ON THE CONVERGENCE OF FOURIER-BESSEL SERIES

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1. *Introduction.* This paper concerns the Fourier-Bessel series of a function $f(x)$ defined for $0 \leqslant x \leqslant 1$, or, more precisely, the Fourier expansion of the function $x^{\frac{1}{2}} f(x)$ in terms of the system of normal and orthogonal functions

$$\sqrt{(2x)}\, J_\nu(j_m x)/J_{\nu+1}(j_m), \tag{1.1}$$

where, in the notation of Watson's book on Bessel functions†, the expression

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \tag{1.2}$$

denotes the Bessel function of order ν of the first kind, and j_m is its m-th positive zero in ascending order of magnitude. The index ν is fixed in (1.1) and we suppose, with Watson‡, that

$$\nu + \tfrac{1}{2} \geqslant 0. \tag{1.3}$$

It is convenient to stipulate further, unless the contrary is explicitly stated, that we have

$$x^{\frac{1}{2}} f(x) = 0 \textit{ when } x = 0 \textit{ and when } x = 1, \tag{1.4}$$

in view of the fact that the functions (1.1) vanish at the end points 0 and 1.

The problem of the convergence of our series in the partial interval $\epsilon \leqslant x \leqslant 1-\epsilon$, where $\epsilon > 0$, may be regarded as settled by the memoir of

† Watson, 1, [p. 40 (8)], hereafter cited as W.

‡ W., 582.

W. H. Young†, written some twenty years ago, in which it is reduced to the problem for trigonometric Fourier series. There remain, however, the problems of convergence for $0 \leqslant x \leqslant \epsilon$ and for $1-\epsilon \leqslant x \leqslant 1$, and it is with these that the present paper is primarily concerned. A discussion of these two problems is given by Watson in his book‡. He supposes that the function $x^{\frac{1}{2}}f(x)$ is of bounded variation and that it satisfies

(i) at the point 0 a *Lipschitz condition*, Lip $(\nu+\frac{1}{2})$, which requires $x^{-\nu}f(x)$ to tend to 0 with x,

(ii) at the point 1 a mere *continuity condition*.

It is the object of the present note to extend Watson's results, first by weakening the condition (i) to a mere continuity condition and thus restoring the *symmetry* between (i) and (ii), and secondly by interpreting the *notion of bounded variation* in a much more general sense, by means of the notions of p-th power and exponential variation, which have already proved important in the convergence theory for trigonometric Fourier series§.

We state and prove our main results for the whole interval $0 \leqslant x \leqslant 1$, rather than for the parts $0 \leqslant x \leqslant \epsilon$ and $1-\epsilon \leqslant x \leqslant 1$. This is because those of our results which bear on the remaining part $\epsilon \leqslant x \leqslant 1-\epsilon$ appear to follow most directly from the fundamental inequalities for higher variations and Stieltjes integrals|| without employing the corresponding results which I have established previously for trigonometric Fourier series.

2. *Notation and relevant formulae.* The sum to n terms of the Fourier expansion of $x^{\frac{1}{2}}f(x)$, in the generalized sense, as a series of the normal and orthogonal functions (1.1), is $x^{\frac{1}{2}}s_n(x)$, where

$$(2.1) \qquad s_n(x) = \sum_{m=1}^{n} J_\nu(j_m x) \frac{\int_0^1 tf(t)\, J_\nu(j_m t)\, dt}{\frac{1}{2}J_{\nu+1}^2(j_m)}$$

$$= \int_0^1 tf(t)\, T_n(t, x)\, dt,$$

† W. H. Young, 2.

‡ W., 594 and 615.

§ L. C. Young, 4. We do not use these notions until § 7.

|| L. C. Young, 1, 2, 3.

and it is shown in Watson's book† that

$$(2.2)\quad T_n(t,x)=\sum_{m=1}^{n}\frac{2J_\nu(j_m x)\,J_\nu(j_m t)}{J_{\nu+1}^2(j_m)}$$

$$=\frac{1}{\pi i(t^2-x^2)}\int_{A_n-i\infty}^{A_n+i\infty}\{tJ_\nu(xw)\,J_{\nu+1}(tw)-xJ_\nu(tw)\,J_{\nu+1}(xw)\}\frac{dw}{J_\nu^2(w)},$$

provided that we choose A_n so that $j_n < A_n < j_{n+1}$; moreover‡

$$(2.3)\quad \int_0^t t^{\nu+1}\,T_n(t,x)(t^2-x^2)\,dt$$

$$=\frac{t^{\nu+1}}{\pi i}\int_{A_n-i\infty}^{A_n+i\infty}\{tJ_\nu(xw)\,J_{\nu+2}(tw)-xJ_{\nu+1}(xw)\,J_{\nu+1}(tw)\}\frac{dw}{wJ_\nu^2(w)}.$$

