# CHAPTER 1 Preliminary Concepts

## 1-1 Historical Outline

* READ!

## 1-2 Some Examples of Viscous-Flow Phenomena

* Viscous flow theory has limitations.
  - → in the high Re turbulent flow regime where random fluctuations exist
    - △ flow is modeled on a semi-empirical time-mean or statistical basis

* Reynolds number : primary controlling parameter

$$Re_L = \rho U L / \mu \qquad (1-1)$$

Factors : (1) fluid properties
(2) size & speed

For a given geometry, as $Re_L$ increases, the flow pattern changes
laminar → transition → turbulent

* Example 1 Flow past a thin airfoil
  - low $\alpha$ : lift by inviscid theory
    drag by BL theory
  - larger $\alpha$ : separation & stall
  - streamlines @ 12°, 15°, & 20° $\alpha$

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* Example 2 Flow past a circular cylinder
  - inviscid theory : d'Alembert paradox
  - in reality, pressure on the rear or lee side is "everywhere" less than the free stream pressure
  - The effect of Re : Fig 1-7
    alternate shedding up to $Re = 10^7$
  - $St = \frac{fD}{U} \approx 0.2$ for $10 < Re < 10^5$ Fig. 1-8
  - Drag crisis @ $Re_D \approx 2 \times 10^5$
    when BL on the front surface becomes turbulent
    → dimples on golf balls.
  - The real fluid flow past a bluff body can differ markedly from the inviscid-flow prediction.
  - BL theory can predict the onset of separation only.

* Example 3 Flow in a circular pipe
  - Internal flow → viscous layers must coalesce at some distance → further downstream, the flow profile ceases to change with axial position : fully developed.
  - Tapered flow does not become fully turbulent
  - Subsonic diffuser : separation, backflow, & unsteadiness
  - subsonic nozzle : no separation or unsteadiness

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## 1-3 Properties of a Fluid

* 4 classes of true fluids' properties
  - Kinematic properties : properties of the flow field itself rather than of a fluid (linear velocity, ...)
  - Transport properties (viscosity, ...)
  - Thermodynamic properties (pressure, density, ...)
  - Others (surface tension, ...)

### 1-3.1 The Kinematic properties

* In fluid mechanics
  (1) choose the most convenient origin of coordinates
  (2) study only the velocity field as a function of position & time, not trying to follow any specific particle paths
  → Eulerian formulation
* Eulerian velocity vector

$$\vec{V}(\vec{r},t) = \vec{V}(x,y,z,t)$$
$$= \hat{i}\,u(x,y,z,t) + \hat{j}\,v(x,y,z,t) + \hat{k}\,w(x,y,z,t)$$

  u, v, w as functions of (x, y, z, t)
  → solution to a fluid mechanics problem
* One conflict : 3 fundamental laws of mechanics
  - conservation of mass, momentum, and energy -
  are formulated for particles of fixed identity
  → Lagrangian
* All 3 laws relate to the time rate of change of some property of a fixed particle

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* Total differential change in Q

$$dQ = \frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz + \frac{\partial Q}{\partial t}dt \qquad (1\text{-}5)$$

* we are following an Infinitesimal particle of fixed identity

$$dx = u\,dt \qquad dy = v\,dt \qquad dz = w\,dt \qquad (1\text{-}6)$$

* (1-6) → (1-5)

time derivative of Q of a particular element particle

$$\frac{DQ}{Dt} = \frac{dQ}{dt} = \frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} + v\frac{\partial Q}{\partial y} + w\frac{\partial Q}{\partial z} \qquad (1\text{-}7)$$

- Substantial, particle, material derivative : suggest that we are following a fixed fluid particle
- Convective derivative : zero if 0-velocity or Q has no spatial change
- $\partial Q/\partial t$ : local derivative

* In vector form

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + (\vec{V}\cdot\nabla)Q \qquad (1\text{-}8)$$

Where $\nabla$ : gradient operator

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

## 1-3.2 Acceleration of a fixed-identity particle

* If $Q = \vec{V}$, we get the particle acceleration vector

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = \hat{i}\frac{Du}{Dt} + \hat{j}\frac{Dv}{Dt} + \hat{k}\frac{Dw}{Dt}$$

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* convective terms present mathematical difficulty
  ∵ non linear products
* Note these non-linear terms are accelerations, not viscous stresses → Irony
* In frictionless flow, these terms still exist, but behave well.

$$(\vec{V}\cdot\nabla)\vec{V} \equiv \nabla\frac{V^2}{2} - \vec{V}\times(\nabla\times\vec{V})$$

$\nabla\frac{V^2}{2}$: familiar in Bernoulli's

$(\nabla\times\vec{V})$: $=0$ when irrotational, usually invoked

## 1-3.3 Other Kinematic properties

* Like its solid counterpart, a fluid element undergoes 4 types of motion or deformation
  (1) translation
  (2) rotation
  (3) extensional strain, or dilation
  (4) shear strain
* Fig. 1-14
  (1) translation : B → B'
  (2) counter-clockwise rotation : BD → B'D'
  (3) dilation : little bigger
  (4) shear strain : square → rhombic
* Now, quantitative basis. The final result is a rate, i.e., a time derivative

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* Rate of translation : $u, v$
* Angular rotation about $z$-axis
  → average counter-clockwise rotation of BC & BA
  BC: $d\alpha$, BA: $-d\beta$

$$d\Omega_z = \frac{1}{2}(d\alpha - d\beta) \qquad (1\text{-}11)$$

$$d\alpha = \lim_{dt \to 0} \left( \tan^{-1} \frac{\frac{\partial v}{\partial x} dx\,dt}{dy + \frac{\partial u}{\partial x} dx\,dt} \right) = \frac{\partial v}{\partial x} dt \qquad (1\text{-}12)$$

$d\beta$ = Similar

(1-12) → (1-11)

$$\frac{d\Omega_z}{dt} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\frac{d\Omega_x}{dt} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\frac{d\Omega_y}{dt} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

} 3 components of angular velocity vector $\frac{d\vec{\Omega}}{dt}$

* Take ½ away : vorticity of the fluid

$$\vec{\omega} = 2\frac{d\vec{\Omega}}{dt} \qquad (1\text{-}15)$$

* Vorticity & velocity are related

$$\vec{\omega} = \text{curl}\ \vec{V} = \nabla \times \vec{V} \qquad (1\text{-}16)$$

$\therefore$ div $\vec{\omega} = \nabla \cdot \vec{\omega} =$ div curl $\vec{V} = 0$

→ Connected with convective acceleration (1-10)

$\vec{\omega} = 0$ : irrotational

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* 2D shear strain
  → average decrease of angle between 2 initially perpendicular lines AB & BC

$$= \frac{1}{2}(d\alpha + d\beta)$$

Shear-strain rate is

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad (1\text{-}12)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Symmetric, i.e., $\varepsilon_{ij} = \varepsilon_{ji}$

* Dilatation (extensional strain)
  → fractional increase in length of the horizontal side of the element (x-direction)

$$\underbrace{\varepsilon_{xx}\,dt}_{\text{rate} \times \text{time}} = \frac{\left(dx + \frac{\partial u}{\partial x}dx\,dt - dx\right)}{dx} = \frac{\partial u}{\partial x}dt$$

$$\therefore \varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \quad (1\text{-}21)$$

* Taken as a whole, strain rates constitute a symmetric 2nd order tensor

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \quad (1\text{-}22)$$

$$I_1 = \ldots$$
$$I_2 = \ldots \quad (1\text{-}23)$$
$$I_3 = \ldots$$

* Let's adopt $u_{i,j} = \frac{\partial u_i}{\partial x_j}$

$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i})$$

Incorporating $\frac{d\vec{\Omega}}{dt}$ & $\varepsilon_{ij}$

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \frac{d\Omega_{ij}}{dt} \qquad (1-27)$$

→ Each velocity derivative can be resolved into (strain rate) + (angular velocity.)

## 1-3.4 The transport properties of a fluid

* Viscosity, thermal conductivity, diffusion
  → related to transport of momentum, heat, and mass
* Viscosity : flux to velocity gradient
  thermal conductivity : heat flux to temperature gradient
  diffusion coefficient : mass transport to concentration gradient
* All three are genuinely analogous.

## 1-3.5 The coefficient of viscosity

* property of a fluid that relates applied stress to the resulting strain rate.

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* See Fig 1-15.

$\tau_{xy}$ : constant throughout the fluid

$u = u(y)$ only.

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\frac{\partial u}{\partial y} = \frac{1}{2}\frac{du}{dy} \qquad (1\text{-}28)$$

($\frac{\partial v}{\partial x}$ struck through; $\frac{\partial u}{\partial y} = \frac{du}{dy}$ since $u = u(y)$)

$$\therefore \tau_{xy} = f(\varepsilon_{xy}) \qquad (1\text{-}29)$$

* For simple fluids, the relationship is linear or newtonian

$\tau \propto \varepsilon_{xy}$

$$\tau_{xy} = \mu \frac{V}{h} = 2\mu\varepsilon_{xy} = \mu\frac{du}{dy} \qquad (1\text{-}30)$$

($\mu$: viscosity coefficient)

$N \cdot s/m^2$

* $\mu$ : a thermodynamic property

varies with T & p

## 1-3.8 The coeff. of thermal conductivity

* Fourier's law

$$\vec{q} = -k\nabla T \qquad (1\text{-}37)$$

($\vec{q}$: vector rate of heat flow per unit area)

* k's unit: $W/(m \cdot k)$

## 1-3.9 The coeff. of Mass Diffusivity

* $\frac{\vec{m}_i}{A} = -D \nabla(\rho_i)$ (1-45)

## 1-4 Boundary Conditions for Viscous-Flow Problems

* The eqns of motion require mathematically tenable and physically realistic BC's.

### 1-4.1 Conditions at a solid surface:

* Macroflows → system dimensions ≫ molecular spacing
  → liquid/gas particles contacting the wall must be in equilibrium with the solid.
* No-slip / no-temperature-jump

$$\vec{V}_{fluid} = \vec{V}_{solid}$$

$$T_{fluid} = T_{solid}$$

### 1-4.5 Conditions at a free liquid surface

* Classic FS in Fig. 1-29
  Assume $z = \eta(x, y)$
* 2 required conditions
  (1) fluid particles @ the surface must remain attached (kinematic condition)
  (2) the liquid and the atmospheric pressure must balance except for surface-tension effects

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* Kinematic condition : particle's upward velocity = motion of the free surface

$$w(x,y,\eta) = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \qquad (1\text{-}105)$$

* pressure equilibrium

$$p(x,y,\eta) = p_a - \tau\left(\frac{1}{R_x} + \frac{1}{R_y}\right)$$

$\tau$ : surface tension coeff.

## 1-4.6 Conditions @ a Liquid-Vapor or Liquid-Liquid Interface

* Free surface : the gas laying over the liquid has no effect except to impose pressure on the liquid.
* In a true l-v or l-l interface, however, the upper fluid is strongly coupled and exerts kinematic, stress, and energy constraints on the lower fluid.
* The motions of the 2 fluids are solved simultaneously and must match
* $\vec{V}_1 = \vec{V}_2 \qquad \tau_1 = \tau_2 \qquad T_1 = T_2$
* $V_{n1} = V_{n2}$ : interface motion (1-105)

  $V_{t1} = V_{t2}$ : no-slip condition
* $p_2 = p \qquad p_1 = p_a$

  unless surface tension is negligible

* $\tau_1 = \mu_1 \frac{\partial V_{t1}}{\partial n} = \tau_2 = \mu_2 \frac{\partial V_{t2}}{\partial n}$

$$q_1 = -k_1 \frac{\partial T_1}{\partial n} = q_2 = -k_2 \frac{\partial T_2}{\partial n}$$

Note the slopes are not equal

* Since * $\Delta \mu$ for a vapor

$\ll$ * $\Delta \mu$ for a liquid

$$\left.\frac{\partial V_t}{\partial n}\right|_{liq} \approx 0 \qquad \left.\frac{\partial T}{\partial n}\right|_{liq} \approx 0$$

## 1-4.8 Inlet & Exit BC's

* Usualy no BC for p @ exit
  → part of the solution

* Unbounded fluid → downstream approximation

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# CHAPTER 2
# Fundamental Equations of Compressible Viscous Flow

## 2-1 Introduction

* have been known for more than 100 years
* very difficult to solve in their complete form
* At high Re, the eqns are impossible to solve with present mathematical techniques.

## 2-2. Classification of the Fundamental Eqns

* 3 laws of conservation
  1. Conservation of mass (Continuity)
  2. Conservation of momentum (Newton's 2nd law)
  3. Conservation of energy (1st law of thermodynamics)
* 3 unknowns must be obtained simultaneously : $\vec{V}$, $p$, $T$

* 2 extra basic relations

4. Conservation of species

5. Laws of chemical reaction

## 2-3 Conservation of mass : The eqn of continuity

* Remember that in the Eulerian system, 3 laws must utilize the particle derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\nabla) \qquad \text{"(1-8)"}$$

* Note, in Lagrangian terms,

$$m = \rho\forall = \text{const.} \qquad (2\text{-}2)$$

* In Eulerian terms

$$\frac{Dm}{Dt} = \frac{D}{Dt}(\rho\forall) = 0 = \rho\frac{D\forall}{Dt} + \forall\frac{D\rho}{Dt} \qquad (2\text{-}3)$$

↳ relate this to the fluid velocity

* How? (total dilation rate)
= (rate of volume increase of a particle per unit volume)

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1}{\forall}\frac{D\forall}{Dt} \qquad (2\text{-}4)$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div}\,\vec{V} = \nabla\cdot\vec{V} \qquad (2\text{-}5)$$

Eq (1-21)

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Combining Eqs (2-31 to (2-5)

$$\rho \cancel{\forall}\, \text{div}\,\vec{V} + \cancel{\forall}\,\frac{D\rho}{Dt} = 0$$

or

$$\frac{\partial \rho}{\partial t} + (\vec{V}\cdot\nabla)\rho + \rho\,\nabla\cdot\vec{V} = 0$$

or

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho\vec{V}) = 0 \qquad (2\text{-}6)$$

* If $\rho$ = const (incompressible flow)

$$\text{div}\,\vec{V} = 0 \qquad (2\text{-}7)$$

## 2-3.1 A Useful Strategy : The Stream Function

* Introduce stream function $\psi$
* 2-D steady incompressible flow in the xy plane

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad (2\text{-}8)$$

* If we define $\psi$ such that

$$\rho u = \frac{\partial \psi}{\partial y} \qquad \rho v = -\frac{\partial \psi}{\partial x} \qquad (2\text{-}9)$$

* By direct substitution, assuming that $\psi$ is continuous to 2nd-order derivatives, Eq (2-8) is satisfied identically.

* Thus continuity eqn is no use.
  → # of dependant variables = 1
* Penalty : the remaining velocity derivatives are increased by one order.
* Physical significance

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$= -\rho v dx + \rho u dy$$ ← area vector's components

$$= \rho \vec{V} \cdot d\vec{A} = d\dot{m} \qquad (2\text{-}10)$$

* lines of constant $\psi$ ($d\psi = 0$) are lines across which there is no mass flow ($d\dot{m} = 0$)
  → streamlines of the flow
* (difference between the values of $\psi$ of any 2 streamlines)
  = (mass flow between those streamlines)

## 2-4 Conservation of Momentum : The Navier-Stokes Equations

* Newton's 2nd law.
  - proportionality between applied force and the resulting acceleration of a particle of mass m

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$$\vec{F} = m\,\vec{a} \qquad (2\text{-}11)$$

* For a fluid particle, work with density instead of mass

$$\rho \underbrace{\frac{D\vec{V}}{Dt}}_{\text{acceleration}} = \vec{f} = \vec{f}_{body} + \vec{f}_{surface} \qquad (2\text{-}12)$$

* Body forces

that apply to the entire mass of the fluid element

$$\vec{f}_{body} = \rho \vec{g} \qquad (2\text{-}13)$$

* Surface forces

those applied by external stresses on the sides of the element

Fig 2-1 : Sign convention

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \qquad (2\text{-}14)$$

* Total force in each direction due to stress

$$\begin{aligned} dF_x &= \tau_{xx}\,dy\,dz + \tau_{yx}\,dx\,dz + \tau_{zx}\,dx\,dy \\ dF_y &= \tau_{xy}\,dy\,dz + \tau_{yy}\,dx\,dz + \tau_{zy}\,dx\,dy \\ dF_z &= \tau_{xz}\,dy\,dz + \tau_{yz}\,dx\,dz + \tau_{zz}\,dx\,dy \end{aligned} \qquad (2\text{-}15)$$

* In equilibrium, these forces would be balanced. If the element is accelerating,

$$\tau_{xx,\,front} = \tau_{xx,\,back} + \frac{\partial \tau_{xx}}{\partial x} dx \qquad (2\text{-}16)$$

* The net force in the x-direction

$$dF_{x,net} = \left(\frac{\partial \tau_{xx}}{\partial x} dx\right) dy\,dz + \left(\frac{\partial \tau_{yx}}{\partial y} dy\right) dx\,dz + \left(\frac{\partial \tau_{zx}}{\partial z} dz\right) dx\,dy$$

or on a unit volume basis, i.e, divide by $dx\,dy\,dz$

$$f_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \qquad (2\text{-}17)$$

$$= \text{div (upper row of the stress tensor)}$$

$$f_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

$$f_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

* Total vector surface force

$$\vec{f}_{sur} = \nabla \cdot \tau_{ij} = \frac{\partial \tau_{xj}}{\partial x_j} \qquad (2\text{-}18)$$

* Combining (2-13) & (2-18)

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} + \nabla \cdot \tau_{ij} \qquad (2\text{-}19)$$

## 2-4.1 The Fluid at Rest : Hydrostatics

* Viscous stresses vanish
* Velocity and shear stresses are zero
* Normal stresses = hydrostatic pressure.

