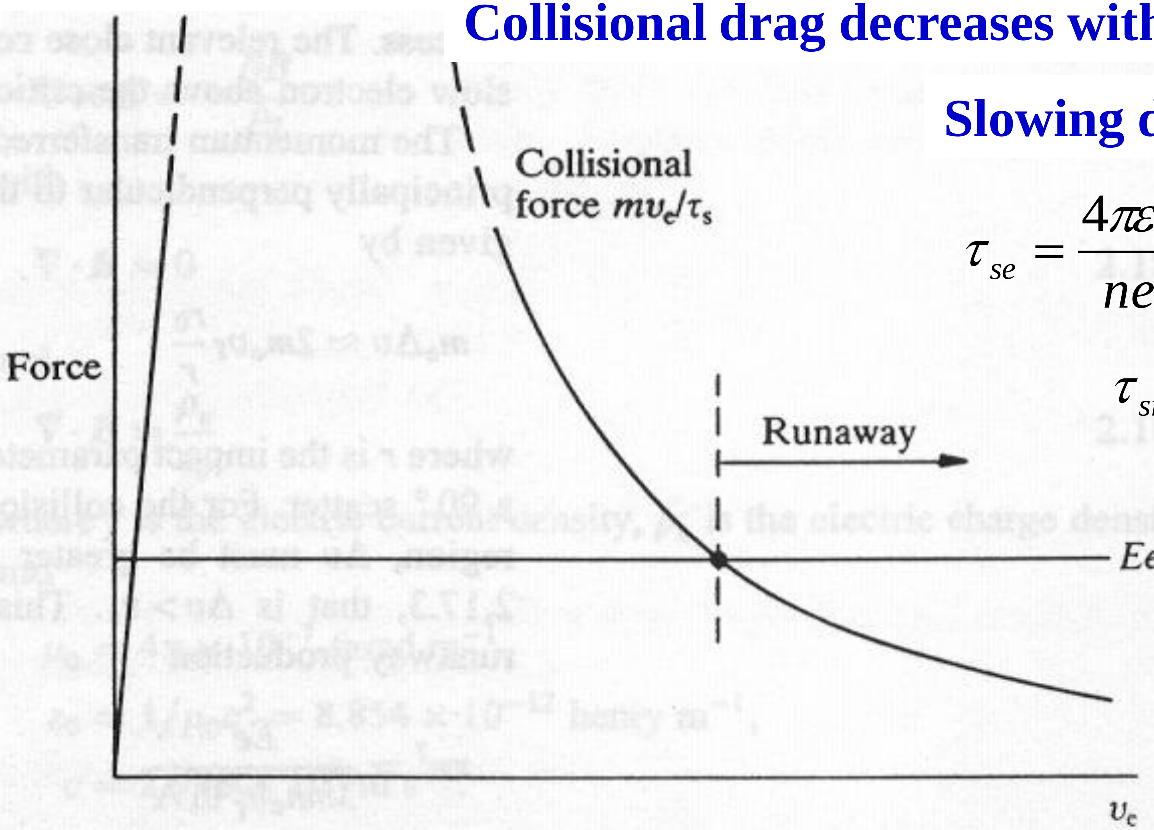
$$= \eta_S \frac{\sum_j n_j Z_j^2}{\sum_j n_j Z_j} = \eta_S Z_{eff}$$

Runaway Electrons

When the electric field force exceeds the collisional force for an electron, the electron would runaway.

$$Ee > \frac{m_e v_e}{\tau_s} \qquad \tau_c = \frac{3(2\pi)^{3/2}\varepsilon_o^2 m^{1/2} T^{3/2}}{ne^4 \ln\Lambda}$$

Collisional drag decreases with increasing velocity as $1/v_e^2$



Slowing down time for fast electrons

$$\tau_{se} = \frac{4\pi\varepsilon_o^2 m^2 v_e^2}{ne^4 \ln\Lambda} \qquad \tau_s^{-1} = \tau_{se}^{-1} + \tau_{si}^{-1}$$

$$\tau_{si} = \tau_{se}/2 \qquad \tau_s = \frac{1}{3}\frac{4\pi\varepsilon_o^2 m^2 v_e^2}{ne^4 \ln\Lambda}$$

Critical velocity

$$v_c^2 = \frac{3ne^3 \ln\Lambda}{4\pi\varepsilon_o^2 m_e E}$$

Electromagnetism and Plasma Fluid

- Maxwell’s Equations in General Form
- Scalar and Vector Potentials
- Poynting’s Theorem
- Fluid Equations
- Magnetohydrodynamics (MHD)

Maxwell's Equations in General Form

$$\nabla \cdot \vec{D} = \rho_f \qquad \nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

In a medium, $\vec{D} = \varepsilon_o \vec{E} + \vec{P} \qquad \vec{H} = \vec{B}/\mu_o - \vec{M}$

In linear isotropic homogeneous media, $\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H} \qquad \vec{J}_f = \sigma \vec{E}$

dielectric constant (→ ε) permeability (→ μ)

In a plasma, $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} = \varepsilon_o \overleftrightarrow{K} \cdot \frac{\partial \vec{E}}{\partial t}$

dielectric tensor (→ $\overleftrightarrow{K}$)

Maxwell's Equations in Integral Form

$$\oint_S \vec{D} \cdot d\vec{a} = \int_V \rho_f \, d\tau = q$$

$$\oint_S \vec{B} \cdot d\vec{a} = 0$$

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{a} = -\frac{\partial \Phi_B}{\partial t}$$

$$\oint_C \vec{H} \cdot d\vec{s} = \int_S \vec{J}_f \cdot d\vec{a} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{a} = \int_S \vec{J}_f \cdot d\vec{a} + \frac{\partial \Phi_E}{\partial t}$$

Scalar and Vector Potentials

$$\nabla \cdot \vec{B} = 0 \longrightarrow \vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t)$$ Vector potential

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial \nabla \times \vec{A}}{\partial t} \longrightarrow \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0 \longrightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi$$

Scalar potential

Gauge transform $\vec{A}' = \vec{A} + \nabla f \qquad \phi' = \phi - \frac{\partial f}{\partial t}$

Lorentz gauge $\nabla^2 f - \mu \varepsilon \frac{\partial^2 f}{\partial t^2} = 0$

Satisfying Lorentz condition

$$\nabla \cdot \vec{A} + \mu \varepsilon \frac{\partial \phi}{\partial t} = 0$$

$$\nabla^2 \phi - \mu \varepsilon \frac{\partial^2 \phi}{\partial t^2} = -\rho_f / \varepsilon$$

$$\nabla^2 \vec{A} - \mu \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}_f$$