We shall take over, from the convergence theory for the partial interval $\epsilon \leqslant x \leqslant 1-\epsilon$, the following results which are obtained by choosing two particular functions§ of bounded variation $x^{\frac{1}{2}}f(x)$:

(2.4) *As* $n\to\infty$, *we have, for* $0<x<1$,

$$\int_0^x t^{\nu+\frac{1}{2}}\sqrt{(xt)}\,T_n(t,x)\,dt \to \tfrac{1}{2}x^{\nu+\frac{1}{2}},$$

and we have uniformly, for $\epsilon \leqslant x \leqslant 1-\epsilon$,

$$\int_0^1 t^{\nu+\frac{1}{2}}\sqrt{(xt)}\,T_n(t,x)\,dt \to x^{\nu+\frac{1}{2}}.$$

Actually, in Watson's book, these results are proved directly‖.

Finally, we quote for reference some simple consequences of the asymptotic expansion of a Bessel function¶. Here and in the rest of the present paper, K denotes a positive constant, whose precise value does not interest us and is not necessarily the same in different formulae. Also $R(z)$ and $I(z)$ denote the real and imaginary parts of a complex variable z.

† W., 581–585.

‡ See the preceding footnote.

§ The first of these, strictly, depends on an additional parameter: we choose $t^{\frac{1}{2}}f(t)$ to be $t^{\nu+\frac{1}{2}}$ for $0<t<x$ and to be 0 elsewhere. The second is simply the function $t^{\frac{1}{2}}f(t)=t^{\nu+\frac{1}{2}}$ for $0<t<1$ and 0 for $t=0$ and for $t=1$.

‖ W., 585, § 18 . 22.

¶ W., 199 (1).

We have

(2.5) $$\left| z^{\frac{1}{2}} J_\nu(z) - \sqrt{\left(\frac{2}{\pi}\right)} \cos\left[z - \tfrac{1}{2}(\nu+\tfrac{1}{2})\pi\right] \right| < \frac{K}{|z|} \exp\left(|I|(z)\right)$$

for large $|z|$ provided that $R(z) > 0$ (or, more generally, provided that $|\arg z| < \pi - \epsilon$). Moreover

(2.6) $$j_m = m\pi + \tfrac{1}{2}(\nu - \tfrac{1}{2})\pi + o(1), \quad J^2_{\nu+1}(j_m) \sim 1/j_m,$$

and in (2.2) and (2.3) we may take

(2.7) $$A_n = n\pi + \tfrac{1}{2}(\nu + \tfrac{1}{2})\pi.$$

With this value for A_n, we have, on account of (2.5), the following two inequalities†, valid for $0 \leqslant t \leqslant 1$ and $R(w) = A_n$, *uniformly* when n is sufficiently large,

(2.8) $$|\sqrt{(tw)}\, J_\nu(tw)| < K \exp(|I(tw)|), \quad |w^{\frac{1}{2}} J_\nu(w)| > K \exp\left(I\,|(w)|\right).$$

3. *Some estimates of magnitude.* An essential part will be played by the following rough estimates of magnitude, which are valid uniformly for $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 1$:

(3.1) $$\sqrt{(xt)}\,|T_n(t, x)| \leqslant Kn,$$

(3.2) $$\sqrt{(xt)}\,|T_n(t, x)| \leqslant K/|t-x|,$$

(3.3) $$\left| \int_0^t t^{\nu+1}\, T_n(t, x)(t^2 - x^2)\, dt \right| \leqslant K \frac{(t+x)\, t^{\nu+1}}{n\sqrt{(xt)}}.$$

The corresponding inequalities given by Watson‡ are just not good enough for our purpose: they are based only on (2.8) and so contain factors which the present more detailed calculations show to be superfluous.

Proof of (3.1). This inequality is obvious when we use the expression, given in (2.2), of $T_n(t, x)$ as a *sum* of n terms. For the modulus of each term is less than $K/\sqrt{(xt)}$ on account of (2.6), since $z^{\frac{1}{2}} J_\nu(z)$ is bounded for real positive z.

Proof of (3.2). We now employ the *integral* representation of $T_n(t, x)$, also given in (2.2), and we choose A_n as in (2.7). We may suppose, in view of (3.1), that $|t-x| > 1/n$. This implies, by (2.7),

(3.4) $$(2 - x - t)\, A_n > K,$$

† W., 584 (9).