* If $\vec{V}=0$, (2-19) reduces to

$\tau_{xx} = \tau_{yy} = \tau_{zz} = -p$

$\tau_{ij} = 0$ for $i \neq j$

$\nabla p = \rho \vec{g}$

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## 2-4.2 Deformation Law for a Newtonian Fluid

* Stokes' 3 postulates

1. The fluid is continuous, $\tau_{ij}$ is at most a linear function of $\varepsilon_{ij}$ (strain rates)
2. The fluid isotropic i.e. the deformation law is independent of the coordinate axes
3. When $\varepsilon_{ij} = 0$, the deformation law must reduce to the hydrostatic pressure condition $\tau_{ij} = -p\delta_{ij}$

* General deformation law for a Newtonian viscous fluid

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda \operatorname{div}\vec{V}$$

## 2-4.3 Thermodynamic Pressure vs. Mechanical Pressure

* Define mechanical pressure $\bar{p}$

$$\bar{p} = -\frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz})$$

put xx, yy, zz in (2-27)

$$= p - (\lambda + \tfrac{2}{3}\mu)\operatorname{div}\vec{V}$$

* (mean pressure in a deforming viscous fluid) $\neq$ (thermodynamic property called pressure)

* Rarely important $\because$ $\operatorname{div} \vec{V} \simeq 0$
* Stokes' hypothesis $\lambda + \frac{2}{3}\mu = 0$
* Incompressible flow $\operatorname{div} \vec{V} = 0$

If not

## 2-4.4 The Navier - Stoke Equations

* Substitute (2-27) into (2-19)

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x} + \lambda \operatorname{div} \vec{V}\right)$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

$$+ \frac{\partial}{\partial y}\left(2\mu \frac{\partial v}{\partial y} + \lambda \operatorname{div} \vec{V}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left(2\mu \frac{\partial w}{\partial z} + \lambda \operatorname{div} \vec{V}\right)$$

* In vector form

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \delta_{ij} \lambda \operatorname{div} \vec{V}\right]$$

## 2-4.5 Incompressible Flow: Thermal Decoupling

* $\operatorname{div} \vec{V} = 0$ & $\lambda$ vanishes
* If assume that $\mu = \text{const.}$

for constant viscosity and density

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{V} \qquad (2\text{-}30)$$

* These eqns are entirely uncoupled from temperature

## 2-4.6 Inviscid Flow: The Euler and Bernoulli Eqns

* Further assume that viscous terms are negligible

$$\rho \frac{D\vec{V}}{Dt} \cong \rho \vec{g} - \nabla p \qquad (2\text{-}30a)$$

← Euler's eqn : 1st order in $\vec{V}$ & $p$

* Tangential velocity is allowed to slip
* Euler's eqn for steady, incompressible, frictionless flow may be integrated along a streamline between 1 & 2

$$\left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_1 \cong \left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_2$$

← Bernoulli's eqn for steady frictionless flow

## 2-5 The Energy Eqn (1st law of thermodynamics)

* 1st law of thermodynamics
(Sum of the work & heat added to the system) result in
(an increase in the energy of the system)

$$dE_t = dQ + dW \qquad (2\text{-}31)$$

## 2-5.2 Summary of the Basic Equations

* 3 basic laws of conservation

$$\frac{\partial \rho}{\partial t} + \text{div}\, \rho \vec{V} = 0$$

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} + \nabla \cdot \tau_{ij}' - \nabla p$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \text{div}(k \nabla T) + \tau_{ij}' \frac{\partial u_i}{\partial x_j}$$

where, for a Newtonian fluid

$$\tau_{ij}' = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \, \text{div}\, \vec{V}$$

* 7 variables in the 3 eqns.

primary variables : $p$, $\vec{V}$, $T$

remaining variables known from auxiliary relations

$\rho = \rho(p, T)$ $\quad$ $\mu = \mu(p, T)$

$h = h(p, T)$ $\quad$ $k = k(p, T)$

← few restrictions apply

(1) the fluid forms a mathematical continuum

(2) the particles are essentially in thermodynamic equilibrium

(3) the only effective body forces are due to gravity

(4) the heat conduction follows Fourier's law

(5) there are no internal heat sources

## 2-6 BCs for Viscous Heat-Conducting Flow

* BCs in Sec 1-4 can be complex and nonlinear
* For analytical solution, simple but realistic approximations
* Fluid-solid interface

  no-slip $\vec{V}_{fluid} = \vec{V}_{solid}$ (2.52)

  no temperature jump $T_{fluid} = T_{solid}$ (2-53)

  or

  equality of heat flux $\left(k\frac{\partial T}{\partial n}\right)_{fluid} = q_0$ (2-54)

* liquid - gaseous atmosphere

  kinematic equivalence $V_{n,liq} \cong \frac{\partial n}{\partial t}$ (2-55)

  equality of normal momentum flux

  $p_{liq} \cong p_{atm}$ (2-56)

  equality of tangential momentum flux & heat flux

  $$\left.\frac{\partial V}{\partial n}\right|_{liq} = \left.\frac{\partial T}{\partial n}\right|_{liq} \cong 0 \qquad (2\text{-}57)$$

  when the atmosphere has negligibly small transport coeffs.
* Inlet, need $\vec{V}$, $p$, $T$
* Exit, need $\vec{V}$ & $T$, not $p$

  difficult ∵ of the formation of wakes and other unknown outflow behavior

  → let streamwise flow gradient vanish far downstream.

* If p can be eliminated, as with the vorticity / stream-function approach, there is no need to impose a pressure BC → vortex method

## 2-7 Orthogonal Coordinate Systems

* The basic eqns of motion are valid for any coord. system
* Problem for non-Cartesian system: derive correct formula for the grad vector $\nabla$ & the related expressions for divergence and curl
* Use metric stretching factors $h_i$

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= (h_1 dx_1)^2 + (h_2 dx_2)^2 + (h_3 dx_3)^2 \qquad (2-58)$$

$h_i$ : functions of $x_i$

HW #1 prob 1 "vector & tensor reference"

* Component of the gradient of $\phi$

$$\frac{1}{h_1}\frac{\partial \phi}{\partial x_1} \quad \frac{1}{h_2}\frac{\partial \phi}{\partial x_2} \quad \frac{1}{h_3}\frac{\partial \phi}{\partial x_3} \qquad (2-59)$$

* Divergence of $\vec{A} = (A_1, A_2, A_3)$

$$\text{div}\,\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 A_1) + \frac{\partial}{\partial x_2}(h_3 h_1 A_2) + \frac{\partial}{\partial x_3}(h_1 h_2 A_3)\right] \qquad (2-60)$$

* Component of $\vec{B} = \text{curl}\,\vec{A}$

$$B_1 = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 A_3) - \frac{\partial}{\partial x_3}(h_2 A_2)\right] \qquad (2-61)$$

Similar for $B_2$ & $B_3$

## 2-7.1 Cartesian Coordinates

* $h_i = (1, 1, 1)$

* $$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\text{div}\ \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (2\text{-}62)$$

$$\text{curl}\ \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{V}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

## 2-7.2 Cylindrical Polar Coordinates

* $(r, \theta, z)$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad (2\text{-}63)$$

$$\rightarrow h_1 = 1 \qquad h_2 = r \qquad h_3 = 1$$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z} \qquad (2\text{-}64)$$

$$\vec{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

## 2-7.3 Spherical Polar Coordinates

* $(r, \theta, \lambda)$

$$x = r\sin\theta\cos\lambda \qquad y = r\sin\theta\sin\lambda \qquad z = r\cos\theta \tag{2-65}$$

$$\rightarrow h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$$

* $$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\lambda}\right)$$

$$\text{div}\,\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\lambda}{\partial\lambda}$$

$$\vec{V}\cdot\nabla = V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + \frac{V_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda} \tag{2-66}$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\lambda^2}$$

* Complete eqns of motion → Appendix B

## 2-8 Mathematical Character of the Basic Eqns

* 2 factors that hinder our analysis (in the character of the basic eqns)
* General theory (developed from the study of quasi-linear 2nd-order eqn)

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} = D \tag{2-67}$$

A, B, C, D : non-linear functions of $x, y, \phi, \partial\phi/\partial x, \partial\phi/\partial y$

* Character of (2-67) changes, depending on the sign of the discriminant function $B^2 - 4AC$

$$\text{If } B^2-4AC \begin{cases} <0 & \text{elliptical} \\ =0 & \text{parabolic} \\ >0 & \text{hyperbolic} \end{cases} \qquad (2\text{-}68)$$

* 1. Elliptical – boundary-value problem
Can be solved only by specifying BCs on a complete contour enclosing the region

2. Parabolic – mixed initial- & boundary-value problem
BCs must be closed in one direction but remain open at one end of the other direction

3. hyperbolic – initial-value problem
Can be solved in a given region by specifying conditions at only one portion of the boundary, the other boundaries remaining open

* Laplace eqn (elliptical)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Heat-conduction eqn (parabolic)

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial\phi}{\partial y} = 0$$

Wave eqn (hyperbolic)

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0$$

* Full viscous-flow eqns are too complicated & display a variable mixture of elliptic parabolic, and hyperbolic behavior

## 2-9 Dimensionless Parameters in Viscous Flow

* Increase usefulness of solutions by casting the eqns in most efficient form → non-dimensionalize the eqns & BCs
* $\vec{V}, p, \rho, T = f(x_i, t, \mu, k, g, c_p, T_w, g_w, \ell, \mathcal{J})$

  known from data or thermodynamic state relationship
* Now select constant reference properties
  1. Ref. velocity $U$ (freestream velocity)
  2. Ref. length $L$ (body length for external flows or duct diameter for internal flows)
  3. Free stream properties $p_\infty, \rho_\infty, T_\infty, \mu_\infty, k_\infty$
* Steady viscous flows have no characteristic time → select particle residence time $L/U$
* If oscillating flow with $\omega$ → select $\omega^{-1}$
* $\left(x_i^* = \frac{x_i}{L}\right) \quad \left(t^* = \frac{tU}{L}\right) \quad \left(\vec{V}^* = \frac{\vec{V}}{U}\right) \quad \left(p^* = \frac{p - p_\infty}{\rho U^2}\right)$

  $\left(\rho^* = \frac{\rho}{\rho_\infty}\right) \quad \vec{\Phi}^* = \frac{L^2}{\mu_\infty U^2}\vec{\Phi} \quad \left(\mu^* = \frac{\mu}{\mu_\infty}\right) \quad k^* = \frac{k}{k_\infty}$

  dissipation function

  $(\nabla^* = L\nabla)$

## 2-9.1 Nondimensionalizing the Basic Eqns

* Continuity

$$\frac{\partial \rho^*}{\partial t^*} + \nabla \cdot (\rho^* \vec{V}^*) = 0$$

Navier - Stokes (2-82)

$$\rho^* \frac{D\vec{V}^*}{Dt^*} = \frac{1}{Fr}\rho^* - \nabla^* p^* + \frac{1}{Re}\nabla^* \cdot \left[\mu^* \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*}\right)\right]$$

Energy

$$\rho^* \frac{DT^*}{Dt^*} = Ec \frac{Dp^*}{Dt^*} + \frac{1}{Re\,Pr}\nabla^* \cdot (k^* \nabla^* T^*) + \frac{Ec}{Re}\Phi^*$$

* Continuity : no parameters
* N-S : $Re = \dfrac{\rho_0 U L}{\mu_0}$

$$Fr = \frac{U^2}{gL}$$

* Re : most important in fluid mechanics
  all viscous-flow phenomena depend on Re
* Fr : Important only if a free surface exists
* Energy : $Pr = \dfrac{\mu_0 C_p}{k_0}$

$$Ec = \frac{U^2}{C_p T_0}$$

* $\rho, \mu, k$ must be specified as thermodynamic functions for the particular fluid

$$\rho^*, \mu^*, k^* = f(p^*, T^*)$$

## 2-9.3 Nondimensionalizing the BCs

* E.g.

$V = U$ & $T = T_0 \rightarrow V^* = 1$ & $T^* = 0$

* @ a fixed wall

$V_w^* = 0 \quad T_w^* = 1$ or $\left(k^* \frac{\partial T^*}{\partial n^*}\right) = \frac{h_w L}{k_0 (T_w - T_0)} = Nu$

Nu: Nusselt #

* @ an interface where surface tension is important

$$p^*_{interface} = C_a + \frac{1}{Fr}\eta^* - \frac{1}{We}\left(\frac{1}{R_x^*} + \frac{1}{R_y^*}\right) \qquad (2\text{-}90)$$

$C_a = \frac{P_a - P_0}{\rho U^2}$ : cavitation #

$Fr = \frac{U^2}{gL}$ : Froude #

$We = \frac{\rho_0 U^2 L}{\sigma}$ : Weber #

* Fr : fundamental to all FS flows
never be neglected

We : Important only if small → usually small L

Ca : appropriate if $p_a = p_v$

* Summary

* 1. All viscous flows : Re
2. Variable-temperature problems : Pr, Ec (or $M_a$)
3. Natural convection : Gr, Pr
△ 4. Wall heat transfer : Nu
* 5. Slip flow : Kn, $\sigma$ (specific heat ratio)
* 6. FS conditions : Fr (always), We (sometimes)
Ca (sometimes)

* 1 more @ high Re, i.e., turbulent flow:
  - degree of surface roughness (departure from geometric similarity)
  - amount of turbulence in the reference velocity $U$ = freestream turbulence (percentage of random fluctuation)

## 2-10. Vorticity Considerations in Incompressible Viscous Flow

* Vorticity $\vec{\omega} = \text{curl } \vec{V}$
  measure of rotational effects
  = 2x(local angular velocity of fluid element)
* Instructive to examine the impact of the vorticity vector on the N-S eqn
* Assume constant $\rho$ & $\mu$, $\vec{g} = -g\hat{k}$
  Then N-S eqn becomes

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p - \rho g \hat{k} + \mu \nabla^2 \vec{V} \qquad (2\text{-}92)$$

HW #1 prob. 2

* Introduce $\vec{\omega}$ into both the acceleration and viscous terms, using

$$(\vec{V}\cdot\nabla)\vec{V} = \nabla\frac{V^2}{2} - \vec{V}\times\vec{\omega} \qquad (1\text{-}10)$$

$$\nabla^2\vec{V} = \nabla(\text{div}\,\vec{V}) - \text{curl}\,\vec{\omega} \qquad (2\text{-}93)$$

+ $\text{div}\,\vec{V} = 0$ if $\rho$ = constant

* Then

$$\rho\frac{\partial\vec{V}}{\partial t} + \nabla\left(p + \frac{1}{2}\rho\vec{V} + \rho g z\right) = \rho\vec{V}\times\vec{\omega} - \mu\,\text{curl}\,\vec{\omega} \qquad (2\text{-}94)$$

Euler terms of inviscid flow ... vanishes if $\vec{\omega} = 0$

* If $\vec{\omega}=0$ (assumption of irrotational flow)
  $\vec{V} = \nabla\phi$ by definition, (2-94) becomes
  → Bernoulli eqn for unsteady incompressible flow

$$\rho\frac{\partial\phi}{\partial t} + p + \frac{1}{2}\rho V^2 + \rho g z = \text{const.} \qquad (2\text{-}95)$$

* Bernoulli's eqn is valid even for viscous fluids if the flow is irrotational
* Every potential flow solution of classical hydrodynamics is exact solution of the full N-S eqns.
* Difficulty? Potential flows cannot satisfy the no-slip condition at a solid wall
  ∵ irrotationality assumption eliminates the 2nd ~~order~~ derivatives, leaving only 1st-order derivatives → only one velocity condition (either .. or ..) can be satisfied at a solid wall
* @ high Re, viscous flow past a solid body is primarily potential flow everywhere except close to the body, where BL exists
* In many cases, BL is so thin. It does not really disturb the outer potential flow.
  ∴ methods of classical hydrodynamics used.
* Derive an expression for the rate of change of vorticity, ← to ascertain when irrotational flow occurs.

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{V} + \nu\nabla^2\vec{\omega} \qquad (2\text{-}96)$$

← maybe by taking curl of (2-94)
→ Helmholtz eqn of hydrodynamics
usually w/o $\nu\nabla^2\vec{\omega}$

* $(\vec{\omega} \cdot \nabla)\vec{V}$ : vortex stretching term from the convective derivative
  0 in special case, low speed diffusion (no convection)
* $\nu \nabla^2 \vec{\omega}$ : viscous diffusion term
  If neglected, (2-96) leads to:
  1. Helmholtz' theorem – Strength of a vortex remains constant
  2. Lagrange's theorem – $\vec{\omega} = 0$ for all time if it is zero everywhere at $t = 0$
  3. Kelvin's theorem – circulation about any closed path moving with the fluid is constant.
* All 3 valid & useful in inviscid approximation
* All 3 fail when viscosity $\neq$ zero
  $\therefore$ $\nu \nabla^2 \vec{\omega}$ is constrained by the no-slip condition not to vanish near solid walls
* Conclusions
  1. In all viscous flows, $\vec{\omega}$ exists, being generated by relative motion near solid walls
  2. @ high Re, $\vec{\omega}$ is swept downstream & remains close to the wall
  3. BL is formed, outside of which $\vec{\omega}$ may be taken as zero

// 3/17/08

2-11 2-D considerations : The Stream Function

* Note
  stream function : exact solution of the continuity eqn when only 2 independent spatial variables are involved
* Simple case of constant $\mu$ & $\rho$
  2D flow : $u(x,y,t)$ $v(x,y,t)$
* Eqns of motion

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.97)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g_x - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (2-98)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = g_y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (2-99)$$

* (2-97) yields to $\psi(x,y,t)$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad (2-100)$$

* Cross-differentiation of (2-98) & (2-99)
  i.e., $\frac{\partial}{\partial x}(2-99) - \frac{\partial}{\partial y}(2-98)$

  p & g are eliminated

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = \nu\left(\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2}\right) \qquad (2-101)$$

  where $\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ (1-13)

* Combining (2-100) & (1-13)

$$\omega_z = -\nabla^2 \psi \qquad (2-102)$$

* (2-101) → 4th order PDE with $\psi$ as the only variable

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2\nabla^2\psi \qquad (2\text{-}103)$$

* BCs in terms of the 1st derivatives of $\psi$
  E.g. flow of a uniform stream in the x direction past a solid body

@ $\infty$ $\quad \frac{\partial\psi}{\partial x} = 0 \qquad \frac{\partial\psi}{\partial y} = U_\infty$

@ body $\quad \frac{\partial\psi}{\partial x} = 0 = \frac{\partial\psi}{\partial y}$ $\qquad (2\text{-}104)$

* Stream-function vorticity approach
  : an alternative to the primitive variable appraach of soluing for (u, v, p)

## 2-12 Noninertial Coordinate Systems

* E.g.
  flow over rotating turbine blades
  geophysical BL on a rotating earth
  → use noninertial coordinates moving with the accelerating system
* In this case, modify Newton's law, i.e., acceleration term

* $\vec{a} = \frac{d^2\vec{R}}{dt^2} + \frac{d\vec{\Omega}}{dt} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$

$$+ \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} \qquad (2\text{-}106)$$

if $\vec{V}$ : noninertial velocity vector, RHS of (2-106) replaces $D\vec{V}/Dt$ in (2-12)

## 2-12.1 Geophysical Flows

* 1st 3 terms on RHS (2-106) are neglected
∵ 1st term : earth's acceleration relative to the fixed stars
2nd term : $d\vec{\Omega}/dt \approx 0$
3rd term : earth's centripetal acceleration, i.e., variation of $\vec{g}$ with latitude
* For geophysical flows

$$\vec{a} \approx \frac{d\vec{V}}{dt} + \underbrace{2\vec{\Omega} \times \vec{V}}_{\text{Coriolis acceleration}} \qquad (2\text{-}107)$$

## 2-13 Control-Volume Formulations

* Use for integral relations to calculate the gross fluxes passing through a finite region of the flow
→ Control volume approach
* For a fixed control volume (Fig 2-4) we wish to calculate the rate of change $dB/dt$ of any gross property $B$ of the system at that instant $t$.

* By limiting process

$$\frac{dB}{dt}\ (\text{system at } t)$$

$$= \lim_{\Delta t \to 0} \frac{(B_{II} + \Delta B_{III})_{t+\Delta t} - (\Delta B_I + B_{II})_t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left[ \frac{(B_{II})_{t+\Delta t} - (B_{II})_t}{\Delta t} + \frac{\Delta B_{III}}{\Delta t} - \frac{\Delta B_I}{\Delta t} \right] \quad (2\text{-}109)$$

* By defining $d\vec{A}$ as having the direction outward normal to CS

$$\frac{dB}{dt} = \frac{d}{dt}\int_{CV} \frac{dB}{dm}\rho\, d(vol) + \int_{CS} \frac{dB}{dm}\rho\vec{v}\cdot d\vec{A} \quad (2\text{-}110)$$

→ Reynolds Transport Theorem

## 2-13.1 Conservation of Mass

* $B = m \qquad dm/dm = 1$

$$\frac{dm}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho\, d(vol) + \int_{CS} \rho\vec{v}\cdot d\vec{A} \quad (2\text{-}111)$$

## 2-13.2 Conservation of Linear Momentum

* $B = m\vec{v} \qquad d(m\vec{v})/dm = \vec{v} \quad (2\text{-}112)$

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}\int_{CV} \vec{v}\,\rho\, d(vol) + \int_{CS} \vec{v}\,(\rho\vec{v}\cdot d\vec{A})$$

# CHAPTER 3.
# Solutions of the Newtonian Viscous-Flow Equations

## 3-1 Introduction and Classification of Solutions

* Almost all the known particular solutions : incompressible newtonian flow with constant transport properties

* Continuity $\operatorname{div} \vec{V} = 0$ (3-1)

Momentum $\rho \frac{D\vec{V}}{Dt} = -\nabla \hat{p} + \mu \nabla^2 \vec{V}$ (3-2)

Energy $\rho C_p \frac{DT}{Dt} = k \nabla^2 T + \Phi$ (3-3)

where $\hat{p} = p + \rho g z$

* Since we assume $\mu = \text{const.}$, solve for $\vec{V}$ & $\hat{p}$ independently of $T$

* 2 types of exact solutions of Eq. (3-2)
1. Linear solutions : convective accel $\vec{V} \cdot \nabla$ vanishes
2. Non-linear " : $\vec{V} \cdot \nabla$ does not vanish

## 3-2 Couette Flows due to Moving Surfaces

### 3-2.1 Steady flow between a Fixed and Moving Plate

* Fig. 3-1
2 infinite plates $h$ apart.
The upper moves at speed $U$ relative to the lower
BCs are independent of $x$ or $z$

* Continuity $\frac{\partial u}{\partial x} = 0$

Momentum $0 = \mu \frac{d^2u}{dy^2}$ (3-4)

Energy $0 = k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2$ (3-5)

* Integrate (3-4) twice

$u = C_1 y + C_2$

* BCs : no slip $u(-h) = 0 \quad u(+h) = U$

$\rightarrow C_1 = U/2h \qquad C_2 = U/2$

Then $u = \frac{U}{2}\left(1 + \frac{y}{h}\right)$ (3-6)

Fig 3-2 a

* Shear stress

$\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\frac{du}{dy} = \frac{\mu U}{2h} = \text{const}$ (3-7)

* Friction coeff (dimensionless shear stress)

$C_f = \frac{\tau}{\frac{1}{2}\rho U^2} = \frac{\mu}{\rho U h} = \frac{1}{Re_h}$ (3-8)

* Introduce Poiseuille #

$Po = C_f Re_h = \frac{2h\tau}{\mu U} = 1$ (3-9)

## 3-2.2 Axially Moving Concentric Cylinders

* Fig 3-3

Either the inner cylinder moves axially at $u = U_0$ or the outer moves at $u = U_1$.