Poynting's Theorem

$$\nabla\cdot(\vec{E}\times\vec{H})=\vec{H}\cdot\nabla\times\vec{E}-\vec{E}\cdot\nabla\times\vec{H}=\vec{H}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right)-\vec{E}\cdot\left(\vec{J}_f+\frac{\partial\vec{D}}{\partial t}\right)$$

$$=-\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}-\vec{E}\cdot\vec{J}_f=-\vec{H}\cdot\mu\frac{\partial\vec{H}}{\partial t}-\vec{E}\cdot\varepsilon\frac{\partial\vec{E}}{\partial t}-\vec{E}\cdot\vec{J}_f=-\frac{\partial}{\partial t}\frac{1}{2}(\mu H^2+\varepsilon E^2)-\vec{E}\cdot\vec{J}_f$$

Integrating over a fixed volume, Poynting's theorem comes out as following:

$$\int_V\nabla\cdot(\vec{E}\times\vec{H})d\tau=\oint_S\vec{E}\times\vec{H}\cdot d\vec{a}$$ rate of energy flow across the boundary

$$=-\frac{\partial}{\partial t}\int_V\frac{1}{2}(\mu H^2+\varepsilon E^2)d\tau-\int_V\vec{E}\cdot\vec{J}_f d\tau$$ energy dissipated into heat

electromagnetic energy contained in the volume V

- Poynting vector : energy current density or power flux $\quad \vec{S}=\vec{E}\times\vec{H}$
- electromagnetic energy density

$$u=u_m+u_e=\frac{1}{2}(\mu H^2+\varepsilon E^2)$$

Fluid Equations

Kinetic equation with distribution function, $f(\vec{x},\vec{v},t)$

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\frac{\partial f}{\partial \vec{x}}+\frac{e_j(\vec{E}+\vec{v}\times\vec{B})}{m_j}\cdot\frac{\partial f}{\partial \vec{v}}=\left(\frac{\partial f}{\partial t}\right)_c$$

$\longrightarrow$ Fluid variables, $n(\vec{x},t)$ $\vec{u}(\vec{x},t)$ $p(\vec{x},t)$

Multiplied by chosen function $\phi(\mathbf{v})$ and integrated over velocity space,

Particle density $n(\vec{x},t)=\int f(\vec{x},\vec{v},t)d\vec{v}$

Fluid velocity $\vec{u}(\vec{x},t)=\frac{1}{n}\int f(\vec{x},\vec{v},t)\vec{v}d\vec{v}$

Pressure tensor $\overleftrightarrow{P}(\vec{x},t)=m\int f(\vec{x},\vec{v},t)(\vec{v}-\vec{u})(\vec{v}-\vec{u})d\vec{v}=\overleftrightarrow{\Pi}+p\overleftrightarrow{1}$

For an isotropic distribution function, the **pressure** is

$$p(\vec{x},t)=\frac{m}{3}\int f(\vec{x},\vec{v},t)(\vec{v}-\vec{u})^2d\vec{v}$$

Fluid Equations

Multiplied by chosen function $\phi(\mathbf{v})$ and integrated over velocity space,

$\phi = 1$ $$n(\vec{x},t) = \int f(\vec{x},\vec{v},t)d\vec{v}$$

$$\frac{\partial(\vec{E}+\vec{v}\times\vec{B})}{\partial\vec{v}} = 0$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial\vec{x}}\cdot\int\vec{v}fd\vec{v} + \frac{1}{m}\int\frac{\partial\vec{F}}{\partial\vec{v}}fd\vec{v} = 0 \longrightarrow \frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0$$

(The term $\frac{1}{m}\int\frac{\partial\vec{F}}{\partial\vec{v}}fd\vec{v}$ is crossed out with an arrow to 0.)

Continuity equation

$\phi = m\vec{v}$ $$\vec{u}(\vec{x},t) = \frac{1}{n}\int f(\vec{x},\vec{v},t)\vec{v}d\vec{v}$$

$$m\frac{\partial(n\vec{u})}{\partial t} + m\frac{\partial}{\partial\vec{x}}\cdot\int\vec{v}\vec{v}fd\vec{v} - \int\frac{\partial(\vec{F}\vec{v})}{\partial\vec{v}}fd\vec{v} = \int m\vec{v}\left(\frac{\partial f}{\partial t}\right)_c d\vec{v}$$

$$mn\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla\cdot\overleftrightarrow{P} + n\vec{F} + \vec{R}$$

Equation of motion

$\phi = m\vec{v}\vec{v}/2$ $$\overleftrightarrow{P}(\vec{x},t) = m\int f(\vec{x},\vec{v},t)(\vec{v}-\vec{u})(\vec{v}-\vec{u})d\vec{v}$$

For closure, approximations such as adiabatic motion are used. $$\frac{\nabla p}{p} = \gamma\frac{\nabla n}{n}$$

Formal validity check with mean free path, Larmor radius, and macroscopic length

Two Fluid Equations

Plasmas are assumed to be composed of two fluids such as electron and ion fluids.

Continuity equations

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{u}_s) = 0 \qquad \text{s = e, i}$$

Equations of motion

$$\vec{F}_s = e_s(\vec{E} + \vec{u}_s \times \vec{B})$$

$$m_s n_s \left(\frac{\partial \vec{u}_s}{\partial t} + \vec{u}_s \cdot \nabla \vec{u}_s\right) = -\nabla \cdot \overleftrightarrow{P}_s + n_s \vec{F}_s + \vec{R}_s = -\nabla p_s + n_s e_s (\vec{E} + \vec{u}_s \times \vec{B}) + \vec{R}_s$$

Equations of state

$$\frac{\nabla p_s}{p_s} = \gamma \frac{\nabla n_s}{n_s}$$

Maxwell's equations

$$\nabla \cdot \vec{\mathrm{E}} = \rho / \varepsilon_o = (n_i e_i + n_e e_e) / \varepsilon_o \qquad \nabla \cdot \vec{\mathrm{B}} = 0$$

$$\nabla \times \vec{\mathrm{E}} = -\frac{\partial \vec{\mathrm{B}}}{\partial \mathrm{t}} \qquad \nabla \times \vec{\mathrm{B}} = \mu_o \vec{\mathrm{j}} = \mu_o (n_i e_i \vec{u}_i + n_e e_e \vec{u}_e)$$

Single Fluid Equations : MHD

Mass density $\rho_m = \sum_s m_s n_s \approx n m_i$ **Charge density** $\rho = \sum_s e_s n_s \approx 0$

Quasi neutrality

Current density $\vec{j} = \sum_s e_s n_s \vec{u}_s \approx ne(\vec{u}_i - \vec{u}_e)$