‡ W., 585 (10) and 585 (11).

since we have $2-x-t \geqslant 1-x \geqslant t-x$ and $2-x-t \geqslant 1-t \geqslant x-t$. It is, moreover, easy to establish (3.2) in the case in which one of the numbers x and t does not exceed $\frac{1}{2}$; for in that case we have $2-x-t \geqslant \frac{1}{2} > K$ and so

$$\pi |(t^2-x^2)\, T_n(t, x)|$$

$$\leqslant t \int_{A_n-i\infty}^{A_n+i\infty} |J_\nu(xw)\, J_{\nu+1}(tw)| \frac{|dw|}{|J_\nu^2(w)|} + x \int_{A_n-i\infty}^{A_n+i\infty} |J_\nu(tw)\, J_{\nu+1}(xw)| \frac{|dw|}{|J_\nu^2(w)|}$$

$$\leqslant (t+x) \frac{K}{\sqrt{(xt)}} \int_0^\infty e^{-(2-x-t)v}\, dv \leqslant K\,(t+x)/\sqrt{(xt)},$$

by a straightforward application of (2.8).

In addition to (3.4), we may therefore assume that

$$(3.5) \qquad x > \tfrac{1}{2} \quad \textit{and} \quad t > \tfrac{1}{2}.$$

This being so, we write

$$T_n(t, x) = T'_n(t, x) + T_n''(t, x),$$

where $T_n'(t, x)$ is obtained by substituting, with suitable square root factors, the cosines occurring in the appropriate forms of (2.5), for the Bessel functions which involve x or t, in the integral representing $T_n(t, x)$. Then

$$T_n'(t, x) = \frac{1}{2\pi i(t^2-x^2)} \int_{A_n-i\infty}^{A_n+i\infty} \frac{2}{\pi} g'(w, x, t) \frac{dw}{\sqrt{(xt)}\, J_\nu^2(w)},$$

where $g'(w, x, t)$ denotes the expression

$$2t \cos[xw - \tfrac{1}{2}(\nu-\tfrac{1}{2})\pi] \cos[tw - \tfrac{1}{2}(\nu+\tfrac{3}{2})\pi]$$
$$- 2x \cos[tw - \tfrac{1}{2}(\nu+\tfrac{1}{2})\pi] \cos[xw - \tfrac{1}{2}(\nu+\tfrac{3}{2})\pi],$$

which may be written simply

$$(t+x) \cos[(t-x)w - \tfrac{1}{2}\pi] - (t-x) \cos[(t+x)w - \nu\pi].$$

Hence, from (2.8) and the corresponding trivial inequalities for the cosines, together with the inequalities $|t-x| < 1$ and $|t-x| < 2-t-x$, we find that

$$2\pi |(t^2-x^2)\, T_n'(t, x)| \leqslant \frac{K}{\sqrt{(xt)}} \left\{ (t+x) \int_0^\infty e^{(|t-x|-2)v}\, dv + |t-x| \int_0^\infty e^{(t+x-2)v}\, dv \right\}$$

$$\leqslant \frac{K}{\sqrt{(xt)}} \left\{ 2 \int_0^\infty e^{-v}\, dv + (2-x-t) \int_0^\infty e^{-(2-x-t)v}\, dv \right\}$$

and, in view of (3.5), we at once deduce that

$$T_n'(t, x) \leqslant \frac{K}{|t-x| \sqrt{(xt)}}. \tag{3.6}$$

On the other hand, we have

$$T_n''(t, x) = \frac{1}{2\pi i(t^2-x^2)} \int_{A_n-i\infty}^{A_n+i\infty} \frac{g''(w, x, t)}{\sqrt{(xt)}\, w} \frac{dw}{J_\nu^2(w)},$$

where $g''(w, x, t)$ denotes the difference

$$2\sqrt{(xt)}\, w\{tJ_\nu(xw)\, J_{\nu+1}(tw) - xJ_\nu(tw)\, J_{\nu+1}(xw)\} - \frac{2}{\pi} g'(w, x, t),$$

so that, by (2.5) and (3.5),

$$|g''(w, x, t)| < \frac{K}{|w|} \exp[(x+t)\, |I(w)|].$$

Hence, by (2.8),

$$|(t^2-x^2)\, T_n''(t, x)| \leqslant K \int_{A_n-i\infty}^{A_n+i\infty} \frac{\exp[(x+t-2)\,|I(w)|]}{\sqrt{(xt)}\, |w|} |dw|$$

$$\leqslant K \int_0^\infty \frac{e^{-(2-x-t)v}\, dv}{A_n \sqrt{(xt)}}$$

and, in view of (3.4) and (3.5), this gives

$$|T_n''(t, x)| < \frac{K}{|t-x| \sqrt{(xt)}}, \tag{3.7}$$

so that, by adding the inequalities (3.6) and (3.7), we obtain (3.2) and the proof is complete.