* No-slip condition → steady motion $u(r)$ with $u_\theta = u_r = 0$
* Eqns of motion in cylindrical coordinates in App. B

Continuity is satisfied if $u = u(r)$

Momentum $\nabla^2(u) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$ (3-17)

$u = C_1 \ln(r) + C_2$

If the inner moves, $u(r_0) = U_0$ & $u(r_1) = 0$

$C_1 = \frac{U_0}{\ln(r_0/r_1)}$ $\quad C_2 = -C_1 \ln(r_1)$

→ $u = U_0 \frac{\ln(r_1/r)}{\ln(r_1/r_0)}$ $\quad \tau = \frac{-\mu U_0}{r \ln(r_1/r_0)}$

If the outer moves, $u(r_0) = 0$ & $u(r_1) = U_1$

$u = U_1 \frac{\ln(r/r_0)}{\ln(r_1/r_0)}$ $\quad \tau = \frac{\mu U_1}{r \ln(r_1/r_0)}$

## 3-3 Poiseuille Flow through Ducts

* Generated by pressure gradients
* Entrance effect:

thin shear layer + core acceleration

The shear layers grow and meet, and the core disappears within a fairly short entrance length $L_e$

Regardless of duct shape,

$\frac{L_e}{D_h} \approx C_1 + C_2 Re_{D_h}$

$C_1 \approx 0.5$. $C_2 \approx 0.05$. Since transition @ $Re \approx 2000$

$L_e \leq 100\, D_h$

* For $x > L_e$, the velocity becomes purely axial
  $u = w = 0$ & $u = u(y,z)$
  → fully developed Fig. 3-6
* Continuity $\frac{\partial u}{\partial x} = 0$

Momentum
$$0 = -\frac{\partial \hat{p}}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (3\text{-}29)$$
$$0 = -\frac{\partial \hat{p}}{\partial y} = -\frac{\partial \hat{p}}{\partial z}$$

* $\hat{p}$ : function of $x$ only
  Since $u$ does not vary with $x$,
  from the $x$-momentum eqn.,
  $\partial \hat{p}/\partial x = \text{const}$
  Then $$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu}\frac{d\hat{p}}{dx} = \text{const} \qquad (3\text{-}30)$$
  subject to $u_w = 0$

## 3-5 Unsteady Flows with Moving Boundaries

* Assume parallel flow
  $u = u(y,z,t)$, $v = 0$ $w = 0$
* Momentum eqn
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{d\hat{p}}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (3\text{-}104)$$
* $d\hat{p}/dx$ : function of time only
  ∵ no $y$ or $z$ - momentum eqn
  no $x$ in (3-104)
  → absorbed into the velocity

$$u' = u + \int \frac{1}{\rho} \frac{d\hat{p}}{dx}\, dt$$

Then $\frac{\partial u'}{\partial t} = \nu \left( \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} \right)$ (3-105)

* Plane @ $y = 0 \rightarrow u = u(y,t)$
  BCs : no slip @ the plane
  + no initial motion
  $u(0,t) = U(t)$ for $t > 0$ (3-106)
  $u(y,0) = 0$ for $y > 0$
* Sudden acceleratin of the plane to constant $U_0$
  ; analogous to a conducting solid whose bottom plane is suddenly changed to a different temperature
  $\frac{u}{U_0} = 1 - \text{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$ (3-107)
  if combine y & t into a similarity variable $\eta = y / [2\sqrt{\nu t}]$, (3-107) becomes an ODE $\frac{u'}{U_0} = f(\eta)$

  Fig. 3-18 : velocity profile
* Shear layer thickness :
  point where the wall effect on the fluid has dropped to 1 percent, i.e.,
  $u/U_0 = 0.01$ or $u/U_0 = 0.99$
  $\rightarrow$ $\text{erfc}(\beta) = 0.01$ or $\beta \approx 1.82$
  $\delta \approx 3.64\sqrt{\nu t}$ (3-109)

## 3-5.2 Unsteady Flow between Two Infinite Plates

* Start-up of linear Couette flow
* The upper ($y=h$) is fixed.
  the lower ($y=0$) begins to move @ uniform $U_0$.
  Final steady flow $u = U_0(1 - y/h)$
* Work with the difference

$$u_1 = u - U_0\left(1 - \frac{y}{h}\right)$$

* Adopt dimensionless variables

$$u_1^* = \frac{u_1}{U_0} \qquad y^* = \frac{y}{h} \qquad t^* = \frac{\nu t}{h^2}$$

$$\frac{\partial u_1^*}{\partial t^*} = \frac{\partial^2 u_1^*}{\partial y^{*2}}$$

subject to $u_1^*(0, t^*) = u_1^*(1, t^*) = 0$

$u_1^*(y^*, 0) = -1 + y^*$

* Final solution by using a separation of variables $u_1^* = f(y^*)\,g(t^*)$

$$\frac{u}{U_0} = \left(1 - \frac{y}{h}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-n^2\pi^2 t^*)\sin\frac{n\pi y}{h} \qquad (3-119)$$

* The linear asymptote is approached @ $t^* \approx 0.3$
  → for air @ 20°C → $t = 2\,s$ if $h = 1\,cm$

## 3-8 Similarity Solutions

* The # of independent variables is reduced by at least one, usually by a coordinate transformation
  ∴ analogous to dimensional analysis

* Instead of parameters, the coordinates are collapsed into dimensionless groups that scale the velocities
* Similarity solutions generally require a semi-infinite or infinite spatial and temporal extent
* 3 examples here reduce a set of PDEs into ODEs.
* However, the mathematical gains are accompanied by a loss in generality i.e., Similarity solutions are limited to certain geometries and certain BCs

## 3-8.1 Viscous Flow near a Stagnation Point

* Fig. 3-24

To satisfy the no-slip for $u(x,0)$ along the wall, a viscous region must develop near the wall. → This shear layer has a constant thickness and has the effect of "displacing" the outer inviscid flow away from the wall.

### 3-8.1.1 Plane Stagnation Flow

* Continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Which can be satisfied by

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad (3\text{-}147)$$

* $\psi$ must satisfy 2D momentum eqn.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2}\right) \qquad (3\text{-}148)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y^2}\right)$$

* If inviscid flow

$\psi = Bxy \qquad u = By \qquad v = -Bx$

* Modify $\psi$ to vary with y, so that the no-slip can be satisfied

$\psi_{visc} = Bx\, f(y) \qquad u = \frac{\partial \psi}{\partial y} = Bx\frac{df}{dy}$

$v = -\frac{\partial \psi}{\partial x} = -Bf$

* For no slip at the wall, $f(0) = f'(0) = 0$
* Substitute $\psi$ in to the y-mom in (3-148)

$\frac{\partial p}{\partial y} = f(y)$ only

$\therefore \frac{\partial^2 p}{\partial x \partial y} = 0$ for this $\psi$

* Substitute this $\psi$ into the x-mom in (3-148)

$$f''' + \frac{B}{\nu}(ff'' - f'^2) = \text{const} = \frac{B}{\nu} \qquad (3\text{-}149)$$

→ Similarity is achieved, i.e., x disappears, leaving only y & ODE

* Appropriate dimensionless variables

$$\eta = y\sqrt{\frac{B}{\nu}} \qquad \psi = xF(\eta)\sqrt{B\nu} \qquad (3\text{-}150)$$

## 3-8.2 The Flow near an Infinite Rotating Disk

* Steady flow that results if the infinite plane $z = 0$ rotates at constant angular velocity $\omega$ about the axis $r = 0$ beneath a newtonian viscous fluid which would otherwise be at rest.
  Fig. 3-2d
  → Viscous drag of the rotating surface would set up a swirling flow toward the disk.
* Because of radial symmetry, the solution would be independent of $\theta$
* Continuity and the momentum eqns.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}(v_z) = 0$$

$$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right) \quad (3\text{-}18\text{i})$$

⋮

BCs: no slip @ wall & no viscous effect far from the wall

@ $z = 0$ : $v_r = v_z = 0$ $\quad v_\theta = r\omega$ $\quad p = \text{const} = 0$

as $z \to \infty$ : $v_r = v_\theta = \frac{\partial v_z}{\partial z} = 0$

* Von Karman's viscous pump
  Rotating disk sets the ner-wall fluid into circumferential motion. $v_\theta$

→ The flow would move in circular streamlines if p increases radially to balance the inward centripetal acceleration.

→ But in fact, $p = p(z)$ only; i.e., radial imbalance causes $v_r > 0$

→ This outward mass flow is balanced by an inward axial flow toward the disk, $v_z < 0$

* Karman found a similarity solution by deducing that $v_r/r$, $v_\theta/r$, $v_z$, and $p$ are all function of $z$ only.

→ 4 coupled ODEs in the single variable $z$

## 3-8.3 Jeffrey-Hamel Flow in a Wedge-Shaped Region

* Fig. 3-32

Flow in polar coordinates $(r, \theta)$, generated by a source (or sink) at the origin and by boundary at $\theta = \pm\alpha$

* Assume purely radial $v_\theta = 0$

Continuity eqn $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0$ (3-192)

$$r v_r = f(\theta)$$

* Convenient non-dimensionalization

$$n = \frac{\theta}{\alpha} \qquad \frac{v_r}{v_{max}} = f(n) \qquad (3\text{-}193)$$

expecting $v_r$ is max @ $\theta = 0$.

* Momentum eqn in polar coords $u_\theta = 0$

$$u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \cdots \qquad (3\text{-}194)$$

* Eliminate p by cross-differentiation and introduce $\eta$ & $f(\eta)$

→ 3rd order non-linear differential eqn for f

$$f''' + 2Re\,\alpha\, f f' + 4\alpha^2 f' = 0 \qquad (3\text{-}195)$$

# CHAPTER 4
# Laminar Boundary Layers

## 4-1 Introduction

* @ large Re, the viscosity effects become increasingly confined to narrow regions near solid walls
* This means that the rate of downstream convection is much larger than the rate of transverse viscous diffusion
* Flow @ speed U past a thin body of length L.

Time a fluid particle spends near the body : $L/U$

Time required for viscous effects to spread across the streamlines : $O(\sqrt{\nu L/U^3})$

Then the viscous region will be thin if

diffusion time $\ll$ residence time

$$\sqrt{\frac{\nu L}{U^3}} \ll \frac{L}{U} \qquad (4\text{-}1)$$

or $$\sqrt{\frac{UL}{\nu}} = \sqrt{Re_L} \gg 1$$

$\therefore$ a thin BL exists if $Re_L$ is large

How large? Depends on the geometry, the accuracy desired, and whether L is a streamwise or a transverse length scale

* BL is laminar at first, then undergoes transition to turbulence.

* A thin layer is not ensured on the rear side of bluff bodies.
* However, BL theory can estimate where the separation point occurs

## 4-1.1 Flat-Plate Integral Analysis

* Fig 4-1
Fluid shear against the plate ∵ no-slip
* To satisfy mass conservation, the streamlines will be deflected away, so that the fluid pressure remains constant
* $\delta_{max}$ : shear layer thickness
$\delta^*$ : displacement thickness - streamlines outside the shear layer will deflect $\delta^*$

$\therefore$ $y = H$ @ $x = 0$

$\rightarrow$ $y = Y = H + \delta^*$ @ $x = x_1$

* CV is enclosed by dashed lines;
velocity distributions are known @ inlet & exit

The other 2 boundaries must be streamlines where no mass or momentum crosses
→ wall (drag force is exposed) &
streamline outside the shear layer
(viscous drag = 0)

## 4-1.2 The Displacement Thickness

* Mass conservation for this CV

$$\iint_{CS} \rho \vec{V} \cdot d\vec{A} = 0 = \underbrace{\int_0^Y \rho u \, dy}_{\text{exit}} - \underbrace{\int_0^H \rho U \, dy}_{\text{inlet}} \qquad (4\text{-}2)$$

* Assuming $\rho$ = const (incompressible flow)

$$UH = \int_0^Y u \, dy = \int_0^Y (U + u - U) \, dy$$

$$= UY + \int_0^Y (u - U) \, dy \qquad (4\text{-}3)$$

* Rearranging & $Y = H + \delta^*$

$$U(Y - H) = U\delta^* = \int_0^Y (U - u) \, dy$$

or

$$\delta^* = \int_0^{Y \to \infty} \left(1 - \frac{u}{U}\right) dy \qquad (4\text{-}4)$$

* formal definition of BL displacement thickness

* $\delta^*$ : function of $x$ only
  ∵ $y$ variations are integrated away

## 4-1.3 Momentum Thickness as Related to Flat-Plate Drag

* x-momentum conservation

$$\Sigma F_x = -D = \iint_{CS} u(\rho \vec{V} \cdot d\vec{A})$$

$$= \int_0^Y u(\rho u) dy - \int_0^H U(\rho U) dy$$

or

$$\text{Drag} = D = \rho U^2 H - \int_0^Y \rho u^2 dy \qquad (4\text{-}5)$$

* Assume $\rho$ = const &

$$H = \int_0^Y \frac{u}{U} dy \quad \text{from (4-3)}$$

per unit depth in to the paper

$$\text{Drag} = \rho \int_0^Y u(U-u)\, dy$$

or

$$\frac{D}{\rho U^2} = \theta = \int_0^{Y\to\infty} \frac{u}{U}\left(1-\frac{u}{U}\right) dy \qquad (4\text{-}6)$$

~~*~~ definition for $\theta$ (momentum thickness)

* Note, if not a flat plate, $\theta \neq \frac{D}{\rho U^2}$

This means that $\theta$ is not as fundamental a quantity as $\delta^*$

* Fig 4-2

$\delta^* > \theta$

Shape factor $H = \frac{\delta^*}{\theta} > 1 \qquad (4\text{-}7)$

* If $\tau_w(x)$ is the wall shear on the plate,

$$D = \int_0^L \tau_w(x)\, dx \qquad (4\text{-}8)$$

* Non-dimensionalize

$$C_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U^2} \quad \text{: friction coeff.} \qquad (4\text{-}9)$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 L} \quad \text{: drag coeff}$$

* For a flat plate, specialized form

$$C_{f,plate} = \frac{d}{dx}[x\, C_D(x)] = 2\frac{d\theta}{dx} \qquad (4-10)$$

$$C_{D,plate} = \frac{1}{L}\int_0^L C_f(x)dx = \frac{2\theta(L)}{L}$$

## 4-1.4 A Guessed Profile Yields Numerical Estimates

* The beauty of Karman's idea:
  We can guess a reasonable $u(y)$ and get reasonable estimates because integration tends to wash out the positive and negative deviations of the assumed profile
* Simple $u(y)$ Fig 4-1.
  3 conditions to be satisfied
  No-slip: $u(0) = 0$
  Smooth merging: $u(\delta) = U$ $\quad \left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$
* For laminar flow

$$u \approx U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \qquad (4-11)$$

→ into (4-4) and (4-6)

$$\delta^* \approx \frac{\delta}{3} \qquad \theta \approx \frac{2\delta}{15} \qquad H \approx \frac{5}{2} \qquad (4-12)$$

* Differentiate (4-11) to relate wall shear to $u(y)$

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} \approx \frac{2\mu U}{\delta} \qquad (4-13)$$

* (4-12) & (4-13) into (4-10)

$$C_f \approx \frac{2\mu U/\delta}{\frac{1}{2}\rho U^2} = 2\frac{d\theta}{dx} \approx 2\frac{d}{dx}\left(\frac{2\delta}{15}\right)$$

or

$$\delta d\delta \approx \frac{15\mu dx}{\rho U}$$

* Integrate, assuming $\delta(0)=0$, i.e., BL begins at LE

$$\delta^2 \approx \frac{30\mu x}{\rho U} \qquad (4-14)$$

or $$\frac{\delta}{x} \approx \frac{5.5}{\sqrt{Re_x}}$$

* Substituting $\delta$ from (4-14) into (4-12) and (4-13) → flat plate estimates

$$\frac{\delta^*}{x} \approx \frac{1.83}{\sqrt{Re_x}}$$

$$\frac{\theta}{x} = C_f \approx \frac{0.73}{\sqrt{Re_x}} \qquad (4-15)$$

* Integrating $C_f(x)$ from (4-10)

$$C_D \approx \frac{1.46}{\sqrt{Re_L}} = 2C_f(L) \qquad (4-16)$$

* Only 10% error compared with the exact solution due to Blasius
* Advantage - By integrating across the shear layer in the y-direction, only an ODE to solve in the x-direction for parameters such as $\delta(x)$

## 4-1.5 Some Insight into BL Approximations

* The essence of the BL approximation:
  $\delta \ll x$
* From (4-14), this is true if $Re_x \gg 1$
* Since $v/u \leq d\delta^*/dx$, from (4-15)
  $v \ll u$ if $Re_x \gg 1$
* Differentiate (4-11), $\partial u/\partial x \ll \partial u/\partial y$ if $Re_x \gg 1$
* Summary

$Re_x \gg 1$ : $\delta \ll x$ $\quad v \ll u$ $\quad \frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$ $\quad \frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y}$

* Prandtl used these in deriving BL eqns.

## 4-2 Laminar Boundary Layer Equations

* Integral analyses yield information about the sizes of various terms

$$u, T, x = O(\text{unity}) \qquad (4\text{-}9)$$
$$v, y = O(Re^{-1/2})$$

→ used to derive the BL eqns.