Center of mass velocity $\vec{v} = \frac{\sum_s n_s m_s \vec{u}_s}{\sum_s n_s m_s} \approx \frac{m_i \vec{u}_i + m_e \vec{u}_e}{m_i + m_e} \approx \vec{u}_i + \frac{m_e}{m_i} \vec{u}_e$

Equation of mass/charge conservation $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$

Equation of motion $\rho_m \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = -\nabla p + \vec{j} \times \vec{B} + \rho \vec{E}$

Ohm's law $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} + \frac{1}{en}(\vec{j} \times \vec{B} - \nabla p_e)$ ⟶ $\vec{E} + \vec{v} \times \vec{B} = 0$ **Ideal MHD**

Conservation of entropy $\frac{d}{dt}(p \rho_m^{-\gamma}) = 0$ or $\frac{dp}{dt} + \gamma p \nabla \cdot \vec{v} = 0$

Maxwell's equations $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_o \vec{j}$

Macroscopic Plasma Behaviors

- Physics of plasma fluid
- Plasma diamagnetism
- Braginskii equations
- Plasma waves
- Landau damping

Physics of plasma fluid

- **Fluid velocity : not the same with the velocity of guiding centers**
- **Pressure balance**

In a steady-state, $n_s e_s(\vec{E}+\vec{u}_s\times\vec{B})=\nabla p_s$ for s = e, i

For complete plasmas, $\vec{E}\sum_s n_s e_s+\sum_s n_s e_s \vec{u}_s\times\vec{B}=\nabla p$ $\vec{j}\times\vec{B}=\nabla p$

Using Ampere's law, $\nabla p=\frac{\nabla\times\vec{B}}{\mu_o}\times\vec{B}=\frac{1}{\mu_o}\vec{B}\cdot\nabla\vec{B}-\nabla B^2/2\mu_o$

For straight magnetic field, $p+B^2/2\mu_o=const.$

- **Pressure tensor**

Force due to pressure tensor, $F_\alpha=-\sum_\beta\frac{d}{dx_\beta}P_{\alpha\beta}$ $P_{\alpha\beta}=p\delta_{\alpha\beta}+\Pi_{\alpha\beta}$

For anisotropic pressure, $\nabla p=\nabla_\perp p_\perp+\nabla_{//}p_{//}$

Physics of plasma fluid

- **'Frozen in' magnetic field**

In a perfectly conducting fluid, $\vec{E}+\vec{v}\times\vec{B}=0$

Magnetic flux through each surface moving with the fluid is constant and consequently that the magnetic flux can be thought of as frozen-in to the fluid and moving with it. $\Phi=\int\vec{B}\cdot d\vec{S}$

$$\frac{d\Phi}{dt}=\int\frac{d\vec{B}}{dt}\cdot d\vec{S}+\oint\vec{B}\cdot(\vec{v}\times d\vec{l})=-\int\nabla\times\vec{E}\cdot d\vec{S}+\oint\vec{B}\times\vec{v}\cdot d\vec{l}$$

$$=-\int\nabla\times\vec{E}\cdot d\vec{S}-\int\nabla\times(\vec{v}\times\vec{B})\cdot d\vec{S}=-\int\nabla\times(\vec{E}+\vec{v}\times\vec{B})\cdot d\vec{S}=0$$

- **Polarization** In a perfectly conducting fluid, $\vec{E}+\vec{v}\times\vec{B}=0$

$$\left.\begin{aligned}\vec{v}_{ExB}&=\frac{\vec{E}\times\vec{B}}{B^2}\\ \rho_m\frac{dv_\perp}{dt}&=\vec{j}\times\vec{B}\end{aligned}\right\}\quad \rho_m\frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right)=\vec{j}\times\vec{B}$$

$$\vec{j}=\frac{\rho_m}{B^2}\frac{d\vec{E}}{dt}$$ **polarization current**

$$K=1+\frac{c^2}{B^2/\mu_o\rho_m}$$ **dielectric constant**

Plasma Diamagnetism

Plasmas in magnetic fields are naturally diamagnetic, the gyro-orbits of the charged particles being such as to reduce the field.

$$\delta i = -\frac{e_j \omega_{cj}}{2\pi}\delta n = -\frac{e_j^2}{2\pi m_j}\delta n B$$

current $\frac{\text{particles}}{\text{length}}$

$$\delta B = \mu_o \delta i$$

$$\delta B = -\frac{\mu_o e_j^2}{2\pi m_j}\delta n B$$

$$di = -\pi\left(\frac{v_\perp}{\omega_{cj}}\right)^2 \frac{e_j \omega_{cj}}{2\pi} f_j d^3v$$

With a distribution of particle velocities,

$$dB_{js} = -\frac{\mu_o m_j}{2B} v_\perp^2 f_j d^3v = -\frac{e_j}{2\omega_{ci}} v_\perp^2 f_j d^3v$$

$$B_{js} = -\frac{\mu_o}{B}\int \frac{m_j}{2} v_\perp^2 f_j d^3v = -\frac{\mu_o p_j}{B} \qquad B_s = -\sum_j \frac{\mu_o p_j}{B} = -\frac{\mu_o p}{B}$$

Or, from pressure balance equation,

$$p + (B_o + B_d)^2 / 2\mu_o = B_o^2 / 2\mu_o$$

$$B_d\left(1 - \frac{1}{2}\frac{B_d}{B_o}\right) = -\frac{\mu_o p}{B_o} = B_s$$

negligible for low β

Plasma Diamagnetism

Full pressure balance $\vec{j}\times\vec{B}=\nabla p$

$$\vec{B}\times\nabla p=\vec{B}\times(\vec{j}\times\vec{B})=\vec{j}B^2-\vec{B}(\vec{j}\cdot\vec{B}) \longrightarrow \vec{j}_\perp=\vec{B}\times\nabla p/B^2$$

Current associated with the magnetic field $\mathbf{B_s}$ arising from the stationary orbits

$$\mu_o\vec{j}_s=\nabla\times\vec{B}_s=-\nabla\times(\frac{\mu_o p}{B}\vec{b})=-\mu_o\nabla\frac{p}{B}\times\vec{b}-\frac{\mu_o p}{B}\nabla\times\vec{b} \qquad \vec{j}_s=\vec{b}\times\nabla\frac{p}{B}$$