Proof of (3.3). We denote by Φ the expression on the right of (2.3) and we argue with it as with $T_n(t, x)$ in the proof of (3.2), except for a trivial simplification.

If at least one of the numbers x, t does not exceed $\frac{1}{2}$, we have, by (2.8),

$$|\Phi| \leqslant Kt^{\nu+1} \int_{A_n-i\infty}^{A_n+i\infty} (t+x) \frac{\exp[-(2-x-t)\, |I(w)|]}{\sqrt{(xt)}} \frac{|dw|}{|w|}$$

$$\leqslant \frac{K(t+x)\, t^{\nu+1}}{\sqrt{(xt)}\, A_n} \int_0^\infty e^{-(2-x-t)v}\, dv$$

and this last integral may be absorbed into the K.

We may, therefore, suppose, as before, that x and t are both greater than $\frac{1}{2}$, in which case, by (2.5), we may write

$$\Phi = \frac{t^{\nu+1}}{2\pi i}\int_{A_n-i\infty}^{A_n+i\infty} \frac{2}{\pi}\{h'(w, x, t)+h''(w, x, t)\}\frac{dw}{\sqrt{(xt)}\, w^2 J_\nu^2(w)},$$

where $h'(w, x, t)$ denotes the expression

$$2t\cos[xw-\tfrac{1}{2}(\nu+\tfrac{1}{2})\pi]\cos[tw-\tfrac{1}{2}(\nu+\tfrac{5}{2})\pi]$$
$$-2x\cos[xw-\tfrac{1}{2}(\nu+\tfrac{3}{2})\pi]\cos[tw-\tfrac{1}{2}(\nu+\tfrac{3}{2})\pi],$$

which we write in the more convenient form

$$(x-t)\cos[(x+t)w-(\nu+\tfrac{1}{2})\pi]-(t+x)\cos[(x-t)w],$$

and
$$|h''(w, x, t)| \leqslant \frac{\exp[(x+t)|I(w)|]}{|w|}.$$

From this, by majorizing crudely and by applying (2.8), it is now easy to see that $|\Phi|$ cannot exceed a constant multiple of the following combination of three integrals

$$|x-t|\int_0^\infty e^{-(2-x-t)v}\frac{dv}{\sqrt{(A_n^2+v^2)}}+(x+t)\int_0^\infty e^{-(2-|x-t|)v}\frac{dv}{\sqrt{(A_n^2+v^2)}}$$
$$+\int_0^\infty e^{-(2-x-t)v}\frac{dv}{A_n^2+v^2},$$

and this clearly cannot exceed

$$\frac{|x-t|}{(2-x-t)A_n}+\frac{K}{A_n}+\int_0^\infty\frac{dv}{A_n^2+v^2};$$

here the three terms are at most K/A_n, which may also be written as

$$\frac{K(x+t)t^{\nu+1}}{A_n\sqrt{(xt)}},$$

since x and t are supposed to exceed $\frac{1}{2}$. This completes the proof of (3.3).

4. *First properties of the integrated Fourier-Bessel kernels* $g_n(t)$. We are not interested in the expressions (2.2) and (2.3) for themselves, but in the related expression

$$(4.1)\qquad g_n(t) = \int_0^t \sqrt{(xt)}\, T_n(t, x)\, dt$$

considered as a function of t depending on the further parameters n and x. We call $g_n(t)$ the "integrated Fourier-Bessel kernel."

The central idea of the present mode of attack on the Fourier-Bessel convergence problem consists in deriving as much information as we can about $g_n(t)$ merely from the rough estimates of the magnitude of the expressions (2.2) and (2.3), obtained in the preceding paragraph.

We begin by establishing the results which constitute a basis for a Fourier-Bessel *localization principle*.

(4.2) *Given* $\delta > 0$, *we have uniformly in* n, t, x:

(4.3) $|g_n'(t)| < K(\delta)$ *when* $|t-x| \geqslant \delta$, *where* $K(\delta)$ *is a positive number depending on* δ *but independent of* n, t, x. Moreover

$$\operatorname*{Osc}_t\,[g_n(t);\ 0 \leqslant t \leqslant x-\delta] < K(\delta)\,\frac{\log n}{n}, \tag{4.4}$$

$$\operatorname*{Osc}_t\,[g_n(t);\ x+\delta \leqslant t \leqslant 1] < \frac{K(\delta)}{n}. \tag{4.5}$$

Of these results, (4.3) and (4.5) are straightforward, but (4.4), which already differs in form, by the intrusion of a logarithm, from the corresponding result for ordinary trigonometric Fourier kernels, requires a slight device which will play an even more important part later.