* By redefining all variables in terms of these estimates, we can quickly spot which terms in the eqns are negligible if Re is large.
* 2D incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x\beta(T - T_\infty) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(4-30)

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y\beta(T - T_\infty) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

* From (4-29), define dimensionless variables, all are of order unity if Re is large

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L}\sqrt{Re} \qquad t^* = \frac{tU}{L} \qquad T^* = \frac{T - T_\infty}{T_w - T_\infty}$$

$$u^* = \frac{u}{U} \qquad v^* = \frac{v}{U}\sqrt{Re} \qquad p^* = \frac{p - p_\infty}{\rho U^2} \qquad (4\text{-}31)$$

Substitute these into (4-30)
& take the limit of these eqns as Re large

* $$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\beta(T_w - T_\infty)T^*}{Fr_x} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$0 = \frac{\partial p^*}{\partial y^*} + \frac{\beta(T_w - T_\infty)}{Fr_y}T^* \qquad (4\text{-}32)$$

All other terms have coeff $1/Re$ or $(1/Re)^2$

* Things to notice

(1) The continuity eqn is unaffected by Re

(2) $\partial^2 u/\partial x^2$ in the x-mom eqn is neglected.
Except for small velocities and large sizes, free convection is negligible in the BL approx

(3) The pressure gradient in the y direction is nearly zero. In the BL
$\partial p/\partial y \approx 0$ $\qquad p = p(x)$ only
Due to Prandtl (1904), pressure is a known variable in BL analysis, with $p(x)$ assumed to be impressed upon BL by an inviscid

outer flow analysis. → $U = U(x)$, free stream outside BL, is related to $p(x)$. by Bernoulli's theorem for incompressible flow. For steady flow

$$\frac{dp}{dx} = -\rho U \frac{dU}{dx} \qquad (4\text{-}34)$$

(5) Re does not even appear in (4-32). In this coord. system, all thicknesses are scaled to unit size; the only role of Re in laminar flows

(6) All 2nd derivatives w.r.t. $x$ are lost in BL approximation: ① eqns are parabolic instead of elliptic, $x$ is a marching variable and computer solutions are relatively easier ② lost certain BCs on $v$ and $x$: $v=0$ @ $y=0$

* BL eqns are far simpler than their parents. Rewrite (4-32) in dimensional form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (4\text{-}35a)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \approx \left(\frac{dU}{dt} + U\frac{\partial U}{\partial x}\right) + g_x \beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} \qquad (4\text{-}35b)$$

where $U = U(x,t)$ assumed known.

* BCs

No slip: $u(x,0,t) = v(x,0,t) = 0 \quad T(x,0,t) = T_w(x,t)$

Inlet: $u(x_0,y,t)$, $v(x_0,y,t)$ & $T(x_0,y,t)$ known

Patching to the outer layer: $u(x,\infty,t) \to U(x,t)$

$T(x,\infty,t) \to T_\infty(x,t)$

Initial: $u(x,y,0)$, $v(x,y,0)$ & $T(x,y,0)$ known.

* limitations are
(1) Re must be large
(2) If the outer flow is decelerating ($dU/dx < 0$, $dp/dx > 0$), a point may be reached where wall shear approaches zero, the separation point. Beyond this point, BL approximations are not accurate.
(3) @ some large $Re = O(10^6)$, the laminar solutions become unstable and transition to turbulence occcurs. The thin-layer approximations still hold the for the turbulent BL.

## 4-2.1 Orthogonal Curvilinear Coordinates

* Fig. 4-4
(4-35) are also valid for flow along the curved wall, subject only to the requirement that BL thickness $\delta$ be much smaller than the radius of curvature R of the wall.
* The chief difference between the curvilinear and the Cartesian eqns lies in pressure gradient normal to the wall, which is no longer negligibly small

$$\frac{\partial p}{\partial y} \sim \frac{\rho u^2}{R} \sim O(\text{unity}) \qquad (4-37)$$

* But if we integrate this from $y=0$ to $y=\delta$, assuming $u = Uy/\delta$

$$p(\delta) - p(0) \approx \frac{\rho U^2 \delta}{3R} \qquad (4\text{-}38)$$

or $\Delta p^* \approx \delta/3R$

← negligibly small if $\delta \ll R$

* ∴ (4-35) are valid for general curved-wall flows as long as $\delta \ll R$

## 4-2.2 General Remarks about Flow Separation

* Mom eqn @ wall, $u = v = 0$ → (4-30)

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = \frac{1}{\mu}\frac{dp}{dx} \qquad (4\text{-}39)$$

* Profile curvature : indicator of possible BL separation → Fig 4-5(a)
* Negative (favorable) $dp/dx$ → no flow separation
* Zero $dp/dx$ → e.g. flat plate flow, no separation

$\frac{\partial u}{\partial y} = ay + b$

THUS

if $a > 0$, adv $\frac{dp}{dx}$

$a = 0$, zero $\frac{dp}{dx}$

$a < 0$, fav $\frac{dp}{dx}$

* Positive (adverse) $dp\,dx$ → curvature changes sign & the profile is S shaped.
  → Increasing downstream pressure slows down the wall flow → make it go backward → flow separation
* Fig 4-5(b), persistent adverse $dp/dx$ makes the profile more S-shaped, reducing the wall shear to zero and then causing backflow, while BL becomes much thicker.
* Laminar flows have poor resistance and separate easily.
* Turbulent BLs can resist separation longer.

PAGE

at the expense of increased wall friction and heat transfer

## 4-2.3 Shear Stress in a Boundary Layer

* Shear stress : a major interest

In 2D $\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

* By the order-of-magnitude analysis, $\partial v/\partial x$ is 2 orders smaller than $\partial u/\partial y$

$\therefore \tau \simeq \mu \frac{\partial u}{\partial y}$ $\quad \tau_w \simeq \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ (4-40)

* Friction drag : integrated value of the local shear stress
* Approximate point of separation : point of zero shear stress
  → the point where $\tau_w$ passes from (+) to (-).

## 4-3 Similarity Solutions for Steady 2D Flow

* Neglect buoyancy → uncoupled from the temperature
* Mom & continuity eqns for steady flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (4\text{-}41a)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (4\text{-}41b)$$

Subject to $u(x,0) = v(x,0) = 0$

$u(x,\infty) = U(x)$

◀ System of PDEs of the parabolic type

* Reduce these eqns to ODEs by similarity transformation; i.e., 2D PDEs of continuity and momentum can be reduced to a single ODE

## 4-3.1 The Blasius Solution for Flat-Plate Flow

* Similarity to "Plane Stagnation Flow" in Sec. 3-8.1.1

* Blasius eqn derivation from 노트

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7.5.1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (7.5.2)$$

BCs $u = v = 0$ @ $y = 0$

$u = U_\infty$ @ $y = \infty$ (7.5.3)

IC $u = U_\infty$ for all $y$ @ $x = 0$

Introduce Streamfunction

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad (7.5.4)$$

The continuity is satisfied automatically.

(7.5.2) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3} \qquad (7.5.5)$$

Infinite flat plate → velocity profile is similar, i.e., dimensionless velocity profiles @ 2 different locations are same

$$\left.\frac{u[\eta(x,y)]}{u_\infty}\right|_{x_1} = \left.\frac{u[\eta(x,y)]}{u_\infty}\right|_{x_2} \qquad (7.5.6)$$

where $\eta$ : similarity variable

(7.5.5) should be an ODE

Assume

$$\eta = f(x,y) = Ay x^\alpha \qquad (7.5.7)$$

$$\psi(x,y) = B x^\beta f(\eta) \qquad (7.5.8)$$

↳ thin BL, 2nd derivative wrt y is zero

Determine $\alpha$ & $\beta$, so (7.5.5) becomes an ODE for $f$ with $\eta$ as independent variable

$$u = \frac{\partial \psi}{\partial y} = B x^\beta f' \frac{\partial \eta}{\partial y} \qquad (7.5.9)$$

$$= AB x^{\alpha+\beta} f'(\eta)$$

$$v = -\frac{\partial \psi}{\partial x} = B\beta x^{\beta-1} f + B x^\beta f' \frac{\partial \eta}{\partial x} \qquad (7.5.10)$$

$$= -B x^{\beta-1}(\alpha \eta f' + \beta f)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)$$

$$= AB(\alpha+\beta) x^{\alpha+\beta+1} f' + AB x^{\alpha+\beta} f'' \frac{\partial \eta}{\partial x} \qquad (7.5.11)$$

$$= AB x^{\alpha+\beta+1}[(\alpha+\beta) f' + \alpha \eta f'']$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = AB x^{\alpha+\beta} f'' \frac{\partial \eta}{\partial y} \qquad (7.5.12)$$

$$= A^2 B x^{2\alpha+\beta} f''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = A^3 B x^{3\alpha+\beta} f''' \qquad (7.5.13)$$

Substitute (7.5.9) – (7.5.13) into (7.5.5)

$$(\alpha+\beta) f'^2 + \alpha\eta f' f'' - f''(\alpha\eta f' + \beta f) = \frac{\nu A}{B} x^{\alpha-\beta+1} f''' \qquad (7.5.14)$$

The above eqn should be an ODE for a function of $\eta$ only. Then RHS terms should be independent of $x$, i.e.,

$$\alpha - \beta + 1 = 0 \qquad (7.5.15)$$

$\nu A / B$ should be a const, because all the terms in (7.5.14) are dimensionless. Let it be an arbitrary constant $1/2$

$$\frac{\nu A}{B} = \frac{1}{2} \qquad (7.5.16)$$

We need 2 more eqns to determine $\alpha$, $\beta$, A & B

Apply $u = U_\infty$ @ $y = \infty$ to (7.5.9)

$$AB x^{\alpha+\beta} f'(\eta) = U_\infty \qquad (7.5.17)$$

Let $f'(\infty)$ be 1, because it can be any finite number

$$f'(\infty) = 1 \qquad (7.5.18)$$

Then $\alpha + \beta = 0$ ($x$-independent) (7.5.19)

$$AB = U_\infty \qquad (7.5.20)$$

From (7.5.15), (7.5.16), (7.5.19) & (7.5.20)

$$\alpha = -\frac{1}{2} \quad \beta = \frac{1}{2} \quad A = \sqrt{\frac{U_\infty}{2\nu}} \quad B = \sqrt{2\nu U_\infty} \qquad (7.5.21)$$

Substitute the above into (7.5.7), (7.5.8) and (7.5.14)

$$\eta = y\sqrt{\frac{U_\infty}{2\nu x}} \qquad (7.5.22)$$

$$\psi = \sqrt{2\nu U_\infty x}\, f(\eta) \qquad (7.5.23)$$

$$u = \frac{\partial \psi}{\partial y} = U_\infty f' \qquad (7.5.24)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{2x}}(\eta f' - f) \qquad (7.5.25)$$

$$f''' + f f'' = 0 \qquad (7.5.26)$$

with BCs

$$f = 0 \;\&\; f' = 0 \quad @\ \eta = 0 \qquad (7.5.27)$$
$$f' = 1 \quad @\ \eta = \infty$$

$$\therefore\ f''' + f f'' = 0 \qquad (7.5.28)$$

$$\text{with } f(0) = f'(0) = 0 \quad f'(\infty) = 1 \qquad (7.5.29)$$

→ Blasius eqn

HW solve Blasius eqn numerically

* With PC, it is a simple matter to program Eqs (4-45) & (4-46)
→ Complete Blasius eqn solution in Table 4-1
* Fig 4-6(a): Blasius function $f$, $f'$ & $f''$
Fig 4-6(b): comparison of $f' = u/U$ with exp.
* $f' = 0.99$ @ $\eta \simeq 3.5$

$$\therefore\ \delta_{99\%} \simeq 3.5\sqrt{\frac{2\nu x}{U}} = 3.5 x\sqrt{\frac{\nu}{Ux}}\cdot\sqrt{2}$$

$$\text{or } \frac{\delta_{99\%}}{x} \simeq \frac{5.0}{\sqrt{Re_x}}$$

* Displacement thickness

$$\delta^*\sqrt{\frac{U}{2\nu x}} = \int_0^\infty (1 - f')\,d\eta = \lim_{\eta\to\infty}(\eta - f) = 1.21678$$

or $\frac{\delta^*}{x} = \frac{1.720}{\sqrt{Re_x}}$

1.2167$\delta$ : $\eta$ intercept of a 45° line that is asymptotic to the curve $f(\eta)$ @ large $\eta$
Since $f$ is proportional to $\psi$, the dashed 45° line represents an inviscid stream function displaced a dimensionless amount 1.2167$\delta$ away from the plate

* Momentum thickness

$$\theta\sqrt{\frac{U}{2\nu x}} = \int_0^\infty f'(1-f')\,d\eta = f''(0) = 0.4696$$

or $\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$

* Wall shear stress

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{\sqrt{2\nu x/U}}$$

or $C_f = \frac{2\tau_w}{\rho U^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}$ " eq (4-10) "

* Integrated drag coeff on a plate of L

$$C_D(L) = \frac{1}{L}\int_0^L C_f\,dx = 2C_f(L) = \frac{1.32\delta}{\sqrt{Re_L}} \quad (4\text{-}53)$$

↳ $\frac{1}{L}\int_0^L \frac{0.664}{\sqrt{U/\nu}}\,x^{-1/2}\,dx$ ↖

← drag on one side of the plate.

* Fig. 4-7
Blasius relation is accurate for $Re_L \geq 1000$
Oseen theory is valid for $Re_L \leq 1$

* $1 < Re_L < 1000$ : fit reasonably by the correction factor given in a perturbation theory by Imai

$$C_D \approx \frac{1.328}{\sqrt{Re_L}} + \frac{2.3}{Re_L} \qquad (4\text{-}54)$$

* Note that $v \neq 0$ @ edge of BL

from (4-44), as $\eta \to \infty$

$$v(x, \infty) = \frac{0.8604 U}{\sqrt{Re_x}} \qquad (4\text{-}55)$$

3/31/08

## 4-4 Free-Shear Flows

* Free-shear layers

Unaffected by walls, develop & spread in open ambient flow

* Possesses vel. gradients created by some upstream mechanism → try to smooth out by viscous diffusion in convective deceleration
* Examples

(1) free-shear layer between parallel moving streams

(2) jet

(3) wake behind a body in a stream

* If $Re \gg 1$, BL approximations will hold.

$v \ll u$, $\partial u/\partial x \ll \partial u/\partial y$, $\partial p/\partial y \approx 0$

Further, if no confining walls → $dp/dx = 0$

→ satisfy the flat-plate eqns.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \cong \nu\frac{\partial^2 u}{\partial y^2} \qquad (4\text{-}91)$$

except no walls to enforce no-slip condition

* Just downstream of the disturbance
  → developing & nonsimilar

Further downstream
  → similar & vel profiles look alike when suitably scaled.

## 4-4.1 The Free-Shear Layer between Two Different Streams

* Fig. 4-17a
  The discontinuity between $U_1$ & $U_2$ is smoothed out by viscosity into an S-shaped free-shear layer
* Simplest - $U_2 = 0$ : plane airflow emerging from a slot into ambient air at rest.
* Lock (1951) - Blasius type similarity variable

$$\eta_j = y\sqrt{\frac{U_1}{2\nu_j x}}, \qquad f_j' = \frac{u_j}{U_1} \qquad j = 1, 2 \qquad (4\text{-}92)$$

* (4-92) → (4-91) : Blasius type eqn. for each layer

$$f_j''' + f_j f_j'' = 0 \qquad J = 1, 2 \qquad (4\text{-}93)$$

* BCs

(1) asymptotic approach to freestream velocities

$$f_1'(+\infty) = 1 \qquad f_2'(-\infty) = U_2/U_1 \qquad (4\text{-}94)$$

(2) kinematic equality $u_1 = u_2$ & $v_1 = v_2$

@ interface $\eta_j = 0$

$$f_1(0) = f_2(0) = 0 \qquad (4\text{-}95)$$

$$f_1'(0) = f_2'(0) \neq 0$$

(3) equality of shear stress @ interface

$$\mu_1 \frac{\partial u_1}{\partial y}(0) = \mu_2 \frac{\partial u_2}{\partial y}(0) \qquad (4\text{-}96)$$

or $f_1''(0) = k^{1/2} f_2''(0)$

where $k = (\rho_2 \mu_2 / \rho_1 \mu_1)$

$k = 1$ : identical fluids

$k \gg 1$ : gas over liquid

$k \approx 60{,}000$ : air–water

* Fig. 4-17c : solutions by Lock

$k = 60{,}000$ → slow wind-driven flow in the surface layer of a lake

## 4-4.2 The Plane Laminar Jet

* Fig. 4-18

plane jet emerging into a still ambient fluid from a slot at $x = 0$

* At constant pressure & no bounding walls

→ (4-91) must be satisfied

+

constant momentum flux across any $x = \text{const}$ cross section

$$J = \rho \int_{-\infty}^{+\infty} u^2 \, dy = \text{const} \qquad (4\text{-}97)$$

→ zero-drag, zero-freestream version of (4-5)

* Schlichting

Jet entrainment spreads as $x^{1/3}$

$$\psi = \nu^{1/2} x^{1/3} f(\eta) \qquad (4\text{-}98)$$

where $\eta = \dfrac{y}{3\nu^{1/2} x^{2/3}}$

* Corresponding vel components

$$u = \frac{f'(\eta)}{3x^{1/3}}$$

$$v = \frac{-\nu^{1/2}}{3x^{2/3}} (f - 2f'\eta) \qquad (4\text{-}99)$$

* (4-98) & (4-99) → (4-91)

$$f''' + ff'' + f'^2 = 0 \qquad (4\text{-}100)$$

* BCs

Symmetry about X-axis : $V = \partial u/\partial y = 0$ @ $y = 0$

Quiescent ambient fluid : $u = 0$ @ $y = \infty$

$$\rightarrow \quad f(0) = f''(0) = 0 \qquad (4\text{-}101)$$
$$f'(\infty) = 0$$

* Schlichting's exact solution

$$f(\eta) = 2a \tanh(a\eta) \qquad (4\text{-}102)$$

or $f'(\eta) = 2a^2 \operatorname{sech}^2(a\eta)$

* 'a' from evaluating J

$$J = \rho \int_{-\infty}^{+\infty} \left(\frac{2a^2}{3x^{1/3}} \operatorname{sech}^2 a\eta\right)^2 3\nu^{1/2} x^{2/3} d\eta = \frac{16}{9} \rho \nu^{1/2} a^3$$

$$\text{or} \quad a = \left(\frac{9J}{16\sqrt{\rho\mu}}\right)^{1/3} \approx 0.8255 \frac{J^{1/3}}{(\rho\mu)^{1/6}} \qquad (4\text{-}103)$$

* Since $\operatorname{sech} 0 = 1$

$$u_{max} = \frac{2a^2}{3x^{1/3}} \approx 0.4543 \left(\frac{J^2}{\rho\mu x}\right)^{1/3} \qquad (4\text{-}104)$$

→ Jet spreads so that the centerline vel drops off as $x^{-1/3}$

* Velocity distribution (4-105)

$$u(x,y) = U_{max}\,\mathrm{sech}^2 a\eta = U_{max}\,\mathrm{sech}^2\left[0.2752\left(\frac{J\rho}{\mu^2x^2}\right)^{1/3} y\right]$$

* $$\text{Width} = 2y\Big|_{1\%} = b \approx 21\delta\left(\frac{x^2\mu^2}{J\rho}\right)^{1/3} \quad (4\text{-}106)$$

↑ where $u = 0.01\,U_{max}$

→ Jet spreads as $x^{2/3}$

* Mass rate of flow across any vertical plane

$$\dot{m} = \rho\int_{-\infty}^{+\infty} u\,dy = (36\,J\rho\mu x)^{1/3} \approx 3.302(J\rho\mu x)^{1/3} \quad (4\text{-}107)$$

* Since jet vel profiles are S-shaped, i.e., a point of inflection,
→ unstable & transition to turbulence early
→ Jets are more likely to be turbulent

## 4-4.3 The Plane Laminar Wake : Far field Approximation

Fig. 4-19

* Wake - defect in stream velocity behind an immersed body
* Slender plane body → smooth wake whose vel defect decays monotonically downstream
* Blunt body → distorted wake by alternating shed vortex structure
* About 3 L downstream, wake becomes developed with similar profiles
* Consider simple farfield approximation for a developed wake

$u_1(x,y) = U_0 - u(x,y) \ll U_0$ (4-108)

* Momentum eqn (4-91)

$$U_0 \frac{\partial u_1}{\partial x} \approx \nu \frac{\partial^2 u_1}{\partial y^2} \quad (4\text{-}109)$$

subject to $u_1(x, \pm\infty) = 0$

$\partial u_1/\partial y = 0$ @ $y = 0$

* $$u_1 = B U_0 x^{-1/2} \exp\left(-\frac{U_0 y^2}{4x\nu}\right) \quad (4\text{-}110)$$

← soln. of linear heat conduction eqn

* In the far field, Gaussian vel distribution whose centerline value drops off as $x^{-1/2}$

* Obtain B by

(Body drag) = (momentum flux defect in wake)

$$F = C_D \tfrac{1}{2}\rho U_0^2 L \qquad F = \int_{-\infty}^{\infty} \rho u u_1 dy \approx \rho U_0 \int_{-\infty}^{\infty} u_1 dy \quad (4\text{-}111)$$

$$= 2\rho U_0^2 B \sqrt{\frac{\pi\nu}{U_0}}$$

* Wake velocity

$$\frac{u_1}{U_0} = C_D \left(\frac{Re_x}{16\pi}\right)^{1/2} \left(\frac{L}{x}\right)^{1/2} \exp\left(-\frac{U_0 y^2}{4x\nu}\right) \quad (4\text{-}112)$$

→ wake defect $\propto$ body drag

* Flat plate wetted on both sides, (4-113)

$$C_D = \frac{2.656}{\sqrt{Re_L}} \leftarrow \frac{u_1(x,0)}{U_0}\bigg|_{\text{flat plate}} = \frac{0.664}{\sqrt{\pi}}\left(\frac{L}{x}\right)^{1/2}$$

## 4-5 Other Analytic 2D Solutions

* Typical flow in practice : nonsimilar & changes from adverse to favorable gradients.