Current caused by the magnetic field gradient drift

$$\vec{j}_d=\sum_j n_j e_j<\vec{v}_{dj}> \qquad <\vec{\mathrm{v}}_{dj}>=\frac{\mathrm{m}_j<\vec{v}_{\perp j}^2>/2}{\mathrm{e_j B^2}}\vec{b}\times\nabla\mathrm{B}=\frac{\mathrm{m}_j\vec{v}_{Tj}^2}{\mathrm{e_j B^2}}\vec{b}\times\nabla\mathrm{B}$$

$$=\sum_j n_j e_j\frac{\mathrm{m}_j\vec{v}_{Tj}^2}{\mathrm{e_j B^2}}\vec{b}\times\nabla\mathrm{B}=\sum_j\frac{p_j}{\mathrm{B}^2}\vec{b}\times\nabla\mathrm{B}=\frac{p}{\mathrm{B}^2}\vec{b}\times\nabla\mathrm{B}=\vec{b}\times\frac{p}{\mathrm{B}^2}\nabla\mathrm{B}$$

Then,

$$\vec{j}_s+\vec{j}_d=\vec{b}\times\nabla\frac{p}{B}+\vec{b}\times\frac{p}{\mathrm{B}^2}\nabla\mathrm{B}=\vec{b}\times\nabla\frac{p}{B}+\vec{b}\times(-p\nabla\frac{1}{B})=\vec{b}\times\frac{1}{B}\nabla p=\vec{j}_\perp$$

Fluid Equations

Continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0$$

Equation of state

$$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$$

Equation of motion

$$mn\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = -\nabla \cdot \overleftrightarrow{P} + n\vec{F} + \vec{R}$$

$$\overleftrightarrow{P}(\vec{x},t) = m\int f(\vec{x},\vec{v},t)(\vec{v}-\vec{u})(\vec{v}-\vec{u})d\vec{v} \qquad \vec{R} = \int m\vec{v}\left(\frac{\partial f}{\partial t}\right)_c d\vec{v}$$

These fluid equations can be closed with the approximated expression of isotropic pressure term from the equation of state. For more accurate estimation, these quantities can be determined by solving the kinetic equation with the collision term given by the Fokker-Planck equation as following:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{e_j(\vec{E} + \vec{v} \times \vec{B})}{m_j} \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_c \qquad \vec{u} = \vec{v} - \vec{v}_j \quad u_{\alpha\beta} = \frac{u^2\delta_{\alpha\beta} - u_\alpha u_\beta}{u^3}$$

$$\left(\frac{\partial f}{\partial t}\right)_c = \sum_j \frac{e^2 Z^2 Z_j^2 \ln\Lambda}{8\pi\varepsilon_o^2 m} - \frac{\partial}{\partial v_\alpha}\int\left(\frac{f_j(\vec{v}_j)}{m}\frac{\partial f(\vec{v})}{\partial v_\beta} - \frac{f(\vec{v})}{m_j}\frac{\partial f_j(\vec{v}_j)}{\partial v_{j\beta}}\right)u_{\alpha\beta}d\vec{v}_j$$

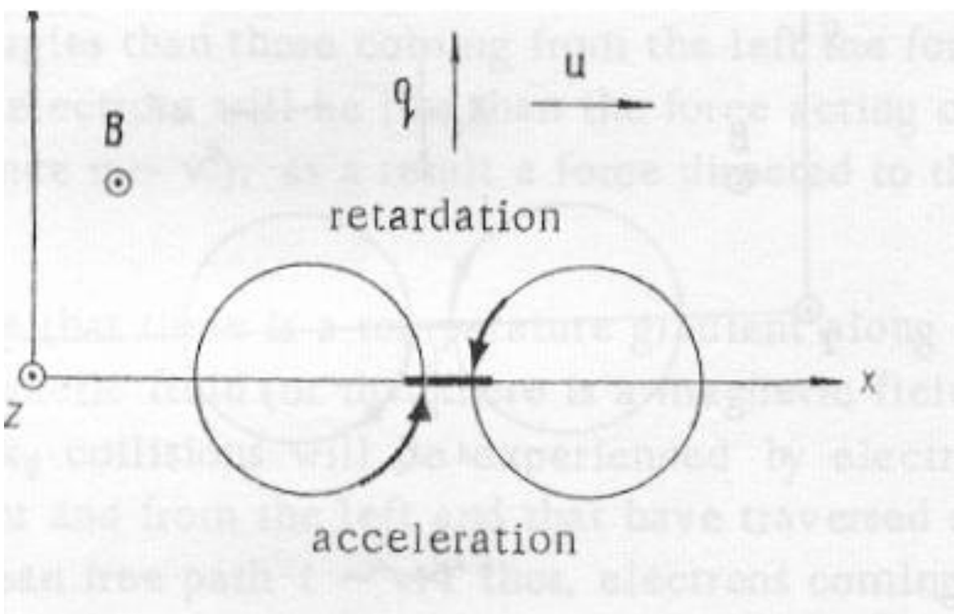
Braginskii Equations

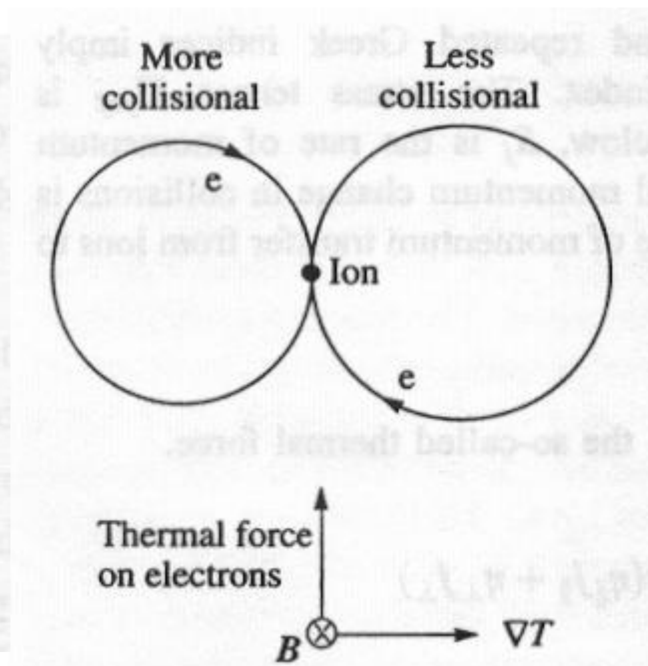
The distribution functions can be expanded about a Maxwellian distribution.

$$f = f_o + \delta f \qquad f_o = \frac{n_j}{(2\pi T_j / m_j)^{3/2}} \exp\left(-\frac{m_j(\vec{v}-\vec{u})^2}{2T_j}\right)$$

In a uniform plasma, $\left(\frac{\partial f}{\partial t}\right)_c = 0$ for Maxwellian distribution, and

$\left(\frac{\partial f}{\partial t}\right)_c \neq 0$ with δf from small gradients and drift velocities.