Proof of (4.3). It is sufficient to remark that this follows at once from (3.2), since the derivative $g_n'(t)$ of $g_n(t)$ is simply $\sqrt{(xt)}T_n(t, x)$.

Proof of (4.5). The factors

$$t^{-(\nu+\frac{1}{2})} \quad \text{and} \quad (t^2-x^2)^{-1}$$

are monotonic and do not exceed $K(\delta)$ when t lies in the interval

$$x+\delta \leqslant t \leqslant 1.$$

Hence, by the second mean-value theorem, for any two values t', t'', where $t' < t''$, in this interval, we have

$$|g_n(t'')-g_n(t')| = \frac{t'^{-(\nu+\frac{1}{2})}x^{\frac{1}{2}}}{t'^2-x^2}\int_{t'}^{t'''} t^{\nu+1}\,T_n(t,x)(t^2-x^2)\,dt \qquad \text{(where } t' < t''' < t''\text{)}$$

$$\leqslant \frac{K(\delta)}{n},$$

by (3.3). The inequality (4.5) follows at once.

Proof of (4.4). Instead of the oscillation of $g_n(t)$, we may, by the second mean value theorem, consider that of the function of t

$$g_n^*(t) = \int_0^t \sqrt{(xt)}\,T_n(t,x)(t-x)\,dt$$

since $x-t$ is monotonic and exceeds $K(\delta)$ for $0 \leqslant t \leqslant x-\delta$.

Now in any interval of the form $\frac{1}{2}u \leqslant t \leqslant u$, by the second mean value theorem applied to the factor $(t+x)^{-1}t^{-(\nu+\frac{1}{2})}$, the oscillation of $g_n{}^*(t)$ cannot exceed

$$K(u+2x)^{-1}u^{-(\nu+\frac{1}{2})}x^{\frac{1}{2}} \operatorname*{Osc}_t \left\{\int_0^t t^{\nu+1} T_n(t,x)(t^2-x^2)\,dt\right\}$$

and therefore, by (3 . 3), cannot exceed K/n. Hence, choosing successively $u = x-\delta$, $u = \frac{1}{2}(x-\delta)$, ..., $u = 2^{-(p-1)}(x-\delta)$ and adding the corresponding oscillations, we see that

$$\operatorname*{Osc}_t [g_n{}^*(t);\ 2^{-p}(x-\delta) \leqslant t \leqslant x-\delta] \leqslant Kp/n. \tag{4.6}$$

Moreover, by (3 . 2), since the oscillation cannot exceed the total variation,

$$\operatorname*{Osc}_t [g_n(t);\ 0 \leqslant t \leqslant 2^{-p}(x-\delta)] < K2^{-p}/\delta. \tag{4.7}$$

Now it follows, from statements already made, that in (4 . 6) we may replace $g_n{}^*(t)$ by $g_n(t)$ provided that we replace on the right K by $K(\delta)$. Hence, by choosing $p = \log n$ and combining (4 . 6) with (4 . 7), we obtain (4 . 4) as desired.

5. *Existence of a bound for the* $g_n(t)$. We now prove that

(5 . 1) *There exists a positive constant K independent of n, x, t such that*

$$|g_n(t)| < K. \tag{5.2}$$

It is clearly sufficient to prove, instead of (5 . 2), the inequality

$$\operatorname{Osc} [g_n(t);\ 0 \leqslant t \leqslant 1] < K, \tag{5.3}$$

which is equivalent to (5 . 2) since $g_n(0) = 0$.

Again, instead of (5 . 3) it is clearly sufficient to prove the three inequalities

$$\begin{cases} \operatorname{Osc} [g_n(t);\ 0 \leqslant t \leqslant \frac{2}{3}x] < K, \\ \operatorname{Osc} [g_n(t);\ \frac{2}{3}x \leqslant t \leqslant \frac{4}{3}x] < K, \\ \operatorname{Osc} [g_n(t);\ \frac{4}{3}x \leqslant t \leqslant 1] < K. \end{cases} \tag{5.4}$$