* 3 basic approaches to analyze nonsimilar BL
  (1) Analytic continuation by series expansion – obsolete
  (2) Approximate integral methods – extension of the integral method
  (3) Numerical modeling on a digital computer

## 4-6 Approximate Integral Methods

* The CV approach is exact in principle, but approximate in practice, using guessed profiles and other correlations
* One by Thwaites

## 4-6.1 The BL Integral Equations

* All integral methods use partially integrated forms of the continuity, momentum, and energy eqns.
* Continuity is eliminated during the derivation, → 2 basic results
  (1) momentum integral relation
  (2) thermal-energy integral relation
* BL eqns

Continuity : $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (4\text{-}35a)$$

Momentum : $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$\simeq \left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \qquad (4\text{-}35b)$$

Thermal energy: $\rho c_v \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$

$$\approx -\frac{\partial q}{\partial y} + \tau\frac{\partial u}{\partial y}$$

Where $\tau = \mu\frac{\partial u}{\partial y}$ $\qquad q = -k\frac{\partial T}{\partial y}$

## 4-6.2 The Mometum - Integral Relation

* (Momentum) − (u − U) × (Continuity)

$$-\frac{1}{\rho}\frac{\partial \tau}{\partial y} = \frac{\partial}{\partial t}(U-u) + \frac{\partial}{\partial x}(uU - u^2) + (U-u)\frac{\partial U}{\partial x} + \frac{\partial}{\partial y}(vU - vu) \qquad (4\text{-}119)$$

* Integrate from the wall to infinity

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial t}\int_0^\infty (U-u)\,dy + \frac{\partial}{\partial x}\int_0^\infty u(U-u)\,dy + \frac{\partial U}{\partial x}\int_0^\infty (U-u)\,dy - U v_w \qquad (4\text{-}120)$$

$\begin{cases} v=0 & @\ \infty \\ u=0 & @\ 0 \end{cases}$

→ Kármán integral relation

* $$\delta^* = \int_0^{Y\to\infty}\left(1-\frac{u}{U}\right)dy \qquad (4\text{-}4)$$

$$\theta = \int_0^{Y\to\infty}\frac{u}{U}\left(1-\frac{u}{U}\right)dy \qquad (4\text{-}6)$$

* $$\frac{\tau_w}{\rho U^2} = \frac{C_f}{2} = \frac{1}{U^2}\frac{\partial}{\partial t}(U\delta^*) + \frac{\partial\theta}{\partial x} + (2\theta+\delta^*)\frac{1}{U}\frac{\partial U}{\partial x} - \frac{v_w}{U} \qquad (4\text{-}121)$$

* For steady flow with an impermeable wall

$$\frac{c_f}{2} = \frac{d\theta}{dx} + (2+H)\frac{\theta}{U}\frac{dU}{dx} \qquad (4-122)$$

where $H = \frac{\delta^*}{\theta} \approx$ 2.0 @ stagnation
3.5 @ separation

## 4-6.5 One-Parameter Integral Methods

* Approximate methods for the 3 relations → 2 basic categories
  (1) guessed velocity & temperature profiles
  (2) empirical correlations among the integral parameters
* For (4-122), approximate $u$ by a one-parameter family of vel. profiles

$$u(x,y) \approx U(x) f[\eta, P(x)] \qquad (4-130)$$

where $\eta = y/\delta$
$P$ : dimensionless parameter

From such a profile, compute $\theta$, $\delta^*$, $H$, and $\tau^*$ the substitute into (4-122) to obtain a 1st order differential eqn for $\delta(x)$

* Polhausen proposed a 4th-order polynomial

$$\frac{u}{U} \approx 2\eta - 2\eta^3 + \eta^4 + \frac{\Lambda}{6}[\eta(1-\eta)^3] \qquad (4-131)$$

where $\Lambda = \delta^2 (dU/dx)/\nu$ varies with the local pressure gradient : Polhausen parameter

* Polhausen's profile : not very accurate

## 4-6-6 The Correlatin Method of Thwaites

* Rewrite (4-122) in terms of a better parameter $\lambda$

$$\lambda = \frac{\theta^2 U'}{\nu} = \left(\frac{\theta}{\delta}\right)^2 \Lambda \qquad (4-132)$$

* $U\theta/\nu \times$ (4-122)

$$\underbrace{\frac{\tau_w \theta}{\mu U}}_{S(\lambda)} = \frac{U\theta}{\nu}\frac{d\theta}{dx} + \frac{\theta^2 U'}{\nu}(2+H) \qquad (4-133)$$

$S(\lambda)$: Shear correlatin

$\theta d\theta = d(\theta^2/2)$

$H(\lambda)$: Shape-factor correlatin

* Rewrite (4-133)

$$U\frac{d}{dx}\left(\frac{\lambda}{U'}\right) \approx 2[S(\lambda) - \lambda(2+H)] = F(\lambda) \qquad (4-135)$$

* Fig 4-22, Thwaites proposed

$$F(\lambda) \approx 0.45 - 6.0\lambda \qquad (4-136)$$

* If $F = a - b\lambda$, (4-135) has a closed-form solutin

$$\frac{\theta^2}{\nu} = a U^{-b}\left(\int_{x_0}^{x} U^{b-1}\,dx + C\right) \qquad (4-137)$$

If $x_0$ is a stagnatin pt., $C$ must be zero to avoid an infinite $\theta$ where $U = 0$.

* $$\theta^2 \approx \frac{0.45\nu}{U^6}\int_0^x U^5\,dx \qquad (4-138)$$

→ calculate $\lambda = \theta^2 U'/\nu$ and

$$\tau_w = \frac{\mu U}{\theta} S(\lambda) \qquad (4-139)$$

$$\delta^* = \theta H(\lambda)$$

* Thwaites' $S(\lambda)$ & $H(\lambda)$ in Table 4-4 & Fig 4-23
* A simple & accurate curve fit

$$S(\lambda) \cong (\lambda + 0.09)^{0.62} \qquad (4\text{-}140)$$

$$H(\lambda) \cong 2.0 + 4.14z - 83.5z^2 + 854z^3 - 3337z^4 + 4576z^5 \qquad (4\text{-}141)$$

Where $z = (0.25 - \lambda)$

* Accuracy : ±5% for favorable or mild adverse gradients
  ±15% near the separation pt.
* Applications Secs 4-6.7 & 4-6.9

## 4-10. Rotationally Symmetrical BLs

* A preliminary to more general 3D BLs.
* Flow in cylindrical polar coordinates $(r, \theta, z)$ where none of $(U_r, U_\theta, U_z, p, T)$ depends on $\theta$
* Swirl : circumferential velocity $U_\theta(r, z)$
  If $U_\theta = 0 \rightarrow$ axisymmetric flow
* Examples
  Pipe flow (inc. uniform swirl), Couette flow between rotating cylinders, axisymmetric stagnation flow, flow near an infinite rotating disk
* BL eqns under rotational symmetry : curvilinear coordinates $(x, y, \theta)$ & vel components $(u, v, w)$
  Fig. 4-34

Continuity: $\frac{\partial}{\partial x}(r_o u) + r_o \frac{\partial v}{\partial y} = 0$ (4-179a)

x-mom: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w}{r}\frac{dr_o}{dx}$

$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$ (4-179b)

y-mom: $\frac{\partial p}{\partial y} = 0$ (4-179c)

θ-mom: $\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{uw}{r_o}\frac{dr_o}{dx}$

$= \nu\frac{\partial^2 w}{\partial y^2}$ (4-179d)

Where $r_o$ : local surface radius

$r_o \gg \delta$

* @ the edge of BL, $U(x)$ & $W(x)$

$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} - \frac{W^2}{r_o}\frac{dr_o}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ (4-180)

$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + \frac{UW}{r_o}\frac{dr_o}{dx} = 0$

* BCs (including a porous surface rotating about its axis @ $\Omega(t)$)

@ $y=0$ : $u=0$, $v=V_w(x,t)$, $w = r_o\Omega(t)$

@ $y\to\infty$ : $u=U(x,t)$, $w = W(x,t)$ (4-181)

@ $x=0$ : known $u(0,y,t)$, $v(0,y,t)$, $w(0,y,t)$

@ $t=0$ : known $u(x,y,0)$, $v(x,y,0)$, $w(x,y,0)$

## 4-10.1 Axisymmetric BLs: Similarity Solutions

* Simpler special case – steady axisymmetric flow ($w=0$)

(4-179) reduces to

$$\frac{\partial}{\partial x}(r_0 u) + r_0 \frac{\partial}{\partial y}(v) = 0 \qquad (4\text{-}182)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

with $u(x,0) = 0$, $v(x,0) = v_w(x)$

$u(x,\infty) \to U(x)$

* Similarity solutions to (4-182) may be found if $U(x) = Cx^n$ (4-183)

→ correspond to potential flow @ $0°$ $\alpha$ past a cone of half-angle $\phi$

→ Table 4-8

* The proper similarity variables

$$u(x,y) = U(x) f'(\eta) \qquad (4\text{-}184)$$

where $\eta = y\left[\frac{(3+n)U(x)}{2x\nu}\right]^{1/2}$

* Substitution into (4-182)

$$f''' + ff'' + \frac{2n}{3+n}(1 - f'^2) = 0 \qquad (4\text{-}185)$$

with $f(0) = f'(0) = 0$ & $f'(\infty) = 1$

← Identical to the Falkner-Skan relation, except that $\beta$ is different

* $$\beta_{cone} = \frac{2n}{3+n} = \beta_{wedge} = \frac{2m}{1+m} \qquad (4\text{-}186)$$

$m_{wedge} = \frac{1}{2} n_{cone}$

## 4-10.2 A Thwaites-Type Method for Axisymmetric Flow

* (4-188) into integral form

$$\frac{C_f}{2} = \frac{1}{U^2}\frac{\partial}{\partial t}(U\delta^*) + \frac{2\theta+\delta^*}{U}\frac{\partial U}{\partial x} + \frac{\partial \theta}{\partial x} + \underbrace{\frac{\theta}{r_0}\frac{dr_0}{dx}} - \frac{V_w}{U} \qquad (4\text{-}188)$$

difference from plane flow

* Steady axisymmetric flow with an impermeable wall

$$\frac{\tau_w}{\rho} = \frac{1}{r_0}\frac{d}{dx}(r_0 U^2\theta) + U\frac{dU}{dx}\delta^* \qquad (4\text{-}189)$$

* Regroup in terms of $\lambda = \theta^2 (dU/dx)/\nu$

$$\frac{U}{r_0^2}\frac{d}{dx}\left(\frac{r_0^2\lambda}{dU/dx}\right) = 2S - 2\lambda(2+H)$$

$$= F(\lambda) \approx 0.45 - 6\lambda \qquad (4\text{-}190)$$

* Integrate (4-190)

$$\theta^2 \approx \frac{0.45\nu}{r_0^2 U^6}\int_0^x r_0^2 U^5\, dx \qquad (4\text{-}191)$$

4/7/08

## 4-10.6 The Narrow Axisymmetric Jet

* A round jet from a circular hole with sufficient momentum, $\partial/\partial r \gg \partial/\partial x$

(radial change) (axial change)

* Axisymmetric BL eqns

Continuity: $\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0 \qquad (4\text{-}201a)$

x-mom: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad (4\text{-}201b)$

* Jet thickness grows linearly, $\therefore$ $r/x$ is the proper similarity variable
* Stream function

$$\psi(r,x) = \nu x F(\eta) \qquad \eta = r/x \qquad (4-202)$$

$$\rightarrow u = \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{\nu F'}{r} \qquad (4-203)$$

$$v = -\frac{1}{r}\frac{\partial \psi}{\partial x} = \frac{\nu}{r}(\eta F' - F)$$

* Substitution into (4-201b) (4-204)

$$\frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right) = \frac{1}{\eta^2}(FF' - \eta F'^2 - \eta F F'')$$

→ devoid of $x$ & $r$

- BCs $F(0) = F'(0) = F'(\infty) = 0$

* Exact solution $F = \dfrac{(C\eta)^2}{4 + (C\eta/2)^2}$ (4-205)

* $C$ is determined from the momentum of the jet

$$J = \rho \int_0^\infty u^2 2\pi r\, dr = \frac{16\pi}{3}\rho C^2 \nu^2$$

$$\text{or} \quad C = \left(\frac{3J}{16\pi\rho\nu^2}\right)^{1/2} \qquad (4-206)$$

* Axial jet velocity, from (4-203)

$$u = \frac{3J}{8\pi\mu x}\left(1 + \frac{C^2\eta^2}{4}\right)^{-2} \qquad (4-207)$$

↳ jet centerline vel drops off as $x^{-1}$

* $\dot{m} = \rho \int_0^\infty u 2\pi r\, dr = 8\pi\mu x$ (4-208)

## 4-10.7 Axisymmetric Wakes

* (4-201b) is linearized to

$$U_0 \frac{\partial u_1}{\partial x} \approx \frac{\nu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_1}{\partial r}\right) \qquad (4\text{-}209)$$

subject to $\frac{\partial u_1}{\partial r}(0,x) = 0$, $u_1(\pm\infty, x) = 0$, $u_1(r,\infty) = 0$

* Gaussian wake-velocity distribution

$$u_1 = \frac{C}{x} U_0 \exp\left(-\frac{U_0 r^2}{4x\nu}\right) \qquad (4\text{-}210)$$

where

$$F \approx \rho U_0 \int_0^\infty u_1 2\pi r\, dr = \rho U_0^2 C \frac{4\pi\nu}{U_0} = C_D \frac{1}{2}\rho U_0^2 L^2$$

$$\rightarrow C = C_D U_0 L^2 / 8\pi\nu$$

* The wake defect velocity

$$\frac{u_1}{U_0} = C_D \frac{U_0 L}{8\pi\nu} \frac{L}{x} \exp\left(-\frac{U_0 r^2}{4x\nu}\right) \qquad (4\text{-}211)$$

## 4-11 Asymptotic Expansions and Triple-Deck Theory

* BL theory has vexing limitations, even for incompressible flows
  1. High Re $= O(1000)$
  2. Not valid in separated-flow regions
  3. Not valid too near LE (where $Re_x \ll Re_L$)
  4. Not valid @ TE, where no slip gives away to wake
  5. Not able to predict the effect of viscous streamline displacement on the outer stream
  6. Not able to predict the effect of wall curvature, whether transverse or longitudinal

7. Not able to predict the effect of finite shear (vorticity) in the outer stream

* Many attempts to extend BL theory

## 4-11.1 Friedrichs' Model of the Boundary Layer

* One trouble with BL momentum eqn → singular perturbation problem
* As $u \to 0$ (inviscid flow), drop back to 1st order derivatives and no-slip is lost. ∴ high Re flow gives a singular change in the near-wall flow pattern
* Friedrichs' model

$$\varepsilon \frac{d^2u}{dy^2} + \frac{du}{dy} = a \qquad \varepsilon \ll 1 \qquad (4\text{-}212)$$

with $u(0) = 0 \qquad u(1) = 1$

→ rough resemblance to BL momentum (4-41) with $\varepsilon = 1/Re$ and $a < 1$ modeling the x-derivative terms

* Exact solution of (4-212)

$$u = ay + (1-a)\frac{1 - \exp(-y/\varepsilon)}{1 - \exp(-1/\varepsilon)} \qquad (4\text{-}213)$$

* In the limit as $\varepsilon \to 0$, (4-212) is $u' = a$ → solution $u = ay + C$

BC $u(1) = 1$ → $u_{outer} = ay + 1 - a$ (4-214)

does not satisfy no-slip @ wall

* The inner region is thin $O(\varepsilon)$ where $\varepsilon \ll 1$. → Define a magnified variable

$Y = y/\varepsilon$ $\quad F(Y) = f(y)$

* Making this change in (4-212)

$$\frac{d^2F}{dY^2} + \frac{dF}{dY} = a\varepsilon \approx 0 \qquad (4\text{-}215)$$

no-slip $F(0) = 0$

$$F_{inner} = C(1 - e^{-Y}) \qquad (4\text{-}216)$$

C: arbitrary ∵ it is not valid in the outer region

## 4-11.2 The Matching Principle

* How can match the two to approximate the desired solution?
* Leave (4-214) alone & rewrite (4-216) in terms of y

$$F = C(1 - e^{-y/\varepsilon}) \approx C \qquad (4\text{-}217)$$

* Comparing (4-217) & (4-214), best-match value of C is (1-a)

$$F_{inner} \approx (1-a)(1 - e^{-Y}) \qquad (4\text{-}218)$$

$$f_{outer} \approx (1-a) + ay$$

→ Fig 4-39 b for $\varepsilon = 1$

: excellent approximation in its respective region, although do not merge to form a complete approximation to the problem

* Composite function that is a good approximation over the entire region :

$u \approx [F(Y) + f(y) - (\text{common terms})]$

$u_{composite} \cong (1-a)(1-e^{-y/\varepsilon}) + ay$ (4-219)

within ±1 % of the exact solution (4-213)

Fig 4-39b for $\varepsilon = 0.1$

* Higher order matching & Application to flat-plate BL :
  Sections 4-11.3 ~ 4.11.5

## 4-12 Three-Dimensional Laminar Boundary Layers

* Fig 4-41
* Choose orthogonal curvilinear coordinates $(x_1, x_2, x_3)$ with $x_3$ everywhere normal to S.
* BL velocities $(u_1, u_2, u_3)$ $u_3 << u_1, u_2$
* Freestream velocity components $U_1$ & $U_2$
* In orthogonal coordinates,

$$(ds)^2 = (h_1 dx_1)^2 + (h_2 dx_2)^2 + (h_3 dx_3)^2 \quad (4-229)$$

Since $x_3$ normal to S, $h_3 \equiv 1$

* In general, h can be found by relating $(x_1, x_2, x_3)$ to $(x, y, z)$

$$h_1^2 = \left(\frac{\partial x}{\partial x_1}\right)^2 + \left(\frac{\partial y}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_1}\right)^2 \quad (4-230)$$

$$= \left[\left(\frac{\partial x_1}{\partial x}\right)^2 + \left(\frac{\partial x_1}{\partial y}\right)^2 + \left(\frac{\partial x_1}{\partial z}\right)^2\right]^{-1}$$

$h_2^2$ = similar

$h_3 = 1$ in this particular case

* In these coordinates, incompressible continuity eqn.

$$\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 u_1) + \frac{\partial}{\partial x_2}(h_1 u_2) + \frac{\partial}{\partial x_3}(h_1 h_2 u_3)\right] = 0 \qquad (4\text{-}231)$$

(By BL approximations, i.e., variation in $x_3$ = small) $\rightarrow \approx \frac{\partial u_3}{\partial x_3}$

* Thin BL $\rightarrow \partial p/\partial x_3 \approx 0$

2 remaining BL momentum eqns.

$$\frac{Du_1}{Dt} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} - \frac{u_2^2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} \approx -\frac{1}{\rho h_1}\frac{\partial p}{\partial x_1} + \nu\frac{\partial^2 u_1}{\partial x_3^2}$$

$$\frac{Du_2}{Dt} - \frac{u_1^2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} \approx -\frac{1}{\rho h_2}\frac{\partial p}{\partial x_2} + \nu\frac{\partial^2 u_2}{\partial x_3^2} \qquad (4\text{-}232)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u_1}{h_1}\frac{\partial}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}$

* Pressure gradients in (4-232): matching by Euler's relations for inviscid flow

$$-\frac{1}{\rho h_1}\frac{\partial p}{\partial x_1} = \frac{\partial u_1}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial x_2} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} - \frac{u_2^2}{h_1 h_2}\frac{\partial h_2}{\partial x_1}$$

$$-\frac{1}{\rho h_2}\frac{\partial p}{\partial x_2} = \frac{\partial u_2}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_2}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_2}{\partial x_2} + \frac{u_1 u_2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} - \frac{u_1^2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} \qquad (4\text{-}234)$$

* BCs

@ $x_3 = 0$: $u_1 = u_2 = 0$ $\quad u_3 = v_w(x_1, x_2, t)$ $\quad T = T_w$ $\qquad (4\text{-}235)$

@ $x_3 \rightarrow \infty$: $u_1 \rightarrow U_1$ $\quad u_2 \rightarrow U_2$ $\quad T = T_e$

* 2 components of shear stress

$$\tau_1 = \mu \frac{\partial u_1}{\partial x_3} \qquad \tau_2 = \mu \frac{\partial u_2}{\partial x_3} \qquad (4\text{-}236)$$

## 4-12.1 Cartesian Coordinates: Secondary Flow and Skewing

* In Cartesian $h_1 = h_2 = 1$
* For steady flow $\rho = \text{const}$ $\mu = \text{const}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = U\frac{dU}{dx} + W\frac{\partial U}{\partial z} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (4\text{-}237)$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2}$$

(pressure gradient)

* Parabolic in $x$ & $z$ → can be marched downstream from known initial values
* ICs: not limited to a single $x=0$, but along the sides
  → "region of influence" in 3D BL is affected by both convection and diffusion, strong when large crossflow convection
* Fig 4-42 a: uniformly positive crossflow pressure gradient → Fig 4-42 c
* Fig 4-42 b: bidirectional crossflow → integral analysis is less successful ∵ hodograph shape (Fig 4-42 d) is not well correlated with the outer stream conditions.