Continuity and Momentum Equations

Continuity equation $\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{u}_j) = \frac{\partial n_j}{\partial t} + \vec{u}_j \cdot \nabla n_j + n_j \nabla \cdot \vec{u}_j = \frac{dn_j}{dt} + n_j \nabla \cdot \vec{u}_j = 0$

Momentum equation $m_j n_j (\frac{\partial \vec{u}_j}{\partial t} + \vec{u}_j \cdot \nabla \vec{u}_j) = -\nabla \cdot \overleftrightarrow{P}_j + n_j e_j (\vec{E} + \vec{u}_j \times \vec{B}) + \vec{R}_j$

$$\vec{R}_j = \int m_j \vec{v} \left(\frac{\partial f_j}{\partial t} \right)_c d\vec{v} \qquad \vec{R}_e = -\sum_i \vec{R}_i = \vec{R}_u + \vec{R}_T$$

Friction force $\vec{R}_u = -\frac{m_e n}{\tau_e}(0.51\vec{v}_{//} + \vec{v}_\perp) = ne(\eta_{//} \vec{j}_{//} + \eta_\perp \vec{j}_\perp)$ $\quad \vec{v} = \vec{u}_e - \vec{u}_i$

Thermal force $\vec{R}_T = -0.71 n \nabla_{//} T_e - \frac{3}{2} \frac{n}{|\omega_{ce}|\tau_e} \vec{b} \times \nabla T_e$

$$R_{T//} \sim (\Delta \nu_e) n_e m_e \bar{u}_e \sim (\lambda |\nabla_{//} \nu_e|) n_e m_e u_{Te} \qquad \nu_e = \tau_e^{-1} \sim T_e^{-3/2}$$

$$\nabla_{//} \nu_e \sim \lambda \frac{d\nu_e}{dT_e} \nabla_{//} T_e \sim -\lambda \frac{\nu_e}{T_e} \nabla_{//} T_e \sim -\frac{u_{Te}}{T_e} \nabla_{//} T_e \qquad R_{T//} \sim -n_e \nabla_{//} T_e$$

$$R_{T\perp} \sim \Delta \nu_e n_e m_e u_{Te}$$

$$\nabla \nu_e \sim \rho_e |\nabla_\perp \nu_e| \sim \frac{u_{Te}}{\omega_{ce}} \frac{\nu_e}{T_e} |\nabla_\perp T_e| \qquad \vec{R}_{T\perp} \sim -\frac{n}{|\omega_{ce}|\tau_e} \vec{b} \times \nabla T_e$$

Stress Tensors

$$\overset{\leftrightarrow}{P}_j(\vec{x},t) = m_j \int f_j(\vec{x},\vec{v},t)(\vec{v}-\vec{u}_j)(\vec{v}-\vec{u}_j)d\vec{v} = p_j \overset{\leftrightarrow}{1} + \overset{\leftrightarrow}{\Pi}$$ ← Stress tensor

In a strong magnetic field $\omega_{cj}\tau_j >> 1$

$$\Pi_{zz} = -\eta_o W_{zz} \qquad \Pi_{xx} = -\frac{1}{2}\eta_o(W_{xx}+W_{yy}) - \frac{1}{2}\eta_1(W_{xx}-W_{yy}) - \eta_3 W_{xy}$$

$$\Pi_{yy} = -\frac{1}{2}\eta_o(W_{xx}+W_{yy}) - \frac{1}{2}\eta_1(W_{xx}-W_{yy}) + \eta_3 W_{xy}$$

$$\Pi_{xy} = \Pi_{yx} = -\eta_1 W_{xy} + \frac{1}{2}\eta_3(W_{xx}-W_{yy})$$

$$\Pi_{xz} = \Pi_{zx} = -\eta_2 W_{xz} - \eta_4 W_{yz} \qquad \Pi_{yz} = \Pi_{zy} = -\eta_2 W_{yz} + \eta_4 W_{xz}$$

$$W_{\alpha\beta} = \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\nabla\cdot\vec{u}$$ Rate of strain tensor

Viscosity coefficients

$$\eta_0^i = 0.96 n_i T_i \tau_i \qquad \eta_1^i = \frac{3 n_i T_i}{10\omega_{ci}^2\tau_i} \qquad \eta_2^i = 4\eta_1^i \qquad \eta_3^i = \frac{n_i T_i}{2\omega_{ci}} \qquad \eta_4^i = 2\eta_3^i$$

$$\eta_0^e = 0.73 n_e T_e \tau_e \qquad \eta_1^i = 0.51\frac{n_e T_e}{\omega_{ce}^2\tau_e} \qquad \eta_2^e = 4\eta_1^e \qquad \eta_3^e = \frac{n_e T_e}{2|\omega_{ce}|} \qquad \eta_4^e = 2\eta_3^e$$

Energy Equation

$$\frac{3}{2} n \frac{dT_j}{dt} = -p_j \nabla \cdot \vec{u}_j - \nabla \cdot \vec{q}_j + \Pi_{j\alpha\beta} \frac{\partial u_{j\alpha}}{\partial x_\beta} + Q_j$$

Electron heat flux $\vec{q}^{\,e} = \vec{q}_u^{\,e} + \vec{q}_T^{\,e}$

$$\vec{q}_u^{\,e} = nT_e (0.71 \vec{v}_{//} + \frac{3/2}{|\omega_{ce}|\tau_e} \vec{b} \times \vec{v})$$

$$\vec{q}_T^{\,e} = \frac{nT_e \tau_e}{m_e} (-3.16 \nabla_{//} T_e - \frac{4.66}{\omega_{ce}^2 \tau_e^2} \nabla_\perp T_e - \frac{5/2}{|\omega_{ce}|\tau_e} \vec{b} \times \nabla T_e)$$

$$\vec{q}^{\,i} = \frac{nT_i \tau_i}{m_i} (-3.9 \nabla_{//} T_i - \frac{2}{\omega_{ci}^2 \tau_i^2} \nabla T_i - \frac{5/2}{\omega_{ci} \tau_i} \vec{b} \times \nabla T)$$

Heat exchange between ions and electrons due to collisions

$$Q_i = \frac{3m_e}{m_i} \frac{n}{\tau_e} (T_e - T_i)$$

$$Q_e = -\vec{R} \cdot \vec{v} - Q_i = \eta_{//} j_{//}^2 + \eta_\perp j_\perp^2 + \frac{1}{ne} \vec{j} \cdot \vec{R}_T + \frac{3m_e}{m_i} \frac{n}{\tau_e} (T_e - T_i)$$

Plasma Waves

Solve Maxwell's equations and linearized plasma equations together

$$\nabla\times\nabla\times\vec{E} = -\nabla\times\frac{\partial\vec{B}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \mu_o\vec{j}$$

Perturbed quantities in the form $A_1(\vec{x},t) = \mathrm{Re}[\hat{A}_1 e^{-i(\omega t-\vec{k}\cdot\vec{x})}]$

$$-\vec{k}\times\vec{k}\times\vec{E} = \frac{\omega^2}{c^2}\vec{E} + i\omega\mu_o\vec{j} = \frac{\omega^2}{c^2}(\vec{E} + \frac{i}{\varepsilon_o\omega}\vec{j}) = \frac{\omega^2}{c^2}\overleftrightarrow{K}\cdot\vec{E}$$

When the current density $\boldsymbol{j}$ is determined in terms of $\boldsymbol{E}$ from the linearized plasma equations by setting the variables in the form of $A=A_o + A_1$.