The first of these three inequalities is trivial. In fact, by (3 . 2),

$$\int_0^{\frac{2}{3}x} |g_n{}'(t)|\,dt < K. \tag{5.5}$$

In order to establish the second inequality (5.4), we write, as in the proof of (4.4) in the preceding paragraph,

$$g_n^*(t) = \int_0^t \sqrt{(xt)}\, T_n(t, x)\,(t-x)\,dt.$$

Evidently, by the second mean-value theorem applied to the factor $t^{-(\nu+\frac{1}{2})}(x+t)^{-1}$, (3.3) implies that

$$\text{Osc}\,[g_n^*(t);\ \tfrac{2}{3}x \leqslant t \leqslant \tfrac{4}{3}x] < K/n.$$

Consequently, if $0 < u < \frac{1}{3}x$, the same theorem gives

$$(5.6)\qquad \begin{cases} \text{Osc}\,[g_n(t);\ x-u \leqslant t \leqslant x-\frac{1}{2}u] < K/(nu), \\ \text{Osc}\,[g_n(t);\ x+\frac{1}{2}u \leqslant t \leqslant x+u] < K/(nu). \end{cases}$$

If we now add the inequalities obtained from (5.6) by choosing successively for u those values of the form $\frac{1}{3}x2^{-r}$ which exceed $1/n$, where r denotes a non-negative integer, we see that the sum of the right-hand sides is less than a fixed constant K, since†

$$\sum_{u>1/n}{}' \frac{1}{nu} < \sum_{m=0}^{\infty} 2^{-m} < K.$$

The second inequality (5.4) now follows at once, if we observe that, by (3.1), the oscillation of $g_n(t)$ in the remaining part of $\frac{2}{3}x \leqslant t \leqslant \frac{4}{3}x$ is trivial, since we actually have

$$(5.7)\qquad \int_{x-1/n}^{x+1/n} |g_n'(t)|\,dt < K.$$

In order, finally, to establish the third inequality (5.4), we first remark that, in any sub-interval of $\frac{4}{3}x \leqslant t \leqslant 1$, the oscillation of the function

$$\bar{g}_n(t) = \int_0^t t^{\nu+2\frac{1}{2}} \sqrt{(xt)}\, T_n(t, x)\,dt$$

cannot exceed a certain fixed constant K multiplied by the corresponding oscillation of the function

$$\int_0^t t^{\nu+1} x^{\frac{1}{2}}\, T_n(t, x)(t^2-x^2)\,dt;$$

this is an immediate consequence of the second mean value theorem applied with the factor $t^2/(t^2-x^2)$ to this last integral. Hence, for $\frac{8}{3}x \leqslant u \leqslant 1$,

† The index m of the second summation is connected with our former index r by the equation $r+m = N$, where N is the number of relevant values of u.

we have, by (3.3),

$$\operatorname{Osc}\,[\bar{g}_n(t);\ \tfrac{1}{2}u \leqslant t \leqslant u] < Ku^{\nu+\frac{3}{2}}/n,$$

and therefore

$$\operatorname{Osc}\,[g_n(t);\ \tfrac{1}{2}u \leqslant t \leqslant u] < K/(nu), \tag{5.8}$$

since, by the second mean-value theorem, the left-hand side of (5.8) is at most $Ku^{-(\nu+2\frac{1}{2})}$ times the corresponding oscillation of $\bar{g}_n(t)$.

If we now add the inequalities obtained from (5.8) by choosing successively $u = 2^{-r}$, where r denotes a non-negative integer and where $u \geqslant a = \max(1/n,\ \tfrac{8}{3}x)$, the sum of the right-hand sides is less than a fixed constant K, as before. The third inequality (5.4) now follows at once, if we observe that the inequalities (5.7) and

$$\int_{\frac{4}{3}x}^{\frac{8}{3}x} |g_n'(t)|\,dt < K, \tag{5.9}$$

which is an immediate consequence of (3.2), show that the oscillation of $g_n(t)$ for $\tfrac{4}{3}x \leqslant t \leqslant a$ is trivial.

6. *First applications.* We now establish the following result.

(6.1) *Let $x^{\frac{1}{2}}f(x)$ denote a function of bounded variation in $0 \leqslant x \leqslant 1$, vanishing for $x = 0$ and for $x = 1$. Then* (i), *as $n \to \infty$, we have, for $0 < x < 1$,*

$$x^{\frac{1}{2}} s_n(x) \to x^{\frac{1}{2}} \left\{ \frac{f(x+0)+f(x-0)}{2} \right\}$$

boundedly; *moreover* (ii), *if $x^{\frac{1}{2}}f(x)$ is, furthermore, continuous, we have, uniformly in x, for $0 \leqslant x \leqslant 1$,*

$$x^{\frac{1}{2}} s_n(x) \to x^{\frac{1}{2}} f(x).$$

To prove this, we require a few simple preliminary lemmas.