* Surface streamline angle

$$\tan\beta = \frac{\tau_{zy,\,wall}}{\tau_{xy,\,wall}} \qquad (4\text{-}23d)$$

## 4-12.4 Separation Geometry in Three-Dimensional Flow

* 2D separation – simple geometry
  entire BL breaks away @ the point $\tau_w = 0$
  → has to go up and over the resulting separation bubble
* Fig 4-4f : 3D separation example
* 4 special points in separation
  1. nodal point
     infinite # of surface streamlines merge tangentially to the separation line
  2. Saddle point
     only 2 surface streamlines intersect & all others divert to either side
  3. focus
     near a saddle point, around which an infinite # of surface streamlines swirl
  4. 3D singular point
     velocity is zero, serving as the center for a horseshoe vortex
* Fig 4-4g
  Obstacle → adv. pressure gradient → flow must separate → horseshoe vortex : bubble wraps itself around the obstacle

S : nodal point of separation
3D singular point : inside the bubble, on the center plane between S and the obstacle
Nodal attachment point & twin foci : rear of obstacle

* Fig 4-49 : flow past a round-nosed body at $\alpha$ — open, non-bubble separation
A : nodal attachment point
S : saddle point
F : focus of separation
Instead of forming a bubble, the separated flow has broken away "from" the surface in a tornado-like vortex

## 4-13 Unsteady Boundary Layers : Separation Anxiety

### 4-13.1 Start-Up Flows

* Fig 4-50a
flow is inviscid & matches the classical potential-flow pattern
* Fig 4-50b
@ $Ut/R \approx 0.35$, $\tau_w$ vanishes @ rear stagnation point → separation begins
* Fig 4-50c
@ $Ut/R \approx 1.5$, large "standing" double-vortex layer has formed

* Fig 4-50d
alternating Kármán vortex street, with separation @ 81° from the stagnation point

* MRS criterion
Unsteady separation is signaled by a singular point $x_s(t)$, $y_s(t)$ in the flow where the velocity relative to the moving point s, $\tau_w$, $\omega_s$ all vanish

$$u_s = \frac{dx_s}{dt} = 0 \qquad \tau_s = \omega_s = \left.\frac{\partial u}{\partial y}\right|_s = 0$$

s need not be @ the wall
There need not be any reverse flow

# CHAPTER 5

# THE STABILITY OF LAMINAR FLOWS

## 5-1 Introduction: The Concept of Small-Disturbance Stability

* Critical Re - For any laminar flow, there is a finite value of Re which threatens its very existence
* Laminar flows: exception rather than the rule in most engineering situations
* 2 key words - stability & transition
* Stable?
  A physical state withstand a disturbance & still return to its original state
* Transition?
  Change over space, time, and a certain Re range
* Turbulence: experimentally observed fact. never been proved mathematically that turbulent flow is the proper stable state at high Re

## 5-1.1 Outline of a Typical Stability Analysis

* General line of attack (7 steps)

1. Stability of a basic solution to the physical problem, $Q_0$
2. Add a disturbance variable $Q'$ and substitute $(Q_0 + Q')$ into the basic eqns.
3. Subtract the basic terms that $Q_0$ satisfies identically → disturbance eqn.

4. Linearize the disturbance eqn by assuming small disturbances, i.e., $Q' \ll Q_0$ & neglect $Q'^2, Q'^3, \ldots$
5. If the linearized disturbance eqn is complicated & multidimensional, it can be simplified by assuming a form for the disturbances, e.g., a traveling wave or a perturbation in only one direction.
6. The linearized disturbance eqn should be homogeneous and have homogeneous BCs. → be solved only for certain specific values of the eqn's parameters. ∴ eigenvalue problem
7. The eigenvalues are examined to determine when they grow (unstable), decay (stable) remain constant (neutrally stable). Typically the analysis ends with a chart showing regions of stability separated from unstable regions by neutral curves.

## 5-1.2 Wind-Generated Waves : The Kelvin-Helmholtz Instability

Illustrate how to use the 7 steps

* Fig 5-2 — incompressible, irrotational, inviscid
* Step 1
solution to the basic flow

$z<0$ : $\Phi_1 = U_1 z$ $\quad p_1 = p_0 - \rho_1 g z$ (5-1)

$z>0$ : $\Phi_2 = U_2 z$ $\quad p_2 = p_0 - \rho_2 g z$

→ vortex sheet with a discontinuity in velocity

* Step 2 - apply a disturbance to $\Phi_1$ & $\Phi_2$

$$\Phi_1 = U_1 z + \hat{\Phi}_1(x, z, t)$$
$$\Phi_2 = U_2 z + \hat{\Phi}_2(x, z, t) \quad (5-2)$$

Pressure satisfies the unsteady Bernoulli relation

$$p_i = C_i - \rho_i \frac{\partial \Phi_i}{\partial t} - \frac{\rho_i}{2} |\nabla \Phi_i|^2 - \rho_i g z$$

When no disturbance, @ $z=0$

$$C_1 - \frac{\rho_1}{2} U_1^2 = C_2 - \frac{\rho_2}{2} U_2^2$$

BC #1 : pressure continuous across the interface

@ $z=\eta$ : $C_1 - \rho_1 \frac{\partial \Phi_1}{\partial t} - \frac{\rho_1}{2} |\nabla \Phi_1|^2 - \rho_1 g \eta$

$$= C_2 - \rho_2 \frac{\partial \Phi_2}{\partial t} - \frac{\rho_2}{2} |\nabla \Phi_2|^2 - \rho_2 g \eta$$

BC #2 : vertical velocities match

@ $z=\eta$ : $W_i = \frac{\partial \Phi_i}{\partial z} = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \frac{\partial \Phi_i}{\partial x} \frac{\partial \eta}{\partial x}$

BC #3 : disturbance dies out far from the interface

$$\nabla \hat{\Phi}_2 \to 0 \quad \text{as} \quad z \to +\infty$$
$$\nabla \hat{\Phi}_1 \to 0 \quad \text{as} \quad z \to -\infty \quad (5-3)$$

* Step 3 – disturbance eqn.

For inviscid flow, also Laplace's eqn.

$$\nabla^2 \hat{\Phi}_1 = 0 \qquad \nabla^2 \hat{\Phi}_2 = 0 \tag{5-4}$$

* Step 4 – linearize

(1) small displacements & slopes

$$g\eta \ll U_i^2 \qquad \frac{\partial \eta}{\partial x} \ll 1 \rightarrow z = \eta \approx 0$$

(2) disturbance velocities are small

$$|\nabla \hat{\Phi}_i| \ll U_i$$

→ Linearize the interfacial conditions

Pressure: $$\rho_1 \left(U_1 \frac{\partial \hat{\Phi}_1}{\partial x} + \frac{\partial \hat{\Phi}_1}{\partial t} + g\eta\right) \simeq \rho_2 \left(U_2 \frac{\partial \hat{\Phi}_2}{\partial x} + \frac{\partial \hat{\Phi}_2}{\partial t} + g\eta\right)$$

@ $z \simeq 0$

Kinematic: $$\frac{\partial \hat{\Phi}_1}{\partial x} \approx \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x} \tag{5-5}$$

$$\frac{\partial \hat{\Phi}_2}{\partial x} \approx \frac{\partial \eta}{\partial t} + U_2 \frac{\partial \eta}{\partial x} \quad @\ z \approx 0$$

* Step 5 – assume a simple form for disturbances

i.e., 2D traveling waves or normal modes

$$\eta = \eta_0 e^{i(\alpha x - \sigma t)}$$

$$\hat{\Phi}_j = \phi_j(z)\, e^{i(\alpha x - \sigma t)}$$

$\alpha$ : real

$\sigma$ : complex ∴ disturbance will be unstable if its imaginary part $\sigma_i > 0$

* Step 6 – Solution of the disturbance eqn with the far-field BC

$$\phi_1' = A_1 e^{kz} \quad \& \quad \phi_2' = A_2 e^{-kz} \qquad (5\text{-}6)$$

Substitution into (5-5)

$$A_1 = \bar{x}\eta_0 \left(U_1 - \frac{\sigma}{\alpha}\right)$$

$$A_2 = -\bar{x}\eta_0 \left(U_2 - \frac{\sigma}{\alpha}\right)$$

} coeffs in the disturbance eigenfunctions

$\sigma$ from pressure condition (5-5)

$$\sigma = \alpha \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[ \underbrace{\frac{\alpha^2 \rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{\alpha g (\rho_1 - \rho_2)}{(\rho_1 + \rho_2)}}_{\text{key to stability}} \right]^{1/2}$$

$-/0/+ \rightarrow$ stable/neutral/unstable

The unstable condition

$$\alpha \rho_1 \rho_2 (U_1 - U_2)^2 > g(\rho_1^2 - \rho_2^2) \qquad \alpha = \frac{2\pi}{\lambda} \qquad (5\text{-}7)$$

( $\lambda$: wavelength of disturbance

always true for $\alpha \gg 1$, $\lambda \ll 1$

∴ vortex sheets are always unstable

* Step 7 – chart showing regions of stability and instability

## 5-2 Linearized Stability of Parallel Viscous Flows

* For viscous-flow instability
* Same 7 steps

1. Select a basic solution flow
2. Add a disturbance
3. Find the disturbance eqns.

4. Linearize
5. Simplify
6. Solve for the eigenvalues
7. Interpret the stability conditions & draw a chart showing the neutral curves and the growth and decay rates

11/4/98

## 5-2.1 Derivation of the Orr-Sommerfeld Equation

* 7 steps for Incompressible laminar flow with constant ($\rho, \mu, k, c_p$) & no buoyancy.
* $\nabla \cdot \vec{V} = 0$

(5-10)

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V}$$

* Laminar flow solution
  $\vec{V}_0 = (U, V, W) = \vec{V}_0(\vec{x}, t)$ & $p_0(\vec{x}, t)$
* Small disturbance
  $\vec{v}(\vec{x}, t) = (\hat{u}, \hat{v}, \hat{w})$ & $\hat{p}(\vec{x}, t)$
* Substitute $\vec{V}_0 + \vec{v}$ & $p_0 + \hat{p}$ into (5-10)
  → Subtract original $\vec{V}_0$ & $p_0$ equalities
  → neglect higher powers and products of $\vec{v}$
* Linearized disturbance eqns

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0$$

$$\frac{\partial \hat{u}}{\partial t} + U\frac{\partial \hat{u}}{\partial x} + \hat{u}\frac{\partial U}{\partial x} + V\frac{\partial \hat{u}}{\partial y} + \hat{v}\frac{\partial U}{\partial y} + W\frac{\partial \hat{u}}{\partial z} + \hat{w}\frac{\partial U}{\partial z}$$

$$\approx -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial x} + \nu \nabla^2 \hat{u}$$

$$\frac{\partial \hat{v}}{\partial t} + \cdots \qquad (5\text{-}11)$$

$$\frac{\partial \hat{w}}{\partial t} + \cdots$$

← linear PDE for $\hat{u}, \hat{v}, \hat{w}$, and $\hat{p}$

* Now reduce (5-11) to a single ODE

Assume a locally parallel basic flow

$V \approx 0$

$U \approx U(y) \quad W \approx W(y)$

Assume that disturbances are parallel

→ most general form for $(\hat{u}, \hat{v}, \hat{w})$ is that of a traveling wave, whose amplitude varies with $y$ & which moves along the wall at $\phi$ w.r.t. $x$-axis

$$(\hat{u}, \hat{v}, \hat{w}, \hat{p}) = [u(y), v(y), w(y), p(y)] \qquad (5\text{-}12)$$
$$\exp[i\alpha(x\cos\phi + z\sin\phi - ct)]$$

$\alpha$ : wave #

$c$ : propagation speed

$\omega = \alpha c$ : frequency

Tollmien - Schlichting waves : 1st (infinitesimal) indications of laminar-flow instability

* Substitute (5-12) into (5-11), linear ODE

$$i\alpha u\cos\phi + v' + i\alpha w\sin\phi = 0 \qquad (5-13a)$$

$$i\alpha u(U\cos\phi + W\sin\phi - c) + U'v = -\frac{i}{\rho}\alpha p\cos\phi + \nu(u'' - \alpha^2 u) \qquad (5-13b)$$

$$i\alpha v(U\cos\phi + W\sin\phi - c) = -\frac{1}{\rho}p' + \nu(v'' - \alpha^2 v) \qquad (5-13c)$$

$$i\alpha w(U\cos\phi + W\sin\phi - c) + W'v = -\frac{i}{\rho}\alpha p\sin\phi + \nu(w'' - \alpha^2 w) \qquad (5-13d)$$

* (5-13b) $\times\cos\phi$ + (5-13d) $\times\sin\phi$
& introduce

$$u_0 = u\cos\phi + w\sin\phi \qquad (5-14)$$
$$U_0 = U\cos\phi + W\sin\phi$$

→

$$i\alpha u_0 + v' = 0 \qquad (5-15a)$$

$$i\alpha u_0(U_0 - c) + U_0'v = -\frac{1}{\rho}\alpha p + \nu(u_0'' - \alpha^2 u_0) \qquad (5-15b)$$

$$i\alpha v(U_0 - c) = -\frac{1}{\rho}p' + \nu(v'' - \alpha^2 v) \qquad (5-15c)$$

* Now eliminate any 2 of the 3 variables $u, v, p$.
If eliminate $u$, obtain a simple eqn for the pressure fluctuation

$$p'' - \alpha^2 p = -2i\alpha\rho U'v \qquad (5-16)$$

In the outer freestream, $U' \approx 0$, the solution must be of the form

$$p = ae^{-\alpha y} + \underbrace{be^{+\alpha y}}_{\to 0} \quad (5\text{-}17)$$

$\therefore$ disturbance vanishes @ infinity

* Eliminate $u$ & $p$ $\rightarrow$ Orr-Sommerfeld relation

$$(U-c)(v'' - \alpha^2 v) - U'' v + \frac{i\nu}{\alpha}(v'''' - 2\alpha^2 v'' + \alpha^4 v) = 0 \quad (5\text{-}18)$$

* BCs : $u$ & $v$ must vanish at infinity and at the wall (no-slip)

Note : $u = 0 \rightarrow v' = 0$ continuity

Duct flows : $v(\pm h) = v'(\pm h) = 0$

BLs : $v(0) = v'(0) = 0 \quad v(\infty) = v'(\infty) = 0 \quad (5\text{-}19)$

Free-shear layers : $v(\pm\infty) = v'(\pm\infty) = 0$

* The O-S eqn has 3 parameters : $\alpha, c, \& \nu$

For a given $U(y)$ & $U''(y)$, only a certain continuous but limited sequence of these parameters (eigenvalues) will satisfy the relations (5-18) and (5-19).

The mathematical problem is to find this sequence $\rightarrow$ different for Spatial vs. temporal growth

Temporal growth : $f(Re, \alpha, c_r, c_i) = 0 \quad (5\text{-}20a)$

Spatial growth : $g(Re, \alpha_r, \alpha_i, \omega) = 0 \quad (5\text{-}20b)$

( $c_i = 0$ temporal case neutral )
( $\alpha_i = 0$ spatial case neutral )

$\rightarrow$ locus of these neutral points forms the boundary between stability and instability.

PAGE

## 5-2.2 Inviscid - Stability Theory

* Limiting, but very instructive case
  : infinite Re or negligible viscosity
  → drop $\nu$ in the Orr-Sommerfeld (5-18)
* Inviscid - disturbance relation

$$v'' - \left(\frac{U''}{U-c} - \alpha^2\right) v = 0 \qquad (5\text{-}21)$$

→ Rayleigh eqn.

BL conditions : $v(0) = v(\infty) = 0$ (5-22)

* 5 Important theorems on inviscid stability

Theorem 1: It is necessary for instability that the velocity profile have a point of inflection

Theorem 2: The numerical value $|v'|$ of the vorticity be a maximum at the point of inflection.

Theorem 3: If a point of inflection exists, $U''(U - U_{PI}) < 0$ somewhere on the profile, when $U_{PI}$ = velocity at the point of inflection

Theorem 4: If $U(y)$ possesses a PI @ $y = y_c$, a neutral disturbance ($c_i = 0$) may exist whose velocity is $c_r = U(y_c)$

Theorem 5: The phase velocity $c_r$ of an amplified disturbance must always lie between the minimum and maximum values of $U(y)$

* Theorem 1 → believe that real (viscous) profiles w/o a PI would be stable
→ Prandtl showed that viscosity can indeed be destabilizing for certain wave numbers @ finite Re.

## 5-4 Transition to Turbulence

* Linearized stability theory for prediction of the demise of laminar flow @ finite Re.; however does not predict the onset of turbulence.
* Initial breakdown of laminar flow through amplification of infinitesimal disturbances
→ Complicated sequence of spatial changes
→ unsteady & disorderly, but rational & stable phenomenon : turbulence

transition ↓

* The transition mechanisms are still not completely understood.
* Fig. 5-24 : original transition exp.
→ still not well understood.
* $Re < 2000$, laminar : dye filament stay along a nealy straight and distinct streamline (Fig 5-24 a)
* $2000 < Re < 13000$ : depending on the smoothness of entrance condition, the filament broke up & mixed with surrounding water. (Fig. 5-24 b).