For non-trivial solutions of $\boldsymbol{E}$, dispersion relation need to be satisfied.
And the refractive index is defined by $n=|\boldsymbol{n}|$

$$\vec{n} = \vec{k}c/\omega \qquad \omega = \omega(\vec{k})$$

For free space without any plasma, $\boldsymbol{j}=0$

$$\frac{\omega^2}{c^2}\vec{E} = -\vec{k}\times\vec{k}\times\vec{E} = k^2\vec{E} - \vec{k}(\vec{k}\cdot\vec{E}) = k^2\vec{E} \rightarrow \begin{cases}\omega^2 = k^2c^2 \\ n = kc/\omega = 1\end{cases}$$

Plasma Oscillations

Linear perturbation theory $n = n_o + n_1$ $\vec{u} = \vec{u}_1$ $\vec{E} = \vec{E}_1$ $\vec{k} \,//\, \vec{E}_1$

$$\frac{\partial n_{e1}}{\partial t} + \nabla\cdot(n_o \vec{u}_{e1}) = 0 \longrightarrow -i\omega n_{e1} + i n_o \vec{k}\cdot\vec{u}_{e1} = 0$$

$$m_e n_o \frac{\partial \vec{u}_{e1}}{\partial t} = -n_o e \vec{E}_1 \longrightarrow -i\omega m_e n_o \vec{u}_{e1} = -n_o e \vec{E}_1$$

$$\nabla\cdot\vec{E}_1 = -n_{e1} e/\varepsilon_o \longrightarrow i\vec{k}\cdot\vec{E}_1 = -n_{e1} e/\varepsilon_o$$

$$\Bigg\}\ \omega^2 = \frac{n_o e^2}{m_e \varepsilon_o} = \omega_{pe}^2$$

$$\nabla\times\vec{B}_1 = \mu_o \vec{j}_1 + \varepsilon_o \mu_o \frac{\partial \vec{E}_1}{\partial t} = -\mu_o (n_o e \vec{u}_{e1} + i\omega\varepsilon_o \vec{E}_1)$$

$$= -\mu_o \left(n_o e \vec{u}_{e1} + i\omega\varepsilon_o \frac{i\omega m_e \vec{u}_{e1}}{e}\right) = -\mu_o n_o e \vec{u}_{e1}\left(1 - \omega^2 \frac{m_e \varepsilon_o}{n_o e^2}\right) = 0$$

Longitudinal oscillations : the plasma current and the displacement current cancel.

Transverse Electromagnetic Waves

In a transverse electromagnetic wave, the plasma current and the displacement current do not cancel.

$$\vec{k} \perp \vec{E}_1 \qquad \vec{k} \perp \vec{B}_1$$

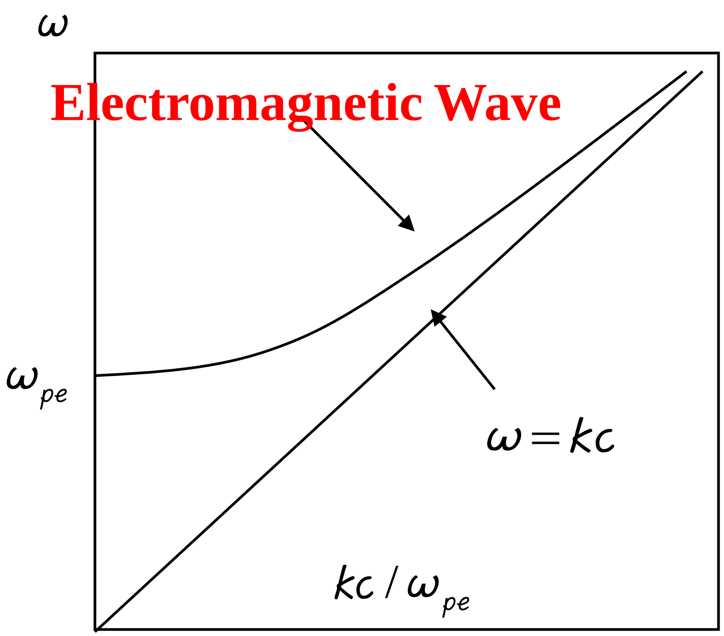
$$-\vec{k} \times \vec{k} \times \vec{E}_1 = k^2 \vec{E}_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) = k^2 \vec{E}_1 = \frac{\omega^2}{c^2} \vec{E}_1 + i\omega\mu_o \vec{j}_1$$

$$\vec{j}_1 = -n_o e \vec{u}_{e1}$$

$$m_e n_o \frac{\partial \vec{u}_{e1}}{\partial t} = -n_o e \vec{E}_1 \longrightarrow -i\omega m_e n_o \vec{u}_{e1} = -n_o e \vec{E}_1$$

$$\omega^2 = k^2 c^2 + \omega_{pe}^2$$

Wave propagates only for $\omega > \omega_{pe}$



Cut-off density

$$n_c = \frac{m_e \varepsilon_o \omega^2}{e^2}$$

Sound Waves

At low frequencies, ion motions should be included

$$m_i n_o \frac{\partial \vec{u}_{i1}}{\partial t} = n_o e\vec{E}_1 - \nabla p_{i1} \longrightarrow -i\omega m_i n_o \vec{u}_{i1} = n_o e\vec{E}_1 - i\vec{k}p_{i1}$$

negligible electron inertia, $0 = -n_o e\vec{E}_1 - \nabla p_{e1} \longrightarrow 0 = -n_o e\vec{E}_1 - i\vec{k}p_{e1}$