* The flow in the well-mixed region is turbulent, but the instantaneous filament is still confined to a relatively distinct set of curls & eddies. (Fig 5-24c)
* These eddies & associated fluid parcels are in rapid 3D motion @ frequencies up to 1000's of hertz
→ color-blurred effect.
* Notes on Fig 5-24
(1) Transition occurs downstream, the actual point tending to move toward the entrance with increasing $Re_x$
(2) There is a clear region of amplification from laminar into turbulence. Far downstream, turbulence reaches a fully developed equilibrium state of balance between production and decay, which is independent of the initial disturbance
(3) In the beginning turbulence zone, the flow is only intermittently turbulent, interspersed with laminar-flow regions.

## 5-4.1 Development of Spanwise Vorticity

* Initial instability occurs as the Tollmien-Schlichting waves, traveling

in the mean flow direction
△ if the initial disturbance spectrum is nearly infinitesimal and random.

* 3D appears as TS waves begin to show spanwise variations.
* Klebanoff : a shear layer in the unstable region has a strong ability to amplify any slight 3D
  → Fig. 5-25 : rapid development of spanwise peaks and valleys in the streamwise velocity fluctuations downstream of a vibrating ribbon containing spacers. ; The flow is still "laminar".
* Fig. 5-26 a : in-phase Λ vortices (peak - following - peak) which have the same wavelength as a TS wave △ arise at fluctuation amplitudes of ≈ 0.01 $U_\infty$
  → K-type vortices (same as those in Fig 5-25)
  $\lambda_z \cong 0.5 \lambda_x$
* Fig 5-26 b : lower amplitudes ($\cong$ 0.003 $U_\infty$) staggered waves (peak - following - valley)
  $\lambda_z \cong 1.5 \lambda_x$
  → C-type vortices, Sub-harmonic unstable wave
* Fig 5-26 c : intermediate amplitudes ($\cong$ 0.006 $U_\infty$) staggered vortex pattern $\lambda_x \cong 2 \lambda_{TS}$
  $\lambda_z \cong 0.7 \lambda_x$

→ H waves arise from secondary instability of TS waves to 3D disturbances

* ∴ Many paths from TS waves to turbulence ← explained by the non-linear theories of flow instability

## 5-4.2 Turbulent Spots : The Final Transition Process

* We have seen

Shear layer develops viscous instability and forms TS waves that grow into finite amplitude (0.01 ~ 0.02 $U_\infty$)

3D fluctuations in $\vec{V}$ & $p$

→ From this, the process is more a breakdown than a growth.

→ The longitudinally stretched vortices begin a cascading breakdown into smaller units, until the relevant frequencies and wave numbers are approaching randomness.

→ In this diffusely fluctuating state, intense local changes occur @ random times and locations in the shear layer near the wall.

→ Turbulence bursts forth in the form of growing and spreading spots.

* Fig. 5-27a : spot looks like an arrowhead moving downstream and spreading at a half-angle from 8° to 12°
  → LE travels @ ≈ 0.9$U_\infty$, TE @ ≈ 0.5$U_\infty$
  → Spot grows in size but its normalized shape remains the same (Fig 5-27b)
* Fig 5-27c : spot is fully turbulent. Its detailed structure becomes finer as Re increases.
* Spots grow and remain intensely turbulent → entrain surrounding laminar fluid to maintain
  : Strongest entrainment is @ LE and on the upper surface of TE
* In BL, spots form randomly and naturally in a narrow transverse band $\Delta x$.
  $\Delta x$ < total length of the transition zone.
* Overall picture of the transition process in quiet BL
  1. Stable laminar flow near LE
  2. Unstable 2D TS waves
  3. Development of 3D unstable waves and hairpin eddies
  4. Vortex breakdown @ regions of high localized shear
  5. Cascading vortex breakdown into fully 3D fluctuations

6. Formation of turbulent spots @ locally intense fluctuations
7. Coalescence of spots into fully turbulent flow
→ Fig 5-28

## 5-4.3 Classification of BL Transition Processes

* Natural transition : gradual process Fig 5-28
  caused by infinitesimal disturbances
  TS waves → 3D waves → vortex breakdown
  → turbulent spots → fully turbulent
  Quiet stream & smooth walls
* Bypass transition : skip the early stages
  → directly to vortex breakdown
  or turbulent spot
  Caused by large disturbances, i.e.,
  rough walls, noisy freestream,
  vibrating surface, stream subjected
  to acoustic waves
* Separated flow transition :
  Laminar BL separates & forms a
  separation bubble → reattach as
  a turbulent flow

## 5-5 Engineering Prediction of Transition

* No fundamental theory
  → only experiments & correlations

* Prediction of the final onset of fully turbulent flow, $Re_{x,tr}$ or $Re_{\theta,tr}$, as a function of
  1. Pressure gradient
  2. Freestream turbulence
  3. Wall roughness
  4. Mach number
  5. Wall suction or blowing
  6. Wall heating or cooling

## 5-5.1 Effect of Pressure Gradient

* Fig 5-29
  $U(x)$ : known
  Wish to predict $x_{tr}$, using $x_i$ as an input

### 5-5.1.1 The 2-Step Method of Granville

* Compute $x_i$ by following $H(x)$ from Thwaites' until it hits $Re_{crit}$ in Fig. 5-12
* Then compute mean Thwaites' parameter

$$\lambda_m = \frac{1}{(x_{tr} - x_i)} \int_{x_i}^{x_{tr}} \lambda(x)\, dx \qquad (5\text{-}36)$$

  where $\lambda = \theta^2 (dU/dx)/\nu$ ← pressure gradient
* Continue to monitor $Re_\theta(x)$
  Assume $x_{tr}$ occurs when $Re_\theta$ strikes

$$Re_\theta(x_{tr}) \cong Re_\theta(x_i) + 450 + 400\, e^{60\lambda_m} \qquad (5\text{-}37)$$

  for $\lambda < 0.04$

* For adverse gradient ($\lambda \simeq -0.1$), the last term is negligible and transition is very near to $x_i$
* For favorable gradient, the last term is very large and transition moves far downstream.

### 5-5.1.2 The 1-Step Method of Michel

* Correlation based on local $\theta$ and position
* Compute $\theta(x)$ from Thwaites' (4-13f)

$$\theta^2 \simeq \frac{0.45\nu}{U^6}\int_0^x U^5\,dx \qquad (4\text{-}13f)$$

* Michel's transition line

$$Re_{\theta,tr} \simeq \frac{U(x)\,\theta(x)}{\nu} \simeq 2.9\,Re_{x,tr}^{0.4} \qquad (5\text{-}3f)$$

### 5-5.1.5 Comparism for Flat-Plate Flow

* Wazzan et al.: the latest. reflects quieter experiment.
* It is doubtful if a simple theory can resolve the effects of various kinds of disturbances.

### 5-5.1.7 Direct Numerical Simulation of Transition

* DNS solves the full time-dependent N-S eqns w/o closure models, accounting for all important scales of the flow.

* As Re increases, # of CFD mesh points must increase rapidly.
→ foreseeable limit is a low-turbulent Re.
: sufficient for a DNS study of transition

## 5-5.2 Effects of Freestream Turbulence

* Noisy freestream hastens the transition process.
* Parameter characterizing freestream turbulence level

$$T = \frac{q}{U} \qquad (5\text{-}43)$$

where $q = \left[\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\right]^{1/2}$

$\overline{u'^2}$ : mean square fluctuation, averaged over a long period of time

* The effect of T on transition : very strong
@ T = 0.35%, $Re_{tr}$ drops 50% from its quiet value
* Different effects from different disturbances
(1) grid-generated turbulence
(2) acoustic noise
(3) excited standing wave
(4) excited traveling wave

### 5-5.2.1 The Correlation of van Driest and Blumer

* Transition occurs when

$Re_\omega = \left(\frac{\omega y^2}{\nu}\right)_{max}$ reaches a critical value to be correlated with freestream turbulence.

* For flat-plate transition

$$Re_{x,tr}^{1/2} = \frac{-1 + \sqrt{1 + 132500\,T^2}}{39.2\,T^2} \qquad (5\text{-}44)$$

→ Fig 5-33 solid line

## 5-5.3 Effect of Surface Roughness on Transition

* Introduction of a roughness or projection at the wall
  → earlier transition because of the additional disturbances
* (1) 2D roughness : wire or cylinder stretched across the flow
  (2) 3D roughness : sphere or spike or a single grain of sand
* (1) Single roughnesses
  (2) Distributed roughnesses : sandpaper rows of cylinders or multiple rivets
* Fig 5-35 a
  If $k \ll \delta^*(x_k)$ : little effect
* As U or k increases & $k/\delta^* > 0.3$, $x_{tr}$ moves forward toward the roughness element.
  → Fig 5-35b : $Re_{\delta^*,tr}$ drops along a hyperbolic curve

* Criterion for a wire to trip
  $\frac{U_k}{\nu} \approx 850$ (5-48)

* Freestream turbulence exists
  → $Re_{\delta^*,tr}$ is lower
  : the effect of freestream turbulence can be removed by normalizing with its "quiet" value.

## 5-5.5 Control of BL Transition

* Although existing theories are not definitive, we know enough about transition.
  → attempts at designing systems that control transition
* Receptivity : the means by which freestream disturbances enter BL and excite its "normal modes" or TS waves.
  Why is the BL so receptive to certain types of grid turbulence?
* By increasing the negative curvature of the profile → BL stabilized, transition delayed
* 1. Favorable pressure gradient
  Shaping the body, e.g., designing the short tail to avoid extensive flow separation

* 2. Wall suction
  keep BL laminar, reduce drag, delay separation → higher lift
  Often less effective for swept wings where turbulence from fuselage tends to spread out along LE
* 3. Cooling of gases
  Very effective, but expensive.
* 4. Surface heating of water flows
  Stabilizes BL, but large heating power requirement (~ 1000 kW/m²)
  ∵ high heat capacity & thermal conductivity of water
* Active control of BL
  (1) Wave cancellation
  Growing disturbances are detected and the control system generates waves 180° out of phase
  (2) Array of piezo electric actuators
  Generate counter rotating motions to cancel the streamwise vortices that feed turbulence
* All suffer from environmental factors.

# CHAPTER 6.
# INCOMPRESSIBLE TURBULENT MEAN FLOW

## 6-1 Physical and Mathematical Description of Turbulence

* Orderly (laminar flow – strictly limited to finite values of Re.
* Beyond that, turbulence: fluctuating & disorderly motion
* Because turbulence is so complex, its complete analysis and quantification will probably never be achieved.

### 6-1.1 Physical Description.

* Fig 6-1
  Air flow at 3.3 m/s past a plate
  2.4 m (L) × 1.2 m (W)
  → transition to turbulence @ x = 90 cm,
  $Re_x \approx 2 \times 10^5$
* Characteristics
  1. Fluctuations in pressure & velocity
     → superimposed on a mean value
  2. Eddies intermingle & fill the shear layer
     → Eddy size from $\delta$ to Kolmogorov scale $L = (\nu^3 \delta / U^3)^{1/4}$
  3. Random variations in fluid properties which have a particular form.
     → Each property has a specific continuous energy spectrum that drops off to zero at high wave numbers. (small eddy size)

4. Self-sustaining motion
   → maintain itself by producing new eddies to replace those lost by viscous dissipation
   Turbulence production not related to the original instability mechanisms, e.g., TS waves
5. Mixing that is much stronger than that due to laminar (molecular) action.
   → associated with a gradient in the time-mean flow

* Turbulence
  a spatially varying mean flow with superimposed 3D random fluctuations that are self-sustaining and enhance mixing, diffusion, entrainment, and dissipation
* Coherent structures (Fig. 6-2)
  Turbulent flows that interface with nonturbulent fluid contains clearly visible patterns.

## 6.1.2 Mathematical Description

* Wish to predict
  (1) velocity and/or temperature profiles
  (2) wall friction and/or heat transfer
  → essentially impossible to predict Fig 6-1. with a theory.

* Direct numerical simulation
  → transitional and low-turbulent flows
  ($Re \leq 10^4$)

## 6-1.3 Fluctuations and Time Averaging

* Actual computation of a raw velocity component $u(x,y,z,t)$ is not possible in high Re flow
  → separate the fluctuating property from time-mean value
* Fig. 6-3
  Trace a: deep in a shear layer
  → continuous random fluctuations of 5-10% about a mean value
  Trace b: near the edge of a jet
  intermittent regime
* Intermittency factor
  $$\gamma = \frac{\text{turbulent time}}{\text{total time}}$$
* $u(t)$: raw vel component @ a particular spot $(x,y,z)$

$$\bar{u} = \frac{1}{T}\int_{t_0}^{t_0+T} u\,dt \qquad (6\text{-}1)$$

$T \gg$ any significant period of the fluctuations in $u$

* Fluctuation

$$u' = u - \bar{u} \qquad (6\text{-}2)$$

$\bar{u}' = 0$

* To characterize the magnitude of the fluctuation

$$\overline{u'^2} = \frac{1}{T}\int_{t_0}^{t_0+T} u'^2\, dt \qquad (6\text{-}3)$$

$$u'_{rms} = (\overline{u'^2})^{1/2}$$

* Statistically stationary
  → if (6-1) & (6-3) are independent of starting time $t_0$
* 2 paths to turbulent-flow analysis
  (1) Statistical theory of turbulent correlation fuctions
  → statistical properties of the fluctuations
  (2) Semiempirical modeling of turbulent mean quantities
  → turbulent properties of most engineering significance

## 6-1.4 Illustration: Turbulence Measurements in Flat-Plate Flow

* $(u, v, w)$
  $u$: parallel to freestream
  $v$: normal to the wall
  $w$: lateral to freestream
* Fig. 6-4: Turbulent BL @ $Re \approx 10^7$
  → $\sqrt{u'^2}$, $\sqrt{v'^2}$, $\sqrt{w'^2}$,
  & covariance $\overline{u'v'}$: Turbulent shear stress

- fluctuations up to 11%.
- wall presence makes the fluctuations different in magnitude (anisotropy)
- $u'$ largest, unimpeded by wall & slightly reinforced by freestream
- $v'$ smallest, directly impeded by wall its maximum farther out than $u'$ or $w'$
- $w'$ intermediate, quite large
  → 2D turb BL (flat plate) has a 3D set of vel. fluctuations but $w'$ does not directly influence the mean flow
- $\overline{u'v'}$ much smaller, but of fundamental importance ∵ determining its characteristics : central problem in our analysis of turbulent shear flows

* Fig. 6-4

- near wall, $(u', v', w')$ should vanish due to no-slip. Turbulence is quite resistant to wall damping, i.e., significant fluctuations even at $y/\delta = 0.0001$
- @ large $y$, fluctuations extend outside the edge $y = \delta$ → associated with the intermittency of the outer layer.
- Superlayer exists between turbulent and nonturbulent regions
  → Fig 6-5b between $0.4\delta$ and $1.2\delta$

* Fig 6-5a : intermittency in BL
  mean position ($\gamma \approx 0.5$) @ $y = 0.78\delta$
* $(u', v', w')$ become approximately equal for $y/\delta \geq 0.8$, i.e., isotropic
  Near the wall, however, the flow is strongly anisotropic → where most of the production & dissipation of turbulence energy takes place
* Fig 6-6 : energy content or spectrum of the turbulence

$$\overline{u'^2} = \int_0^\infty F_u(k_u)\, dk_u$$

For positions near the wall, more energy at a high wave number (small eddies) whereas away from the wall, the large eddies dominate the spectrum.

($k = 2\pi/\lambda$)

## 6-2 The Reynolds Equations of Turbulent Motion

* Most research uses the concept of time averaging
  → attempts to model the fluctuation terms by relating them to mean properties or their gradients
* The Reynolds equations are far from obsolete, and form the basis of most engineering analyses of turbulent flow.

* $\bar{Q} = \frac{1}{T}\int_{t_0}^{t_0+T} Q\,dt$ (6-5)

$\bar{Q}$ may vary slowly in time → unsteady turbulent flow (Fig 6-7)

* Consider incompressible turbulent flow with constant transport properties

$u = \bar{u} + u'$ $\quad p = \bar{p} + p'$

$v = \bar{v} + v'$ $\quad T = \bar{T} + T'$ (6-6)

$w = \bar{w} + w'$

* Rules of averaging for any turbulent quantities $f$ and $g$

$\bar{f}' = 0 \qquad \bar{\bar{f}} = \bar{f} \qquad \overline{f\bar{g}} = \bar{f}\bar{g}$

$\overline{f'\bar{g}} = 0 \qquad \overline{f+g} = \bar{f}+\bar{g} \qquad \overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}$ (6-7)

$\overline{fg} = \bar{f}\bar{g} + \overline{f'g'} \qquad \overline{\int f\,ds} = \int \bar{f}\,ds$

* Incompressible continuity eqn

$\nabla \cdot \vec{V} = 0$ (6-8)

Substitute (6-6) & time averaging

$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$ (6-9)

* Non-linear N-S eqns

$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \mu \nabla^2 \vec{V}$ (6-11)

Rearrangement of the convective-acceleration term

$\vec{V}\cdot\nabla Q = u\frac{\partial Q}{\partial x} + v\frac{\partial Q}{\partial y} + w\frac{\partial Q}{\partial z} \equiv \frac{\partial}{\partial x}(uQ) + \frac{\partial}{\partial y}(vQ) + \frac{\partial}{\partial z}(wQ)$

Substitute this idea in (6-11) and take time average

$$\rho \frac{D\bar{\vec{V}}}{Dt} + \rho \frac{\partial}{\partial x_j}(\overline{u_i' u_j'}) = \rho \vec{g} - \nabla \bar{p} + \mu \nabla^2 \bar{\vec{V}} \quad (6\text{-}12)$$

→ The mean momentum eqn is complicated by a new term involving the turbulent inertia tensor $\overline{u_i' u_j'}$.

← never negligible & the source of analytic difficulties ∵ its analytic form is not known a priori.

* Rearrange (6-12) to display the turbulent inertia terms as if they were stresses

$$\rho \frac{D\bar{\vec{V}}}{Dt} = \rho \vec{g} - \nabla \bar{p} + \nabla \cdot \tau_{ij} \quad (6\text{-}13)$$

where $\tau_{ij} = \underbrace{\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}_{\text{Laminar Newtonian viscous stresses}} \underbrace{- \rho \overline{u_i' u_j'}}_{\text{turbulent turbulent stresses}}$

* In a BL, the dominant term $-\rho \overline{u'v'}$ : turbulent shear

## 6-2.1 The Turbulent Kinetic-Energy Equation

* Turbulence conservation relation

→ Relation for the turbulent kinetic energy $k$

$$k = \frac{1}{2}(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}) = \frac{1}{2}\overline{u_i' u_i'} \quad (6\text{-}14a)$$

* $u_i$ · ($i$-th momentum eqn) : mechanical-energy eqn

* Turbulent kinetic-energy relation

$$\underbrace{\frac{Dk}{Dt}}_{\text{I}} = \underbrace{-\frac{\partial}{\partial x_j}\left[\overline{u_i'\left(\frac{1}{2}u_j'u_j' + \frac{p'}{\rho}\right)}\right]}_{\text{II}} - \underbrace{\overline{u_i'u_j'}\frac{\partial \bar{u}_j}{\partial x_i}}_{\text{III}}$$

$$+ \underbrace{\frac{\partial}{\partial x_j}\left[\nu\,\overline{u_j'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}\right]}_{\text{IV}} - \underbrace{\nu\,\overline{\frac{\partial u_j'}{\partial x_i}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}}_{\text{V}} \qquad (6-17)$$

I : the rate of change of turbulent energy
II : convective diffusion
III : production
IV : work done by turbulent viscous stresses
V : Turbulent viscous dissipation

## 6-2.2 The Reynolds Stress Equation

* Reynolds stresses $S_{ij} = (-\rho\overline{u_i'v_j'})$

$\rightarrow K = -S_{ii}/2\rho$

* Turbulent shear stresses ($i \neq j$) are more important

* Derivation of a conservation eqn for a single Reynolds stress

$u_j \times$ ($i$-th result of (instantaneous) − (time-averaged) momentum eqn)

$+ u_i \times$ ($j$-th result of (instantaneous) − (time-averaged) momentum eqn).