Adding them together, $-i\omega m_i n_o \vec{u}_{i1} = -i\vec{k}p_{i1} - i\vec{k}p_{e1} = -i\vec{k}p_1$

$$\frac{\partial n_{i1}}{\partial t} + \nabla\cdot(n_o\vec{u}_{i1}) = 0 \longrightarrow -i\omega n_{i1} + i n_o \vec{k}\cdot\vec{u}_{i1} = 0$$

$$\frac{\partial n_{e1}}{\partial t} + \nabla\cdot(n_o\vec{u}_{e1}) = 0 \longrightarrow -i\omega n_{e1} + i n_o \vec{k}\cdot\vec{u}_{e1} = 0$$

$\vec{u}_{e1} = \vec{u}_{i1} (= \vec{u}_1)$

from quasi-neutrality

$n_{e1} = n_{i1}$

Equation of state for adiabatic processes

$$\frac{\partial p_1}{\partial t} + \gamma p_o \nabla\cdot\vec{u}_1 = 0 \longrightarrow -i\omega p_1 + i\gamma p_o \vec{k}\cdot\vec{u}_1 = 0$$

$$\omega^2 = k^2\gamma\frac{p_{i1} + p_{e1}}{n_o m_i} = k^2 C_s^2$$

Sound speed

$$C_s = \sqrt{\gamma\frac{p_{i1} + p_{e1}}{n_o m_i}} = \sqrt{\gamma\frac{T_{i1} + T_{e1}}{m_i}}$$

Dielectric Tensors for Low-Frequency Waves

- Linearized fluid equation of motion

$$mn_o \frac{\partial \vec{u}_1}{\partial t} = -en_o(\vec{E}_1 + \vec{u}_1 \times \vec{B}_o) - \gamma T \nabla n_1$$

$$-i\omega\omega m_{x1} = q(E_{x1} + u_{y1}B_o) - ik_x \gamma T n_1/n_o$$

$$-i\omega\omega m_{y1} = q(E_{y1} - u_{x1}B_o)$$

$$-i\omega\omega m_{z1} = qE_{z1} - ik_z \gamma T n_1/n_o$$

- Linearized continuity equation

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (n_o \vec{u}_1) = 0 \qquad -i\omega\omega_1/n_o + ik_x u_{x1} + ik_z u_{z1} = 0$$

with the angle θ between **k** and **B**$_o$,

$$n_1/n_o = (k/\omega k/\omega_{x1} \sin\theta + u_{z1} \cos\theta o$$

- Linearized current density

$$\vec{j}_1 = \sum q n_o \vec{u}_1 = \overleftrightarrow{\sigma} \cdot \vec{E}_1$$

- Wave equation and dielectric tensor

$$k^2\vec{E}_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) = (\omega^2/c^2)(\vec{E}_1 + i\vec{J}_1/\omega\omega_o) = (\omega^2/c^2)(\overleftrightarrow{1} + i\overleftrightarrow{\sigma}/\omega\omega_o) \cdot \vec{E}_1$$

$$= \omega^2 \mu_o \overleftrightarrow{\varepsilon} \cdot \vec{E}_1 \qquad \overleftrightarrow{\varepsilon} = \varepsilon_o(\overleftrightarrow{1} + i\overleftrightarrow{\sigma}/\omega\omega_o)$$

- Dispersion relation with tensor notation

$$\overleftrightarrow{X} = \overleftrightarrow{1} - \vec{k}\vec{k}/k^2$$

$$k^2\vec{E}_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) \equiv k^2 \overleftrightarrow{X} \cdot \vec{E}_1 = \omega^2 \mu_o \overleftrightarrow{\varepsilon} \cdot \vec{E}_1$$

$$k^2 \overleftrightarrow{X} = \omega^2 \mu_o \overleftrightarrow{\varepsilon}$$

Cold-Plasma Wave Equation

- wave equation with tensor notation $k^2 \overleftrightarrow{X} = \omega^2 \mu_o \overleftrightarrow{\varepsilon}$

$$\overleftrightarrow{X} = \overleftrightarrow{1}_1 - \vec{k}\vec{k}/k^2 = \hat{x}\hat{x}\cos^2\theta + 0 - \hat{x}\hat{z}\sin\theta in\theta c$$
$$+ 0 + \hat{y}\hat{y} + 0 - \hat{z}\hat{x}\sin\theta in\theta c + 0 + \hat{z}\hat{z}\cos^2\theta$$

$$[\hat{x}\hat{x}(S - \tilde{n}^2\cos^2\theta) + \hat{x}\hat{y}\, iD + \hat{x}\hat{z}\, \tilde{n}^2\sin\theta in\theta c$$
$$+ \hat{y}\hat{x}\, iD \quad + \hat{y}\hat{y}\,(S - \tilde{n}^2) \quad + 0$$
$$+ \hat{z}\hat{x}\, \tilde{n}^2\sin\theta in\theta c + 0 + \hat{z}\hat{z}\,(P - \tilde{n}^2\sin^2\theta)] \cdot \vec{E}_1 = 0$$

where

$\tilde{n} \equiv ck/\omega = c/v_p$

$c\vec{k}/\omega = \tilde{n}\sin\theta\hat{x} + \tilde{n}\cos\theta\hat{z}$

$R \equiv 1 - (\omega_{pe}^2/\omega)/(\omega - \omega_c) - (\omega_{pi}^2/\omega)/(\omega + \Omega_c)$

$L \equiv 1 - (\omega_{pe}^2/\omega)/(\omega + \omega_c) - (\omega_{pi}^2/\omega)/(\omega - \Omega_c)$

$S \equiv (R + L)/2$

$D \equiv (R - L)/2$

$P \equiv 1 - \omega_{pe}^2/\omega^2 - \omega_{pi}^2/\omega^2$

ω_{pe}, ω_{pe} : electron and ion plasma frequencies

ω_c, Ω_c : electron and ion cyclotron frequencies

Cold-Plasma Dispersion Relation

- Dispersion relation for cold-plasma

$$(S-\tilde{n}^2\cos^2\theta)(S-\tilde{n}^2)(P-\tilde{n}^2\sin^2\theta)$$
$$-\tilde{n}^4\sin^2\theta\cos^2\theta(S-\tilde{n}^2)$$
$$-D^2(P-\tilde{n}^2\sin^2\theta)=0$$

$$(S^2P-D^2P)-\tilde{n}^2(SP\cos^2\theta+SP+S^2\sin^2\theta-D^2\sin^2\theta)$$
$$+\tilde{n}^4(P\cos^2\theta+S\sin^2\theta)=0$$

- Using S^2-D^2=RL and $\cos^2\theta = 1-\sin^2\theta$,

$$\tan^2\theta=\frac{-P(\tilde{n}^2-R)(\tilde{n}^2-L)}{(S\tilde{n}^2-RL)(\tilde{n}^2-P)}$$