→ Reynolds stress equation

$$\underbrace{\frac{D\overline{u_i u_j}}{Dt}}_{\text{I}} = \underbrace{-\left[\overline{u_j u_k}\frac{\partial \bar{u}_i}{\partial x_k} + \overline{u_i u_k}\frac{\partial \bar{u}_j}{\partial x_k}\right]}_{\text{II}} \underbrace{-2\nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}_{\text{III}} \underbrace{+\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}_{\text{IV}} \underbrace{-\frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} - \nu\frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\overline{p'}}{\rho}\left(\delta_{jk}u_i' + \delta_{ik}u_j'\right)\right]}_{\text{V}}$$

I : rate of change of Reynolds stress
II : generation of stress
III : dissipatim
IV : pressure - strain effects
V : diffusion of Reynolds stress

* (6-17) & (6-18) : extremely complex with many unknown correlatins to model

## 6-3 The 2D Turbulent - Boundary -Layer Equations

* x : parallel to freestream
y : normal to wall
$\delta(x) \ll x$
$\bar{v} \ll \bar{u}$ $\quad \partial/\partial x \ll \partial/\partial y$ (6-19)

* Mean flow structure is 2D
$\bar{w} = 0$ $\quad \partial/\partial z = 0$ (6-20)

although $\overline{w'^2} \neq 0$, but its $z$ derivative is assumed to vanish.

* BL approximation for incompressible flow basic turbulent eqns.

Continuity: $\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$ $\quad dp_e \approx -\rho U_e dU_e$ (6-21a)

Bernoulli's

x-momentum: $\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} \approx U_e \frac{dU_e}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$ (6-21b)

Thermal energy: $\rho c_p \left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) \approx \frac{\partial q}{\partial y} + \tau \frac{\partial \bar{u}}{\partial y}$ (6-21c)

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}$$

$$q = k \frac{\partial \bar{T}}{\partial y} - \rho c_p \overline{v'T'} \qquad (6\text{-}22)$$

← difference from the laminar BL eqns. must be modeled

* y-momentum: $\frac{\partial p}{\partial y} \approx -\rho \frac{\partial \overline{v'^2}}{\partial y}$

Integrated across BL

$$p \approx p_e(x) - \rho \overline{v'^2} \qquad (6\text{-}23)$$

→ slight variation of pressure across BL due to velocity fluctuation normal to the wall

Fig 6-4: rms $v' \leq 4\%\ U_e$

∴ pressure difference < 0.4% of the freestream dynamic pressure i.e., negligible variation

* BCs (6-25)

No slip : $\bar{u}(x,0) = \bar{v}(x,0) = 0 \qquad \bar{T}(x,0) = T_w(x)$

Freestream matching : $\bar{u}(x,\delta) = U_e(x) \qquad \bar{T}(x,\delta_T) = T_e(x)$ //

4/28/08

## 6-3.1 Turbulent Energy and Reynolds Stress

* 2D BL form of the turbulence kinetic-energy

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} \approx -\underbrace{\frac{\partial}{\partial y}\left[\overline{v'\left(\frac{1}{2}u'u' + \frac{p'}{\rho}\right)}\right]}_{\text{pressure-strain}} + \frac{\tau}{\rho}\frac{\partial \bar{u}}{\partial y} - \underbrace{\varepsilon}_{\text{dissipation}}$$

$$\varepsilon = -\nu\,\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}} = \text{turbulent dissipation}$$

* 2D BL form of the Reynolds stress

$$\bar{u}\frac{\partial \overline{u'v'}}{\partial x} + \bar{v}\frac{\partial \overline{u'v'}}{\partial y} \approx 2\overline{u'v'}\frac{\partial \bar{u}}{\partial y} - \frac{\partial}{\partial y}\left(\overline{u'v'^2} + \frac{\overline{p'u'}}{\rho}\right) - 2\nu\,\overline{\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial x}} + \overline{\frac{p'}{\rho}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)} + \nu\frac{\partial^2 \overline{u'v'}}{\partial y^2} \qquad (6\text{-}27)$$

## 6-3.2 Turbulent Boundary Layer Integral Relations

* Derived by using (6-21a) to eliminate $\bar{v}$ in favor of $\bar{u}$, then integrate the result across BL

* Identical to the laminar flow relation

$$\frac{d\theta}{dx} + (2+H)\frac{\theta}{U_e}\frac{dU_e}{dx} = \frac{\tau_w}{\rho U_e^2} = \frac{C_f}{2} \qquad (6\text{-}28)$$

where

$\theta$ = momentum thickness $= \int_0^\infty \frac{\bar{u}}{U_e}\left(1-\frac{\bar{u}}{U_e}\right) dy$

$H$ = momentum shape factor $= \frac{\delta^*}{\theta}$

$\delta^*$ = displacement thickness $= \int_0^\infty \left(1-\frac{\bar{u}}{U_e}\right) dy$

* Mechanical energy Integral relation

← $\bar{u} \times$ (6-21b) & integrate across BL

$$\frac{1}{2}\rho\frac{d}{dx}\left[U_e\int_0^\infty \bar{u}(U_e^2-\bar{u}^2)\,dy\right] = \underbrace{\int_0^\infty \tau\frac{d\bar{u}}{dy}\,dy}_{\text{dissipation integral}} \qquad (6\text{-}30)$$

* From continuity

$$v_e = -\int_0^\infty \frac{\partial \bar{u}}{\partial x}\,dy = -\frac{d}{dx}\left(\int_0^\infty \bar{u}\,dy\right) + U_e\frac{d\delta}{dx}$$

$$\rightarrow \frac{d\delta}{dx} - \frac{v_e}{U_e} = \frac{1}{U_e}\frac{d}{dx}\left[U_e(\delta-\delta^*)\right] \qquad (6\text{-}31)$$

: entrainment relation

* Once the mainstay, now replaced by CFD

## 6-4 Velocity Profiles : The Inner, Outer, and Overlap Layers

* Fig 6-8 : experimental profiles $\bar{u}/U_e$ vs. $y/\delta$ for various pressure gradients

* Prandtl & von Kármán
  the profile consists of an inner and an outer layer + an intermediate overlap between the two
  Inner layer : viscous (molecular) shear dominates
  Outer layer : turbulent (eddy)
  Overlap layer : both important ; profile smoothly connects
* Inner law : $\bar{u} = f(\tau_w, \rho, \mu, y)$ (6-32)
  profile depends on wall shear stress, fluid properties, distance from the wall
* Outer law : $u_e - \bar{u} = f(\tau_w, \rho, y, \delta, \frac{dP_e}{dx})$ (6-33)
  Independant of $\mu$
* Overlap law : $\bar{u}_{inner} = \bar{u}_{outer}$ (6-34)
  over some finite intermediate region

## 6-4.1 Dimensionless Profiles

* Functional forms in Eqs (6-32) - (6.34) are determined from experiment after use of dimensional analysis
  3 primary dimensions : (mass, length, time)
  5 variables
  → 5 - 3 = 2 dimensionless parameters
* Inner law
  $$\frac{\bar{u}}{v^*} = f\left(\frac{yv^*}{\nu}\right) \qquad v^* = \left(\frac{\tau_w}{\rho}\right)^{1/2} \quad (6\text{-}35)$$
  wall friction velocity

* Outer law (velocity defect law)

$$\frac{U_e - \bar{u}}{v^*} = g\left(\frac{y}{\delta}, \Xi\right) \qquad \Xi = \frac{\delta}{\tau_w}\frac{dp_e}{dx} \qquad (6-36)$$

@ any $x$, the defect shape $g(y/\delta)$ will depend on the local pressure gradient $\Xi$

* Overlap law

$$\frac{\bar{u}}{v^*} = f\left(\underbrace{\frac{\delta v^*}{\nu}\frac{y}{\delta}}_{v^* y/\nu}\right) = \frac{U_e}{v^*} - g\left(\frac{y}{\delta}\right)$$

for some particular $\Xi$

→ can be true only if both $f$ & $g$ are logarithmic functions

∴ in the overlap layer

Inner variables

$$\frac{\bar{u}}{v^*} = \frac{1}{\kappa}\ln\frac{y v^*}{\nu} + B \qquad (6-38a)$$

Outer variables

$$\frac{U_e - \bar{u}}{v^*} = -\frac{1}{\kappa}\ln\frac{y}{\delta} + A \qquad (6-38b)$$

Where ( $\kappa$ & $B$ : near universal
$A$ : varies with $\Xi$

△ $\kappa \approx 0.41 \qquad B \approx 5.0 \qquad (6-39)$

* Fig 6-9 from replotting Fig 6-8 in terms of $u^+ = \bar{u}/v^*$ and $y^+ = y v^*/\nu$

→ All the curves collapse into a single logarithmic relation in the overlap region

$35 \le y^+ \le 350 \qquad .2\% \le y/\delta \le .20\%$

* Fig 6-10 from replotting Fig 6-8 using $\frac{u_e - \bar{u}}{v^*}$ vs. $y/\delta$

→ Profiles do not collapse into a single "universal" curve

∵ each profile has a different $\Sigma$

→ each has a different A

## 6-4.2 Inner Layer Details : The Law of the Wall

* Very near the wall : turbulence is damped out & BL is dominated by viscous shear.

→ velocity profile is linear

* $\tau_w = \mu \frac{\bar{u}}{\bar{y}}$ $\quad y^+ \leq 5$ (6-40)

or $u^+ = y^+$

→ viscous sublayer : $\delta_{sub} \simeq 5\nu/v^*$

where $(\nu/v^*)$ : viscous length scale

* $5 \leq y^+ \leq 30$ : buffer layer

velocity profile is neither linear nor logarithmic, but is a smooth merge

* Spalding deduced a single composite formula that covers the entire wall-related region

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right] \quad (6\text{-}41)$$

← excellent fit to inner law data until $y^+ > 100$, i.e., above the log curve

## 6-4.3 Outer Layer Details : Equilibrium Turbulent Flows

* The outer law is sensitive to the pressure gradient parameter $\Xi = \frac{\delta}{\tau_w}\frac{dp_e}{dx}$ and its variation with $x$
* $\Xi = \text{const}$ → equilibrium turbulent flows
* Clauser's equilibrium pressure gradient parameter

$$\beta = \frac{\delta^*}{\tau_w}\frac{dp_e}{dx} \qquad (6\text{-}42)$$

* BL with variable $p_e(x)$, but constant $\beta$ : in turbulent equilibrium → all the gross properties can be scaled with a single parameter
* Defect thickness $\Delta$

$$\Delta = \int_0^\infty \frac{U_e - \bar{u}}{v^*}\,dy = \delta^* \lambda \qquad (6\text{-}43)$$

where $\lambda = \sqrt{2/c_f}$ : measure of the local skin friction

* $G$ : shape factor that would remain constant in an equilibrium BL

$$G = \frac{1}{\Delta}\int_0^\infty \left(\frac{U_e - \bar{u}}{v^*}\right)^2 dy \qquad (6\text{-}44)$$

Related to $H$ $\quad H = \left(1 - \frac{G}{\lambda}\right)^{-1} \qquad (6\text{-}45)$

Note $H \neq \text{const}$ : skin friction varies with $x$

## 6-7 Turbulence Modeling

* For arbitrary variable : pressure gradient wall roughness, blowing or suction
* More details than just wall friction : Complete velocity profile, turbulent shear stress, rms turbulence fluctuations
* Hierarchy of turbulence models
  "n - Eqn" model : in addition to the time-mean Continuity & momentum relations, n time-mean PDEs have been added.
* 1. 0-eqn model : + algebraic eddy-viscosity formulas
  2. 1-eqn model : + time-mean eqn of either (1) TKE or (2) eddy viscosity + some algebric formulas
  3. 2-eqn model : + TKE eqn + PDE involving time-mean turbulence dissipation
  4. 2nd-moment closure model : most complex time-mean turbulence model. Skips TKE eqn, keeps dissipation eqn and models the Reynolds-stress relation
  5. LES : full unsteady N-S calculation for motion scales of the order of the grid or larger, & uses a turbulence model for the smaller SGS models
  6. PNS : no models at all. limited to low-turbulent Re. Details for improving turbulence models

## 6-7.1 Zero Equation Models: The Eddy Viscosity

* Boussinesq assumption

$$\tau_x = -\rho \overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

where $\mu_t$ = eddy viscosity
$\overline{u'v'}$ = only add'l unknown in the momentum eqn.

* Note $\mu_t > 0$ & $\overline{u'v'} < 0$
∵ eddy coming down ($v' < 0$) brings with it a $u' > 0$
or slower eddy moving up ($v' > 0$) brings $u' < 0$
→ $\overline{u'v'} < 0$

## 6-7.1.1 Mixing-Length Theory

* Prandtl - each turbulent fluctuation could be related to a length scale and a velocity gradient

$$-\overline{u'v'} \cong (\text{const})\, u'_{rms}\, v'_{rms}$$

$$\cong (\text{const})\left(l_1 \frac{\partial \bar{u}}{\partial y}\right)\left(l_2 \frac{\partial u}{\partial y}\right)$$

mixing lengths

- mean eddy size >> fluid's mean-free path

* Replace $(\text{const})(l_1 l_2)$ with $l^2$
→ Mixing-length model

$$\mu_t \cong \rho l^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \qquad (6\text{-}88)$$

* In the sublayer $\ell \approx y^+$ (6-89a)
  " overlap layer $\ell \approx \kappa y$ (6-89b)
  " outer layer $\ell \approx$ const (6-89c)
* Van Driest's damping factor
  $\ell_{a,b} \approx \kappa y \left[1 - \exp\left(-\frac{y^+}{A}\right)\right]$ (6-90)
* $\ell_{\text{outer layer}} \approx 0.09\,\delta$ (6-91)

## 6-7.1.5 The Baldwin - Lomax Model

* Read through

## 6-7.2 Higher Order Modeling

* 0- eqn models
  Cannot compute TKE K or fluctuating components $(u', v', w')_{rms}$

## 6-7.2.1 One - Equation Model: Turbulent Kinetic Energy

* Repeat 2D BL form of TKE eqn

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} \approx -\frac{\partial}{\partial y}\left[\overline{v'\left(\frac{1}{2}u_i'u_i' + \frac{p'}{\rho}\right)}\right] + \frac{\tau}{\rho}\frac{\partial \bar{u}}{\partial y} - \varepsilon \quad (6\text{-}26')$$

Where $K = \left(\overline{u_i'u_i'}\right)/2$

$\varepsilon$ = turbulent dissipation [(velocity)$^3$/(length)]

* Let $L$ : turbulent length scale
  or effective eddy size

Such an eddy would have a velocity scale $K^{1/2}$

$$\varepsilon \cong (\text{const}) \frac{K^{3/2}}{L} \qquad (6\text{-}103)$$

← if an eddy of $L$ moves with speed $K^{1/2}$, its power dissipated by unit mass

$$\frac{(\text{drag}) \times (\text{velocity})}{\text{mass}} \cong \frac{[(\text{const})\rho K L^2] K^{1/2}}{(\text{const})\rho L^3} \cong (\text{const}) \frac{K^{3/2}}{L}$$

* Production term (2nd on RHS)
  : $\tau = (\mu + \mu_t)(\partial \bar{u}/\partial y)$
* Convective diffusion term (1st on RHS)
  : difficult ∵ cannot measure $p'$
  assume $-v'\left(\frac{1}{2} u_i' u_i' + \frac{p'}{\rho}\right) \cong (\text{const}) \frac{\partial K}{\partial y}$
* Generic model of TKE eqn

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} \cong \frac{\partial}{\partial y}\left[(\text{const})\frac{\partial K}{\partial y}\right] + \nu_t \left(\frac{\partial \bar{u}}{\partial y}\right)^2 - (\text{const})\frac{K^{3/2}}{L} \qquad (6\text{-}104)$$

* Note : closure is not obtained until we model the length scale $L$

## 6-7.2.2 Two Equations : The $k$-$\varepsilon$ model

* TKE eqn (6-104) performs better if coupled with an eqn modeling the rate of change of either (1) dissipation $\varepsilon$ or (2) turbulent length scale $L$
* TKE

$$\frac{Dk}{Dt} \cong \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{D_k}\frac{\partial K}{\partial x_j}\right) + \nu_t \frac{\partial \bar{u}_i}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \varepsilon \qquad (6\text{-}105a)$$

Dissipation $\frac{D\varepsilon}{Dt} \approx \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)$

$$+ C_1 \nu_t \frac{\varepsilon}{K}\frac{\partial \bar{u}_i}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$- C_2 \frac{\varepsilon^2}{K} \qquad (6\text{-}105b)$$

* $\sigma_K = \nu_t / \nu_K \qquad \sigma_\varepsilon = \nu_t / \nu_\varepsilon$

$$\nu_t \approx C_\mu \frac{K^2}{\varepsilon} \qquad (6\text{-}106)$$

* 5 empirical constants
  - note that not universal

## 6-7.2.3 Wall Functions and Low Reynolds Number Modifications

* Standard 2-eqn models are for high Re
  - $K$-$\varepsilon$ model eqns do not contain $\mu$.
* 2 options
  (1) wall function inner law formulas that patch into the 1st node in the overlap region
  (2) add viscosity and damping terms & use a fine mesh right up to the wall.
* Wall function formulas

$$\frac{\bar{u}_p}{U^*} = \frac{1}{\kappa} \ln\left(\frac{U^* y_p}{\nu}\right) + B$$

$$K_p = U^{*2}/C_\mu^{1/2} \qquad \varepsilon_p = U^{*3}/\kappa y_p \qquad (6\text{-}108)$$

$$\omega_p = \frac{C_\mu^{3/4} K^{3/2}}{\kappa y_p}$$

Advantage: increase in grid size

* Near-wall option
  - add $\mu_t$ & damping terms
  fine grid $y^+ < 1$

Modifications to (6-105)

$$\bar{u}\frac{\partial K}{\partial x} + \bar{v}\frac{\partial K}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_K}\right)\frac{\partial K}{\partial y}\right] + \nu_t\left(\frac{\partial \bar{u}}{\partial y}\right)^2 - \varepsilon \quad (6\text{-}109a)$$

$$\bar{u}\frac{\partial \varepsilon}{\partial x} + \bar{v}\frac{\partial \varepsilon}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] + C_1\frac{\varepsilon \nu_t}{K}\left(\frac{\partial \bar{u}}{\partial y}\right)^2 - C_2\frac{\varepsilon^2}{K} \quad (6\text{-}109b)$$

## 6-7.2.4 Reynolds Stress Models

* The Reynolds stresses are computed directly by either (1) an algebraic stress model or (2) a DE for the rate of change of stress.
* Break-down of the Reynolds stress eqn into separate terms

$$\frac{D}{Dt}(\overline{u'v'}) = D_{ij} + P_{ij} + \Pi_{ij} - \varepsilon_{ij} + \nu\nabla^2(\overline{u'v'}) \quad (6\text{-}111)$$

$D_{ij}$ = turbulent transport $= -\frac{\partial}{\partial x_k}\left(\overline{u_i u_j u_k} + \frac{\overline{p'u_i}}{\rho}\delta_{jk} + \frac{\overline{p'u_j}}{\rho}\delta_{ki}\right)$

$P_{ij}$ = production $= -\overline{u_i u_k}\frac{\partial \bar{u}_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial \bar{u}_i}{\partial x_k}$

$\Pi_{ij}$ = pressure-strain $= \overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$

$\varepsilon_{ij}$ = dissipation $= 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}$

* This eqn is to be used with relations for $k$ and $\varepsilon$, and does not replace them.