- For parallel propagation (θ =0),

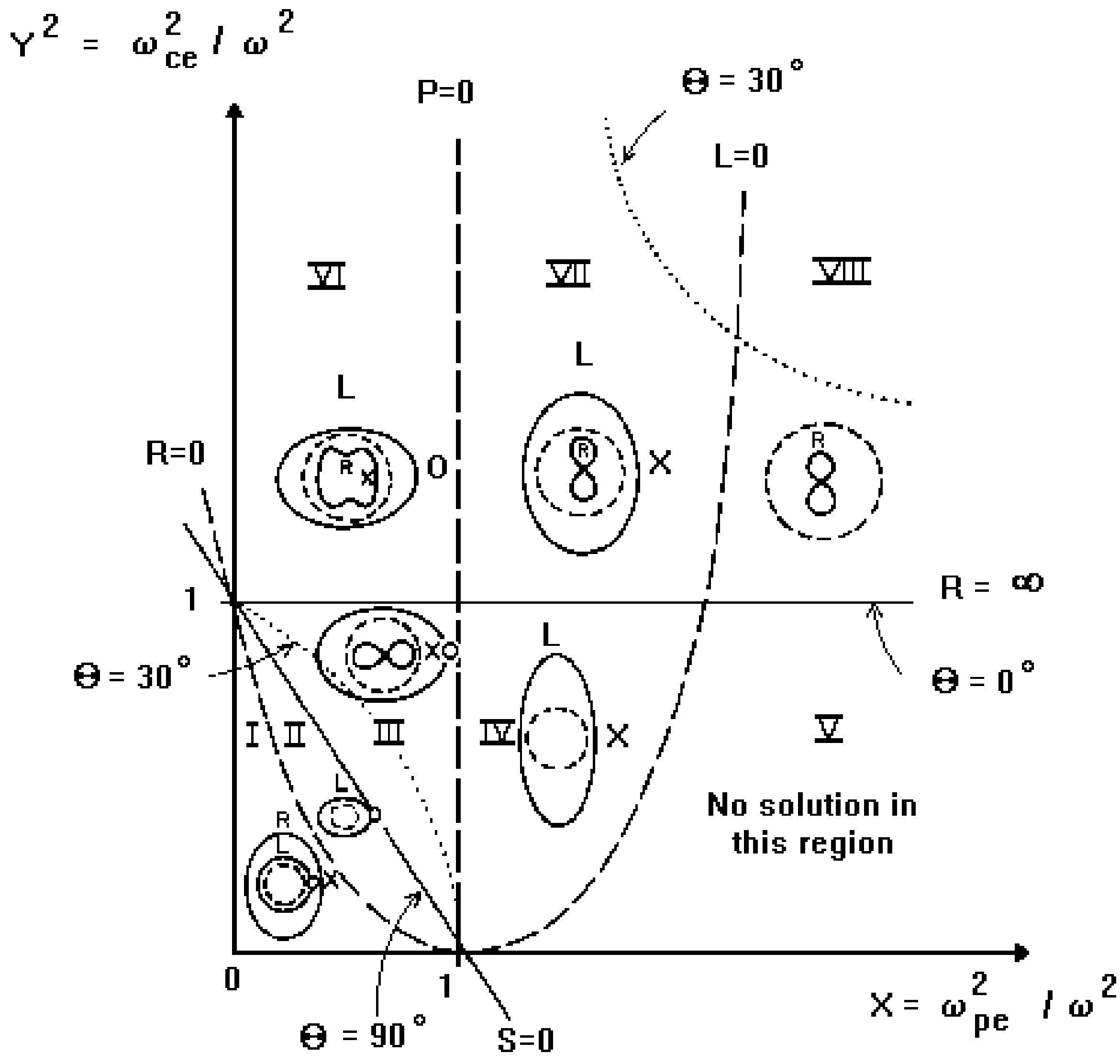
$$\tilde{n}^2=R \quad \text{: R-wave}$$
$$\tilde{n}^2=L \quad \text{: L-wave}$$

- For perpendicular propagation ($\theta = \pi/2$),

$$\tilde{n}^2=RL/S \quad \text{: X-wave}$$
$$\tilde{n}^2=P \quad \text{: O-wave}$$

CMA(Clemmow-Mullaly-Allis) Diagram

- The CMA(Clemmow-Mullaly-Allis) diagram divides the plane into a number of regions such that within each region the characteristic topological forms of the phase velocity surfaces remain unchanged.
- Slow and fast waves
- The solid lines represent the principal resonances and the dashed lines the reflection(cutoff) points.
- The magnetic field increases in the vertical direction, the plasma electron density increases in the horizontal direction.



Summary of Plasma Waves

- R-Wave (**k**//$\mathbf{B}_o$ and $\mathbf{E}_1 \perp \mathbf{B}_o$) : two pass bands

$\omega > \omega_R$ High frequency pass band, v_p --> c as ω --> ∞ .

$\omega < \omega_C$ "Whistler" wave, becoming shear Alfven R-wave at low frequency, v_p --> v_A as ω --> 0 .

- L-Wave (**k**//$\mathbf{B}_o$ and $\mathbf{E}_1 \perp \mathbf{B}_o$) : two pass bands

$\omega > \omega_L$ High frequency pass band, v_p --> c as ω --> ∞ .

$\omega < \Omega_C$ Shear Alfven L-wave, v_p --> v_A as ω --> 0 .

- Langmuir oscillation (**k**//$\mathbf{B}_o$ and $\mathbf{E}_1$//$\mathbf{B}_o$)

$\omega = \omega_{pe}$ Zero group velocity Langmuir oscillation v_p undefined

- For finite temperature (**k**//$\mathbf{B}_o$ and $\mathbf{E}_1 \perp \mathbf{B}_o$)

$\omega > \omega_{pe}$ Langmuir wave, v_p --> $\sqrt{3}\, v_{te}$ as ω --> ∞ .

$\omega < \Omega_C$ Ion sound wave, v_p --> c_S as ω --> 0 .

- O-Mode (**k** $\perp$ $\mathbf{B}_o$ and $\mathbf{E}_1$//$\mathbf{B}_o$) : one pass band

$\omega > \omega_{pe}$ High frequency pass band, v_p --> c as ω --> ∞ .

- X-Mode (**k** $\perp$ $\mathbf{B}_o$ and $\mathbf{E}_1 \perp \mathbf{B}_o$) : three pass bands

$\omega > \omega_R$ High-pass region of X-mode, v_p --> c as ω --> ∞ .

$\omega_L < \omega < \omega_{UH}$ Mid-pass region of X-mode, v_p= c as $\omega = \omega_p$.

$\omega < \omega_{LH}$ Compressional Alfven (Magnetosonic) wave, v_p --> v_A as ω --> 0 .