(6.2) "*Localization*" *principle. Given any fixed $\delta > 0$, the integrals*

$$\int_{0 \leqslant t \leqslant x-\delta} t^{\frac{1}{2}} f(t) \sqrt{(xt)}\, T_n(t, x)\, dt \quad \textit{and} \quad \int_{x+\delta \leqslant t \leqslant 1} t^{\frac{1}{2}} f(t) \sqrt{(xt)}\, T_n(t, x)\, dt$$

tend to zero uniformly in x as $n \to \infty$ for $0 \leqslant x \leqslant 1$, provided that $t^{\frac{1}{2}}f(t)$ is Lebesgue integrable.

This follows at once from (4.2) by a classical Lebesgue theorem†.

(6.3) *With the hypotheses of* (6.1) (i), *the expression*‡

$$x^{\frac{1}{2}} s_n(x) - x^{\frac{1}{2}} f(x-0)\, g_n(x) - x^{\frac{1}{2}} f(x+0)[g_n(1) - g_n(x)]$$

tends to 0 *boundedly as* $n \to \infty$, *for* $0 \leqslant x \leqslant 1$.

The modulus of the expression in question may be written successively

$$\left| \int_0^x \{t^{\frac{1}{2}} f(t) - x^{\frac{1}{2}} f(x-0)\}\, dg_n(t) + \int_x^1 \{t^{\frac{1}{2}} f(t) - x^{\frac{1}{2}} f(x+0)\}\, dg_n(t) \right|$$

$$= \left| \int_{x-\delta}^{x} \{t^{\frac{1}{2}} f(t) - x^{\frac{1}{2}} f(x-0)\}\, dg_n(t) + \int_x^{x+\delta} \{t^{\frac{1}{2}} f(t) - x^{\frac{1}{2}} f(x+0)\}\, dg_n(t) + o(1) \right|$$

$$\leqslant K\tau(\delta;\ x) + K\tau(\delta;\ x) + o(1),$$

by the localization principle and by (5.1), where $\tau(\delta;\ x)$ denotes the total variation§ of $t^{\frac{1}{2}} f(t)$ in $x-\delta \leqslant t < x$ and $x < t \leqslant x+\delta$, which is bounded in x, δ, and, for fixed x, arbitrarily small with δ; and $o(1)$ denotes an expression which tends uniformly to 0 as $n \to \infty$, for fixed δ.

(6.4) *With the hypotheses of* (6.1) (ii), *the expression*

$$x^{\frac{1}{2}} s_n(x) - x^{\frac{1}{2}} f(x)\, g_n(1)$$

tends to 0 *uniformly, as* $n \to \infty$, *for* $0 \leqslant x \leqslant 1$.

It is sufficient to observe that in the preceding proof $\tau(\delta;\ x)$ is now small with δ uniformly in x, and that $f(x+0) = f(x-0) = f(x)$||.

To establish (6.1) it is now sufficient, in view of (5.1), to prove that, for $0 < x < 1$, $g_n(x) \to \frac{1}{2}$ as $n \to \infty$, and that, for $\epsilon \leqslant x \leqslant 1-\epsilon$, $g_n(1) \to 1$ *uniformly* as $n \to \infty$. This is a variant of (2.4) which might have been taken over from the convergence theory for the partial interval $\epsilon \leqslant x \leqslant 1-\epsilon$,

† Hobson, 1, II, 422, Theorem I.

‡ The functions $t^{\frac{1}{2}} f(t)$ and $g_n(t)$ are supposed constant for $t < 0$ and for $t > 1$, if necessary, and we have $f(-0) = f(1+0) = 0$.

§ We use the inequality (valid for $a < \xi < b$)

$$\int_a^b F(t)\, dG(t) < \{|F(\xi)| + V_1(F(t),\ a < t < b)\} \operatorname{osc}[G(t);\ a < t < b],$$

where $V_1(F(t),\ a \leqslant t \leqslant b)$ is the total variation of $F(t)$ in $a \leqslant t \leqslant b$.

|| At the origin these equations must be supposed multiplied through by $x^{\frac{1}{2}}$, since $f(+0)$ by itself need not even exist.

in which case (2.4), which is formally more complicated, could have been dispensed with. With the notation of (2.4) this variant is:

(6.5) *As* $n \to \infty$, *we have*, for $0 < x < 1$,

$$\int_0^x \sqrt{(xt)}\, T_n(t, x)\, dt \to \tfrac{1}{2}.$$

and we have uniformly, for $\epsilon \leqslant x \leqslant 1-\epsilon$,

$$\int_0^1 \sqrt{(xt)}\, T_n(t, x)\, dt \to 1.$$

It is easy to deduce the variant from (2.4) itself, by using (6.3) and (6.4) with two particular functions $t^{\frac{1}{2}}f(t)$. Thus, taking $t^{\frac{1}{2}}f(t) = t^{\nu+\frac{1}{2}}$ for $0 < t < x$ and 0 for $t \geqslant x$, (6.3) asserts that $x^{\frac{1}{2}}s_n(x) - x^{\nu+\frac{1}{2}}g_n(x) \to 0$, and (2.4) that $x^{\frac{1}{2}}s_n(x) \to \frac{1}{2}x^{\nu+\frac{1}{2}}$ for $0 < x < 1$. This gives the first of the relations (6.5). Again, taking $t^{\frac{1}{2}}f(t) = t^{\nu+\frac{1}{2}}$ for $0 < t < 1$, (6.4) asserts that $x^{\frac{1}{2}}s_n(x) - x^{\nu+\frac{1}{2}}g_n(1) \to 0$ and (2.4) that $x^{\frac{1}{2}}s_n(x) \to x^{\nu+\frac{1}{2}}$ both these relations being certainly uniform in x for $\epsilon \leqslant x \leqslant 1-\epsilon$. This gives the second of the relations (6.5).

To establish (6.1) it is sufficient to observe that, by (6.5) and (5.1), combined with the conditions $f(0) = f(1) = 0$, the expressions

$$x^{\frac{1}{2}}f(x-0)\, g_n(x) \quad \text{and} \quad x^{\frac{1}{2}}f(x+0)[g_n(1) - g_n(x)]$$

tend boundedly to

$$\tfrac{1}{2}x^{\frac{1}{2}}f(x-0) \quad \text{and} \quad \tfrac{1}{2}x^{\frac{1}{2}}f(x+0),$$

and that, if $x^{\frac{1}{2}}f(x)$ is continuous at 0 and 1,

$$x^{\frac{1}{2}}f(x)\, g_n(1) \to x^{\frac{1}{2}}f(x)$$

uniformly in x for $0 \leqslant x \leqslant 1$.

7. *Further properties of the* $g_n(t)$. The results so far obtained are still very crude. Nevertheless, the considerations of the preceding paragraphs, for whose length and elementary character we must ask the reader to be indulgent, contain in germ all that is necessary to enable us to apply modern ideas to study the convergence problem more in detail.

In § 5, in particular, in establishing the existence of a bound for the $g_n(t)$, we established incidentally a property almost akin to bounded variation: we proved that, for a particular division of $0 \leqslant t \leqslant 1$ into partial intervals, the oscillations of the $g_n(t)$ are majorized by the terms of three convergent geometric series of the form $\sum_m K2^{-m}$. We might also have remarked, as is at once obvious from the inequality (3.2), that *in each of the partial intervals in question*

$$(7.1) \qquad \int |g_n'(t)|\,dt < K.$$

From these two properties above it is possible to deduce at once, by convexity in p of the logarithm of p-th power variation (or mean variation of order p)† that in the same partial intervals the numbers $V_p(g_n)$ are majorized by K times the $[(p-1)/p]$-th powers of the terms of our three geometric series and therefore are majorized for every choice of $p > 1$, by the terms of three other convergent geometric series. And hence, by repeated application of the formula

$$(7.2) \qquad V_p(f;\ a, b) \leqslant V_p(f;\ a, \xi) + V_p(f;\ \xi, b)$$

(which is an easy consequence of Minkowski's inequality), we see that, in the whole interval $0 \leqslant t \leqslant 1$, $V_p(g_n)$ is less than a fixed constant K provided that we fix $p > 1$.

We can go further still. For a geometric series not only remains convergent when its terms are raised to any fixed power (in this case, the $[(p-1)/p]$-th power), but also when we pass to suitable powers of their logarithms.

This line of thought, which makes it necessary to avoid using (7.2) owing to the fact that Minkowski's inequality is restricted to powers, leads to the following theorem:

(7.3) *Let $\Psi(u)$ denote either the function u^q where $q > 1$ or the function $u\{\log[3M/u]\}^{-q}$ [defined for $u \leqslant 2M$, where M is the common upper bound of the $g_n(t)$ which exists by (5.1)], where $q > 1$. Then the total Ψ-variations of the $g_n(t)$ in the interval $0 \leqslant t \leqslant 1$ are bounded, uniformly in n and x, for fixed q.*

By the total Ψ-variation‡

$$V_\Psi(f;\ a, b)$$

of a function $f(t)$ in an interval $a \leqslant t \leqslant b$ we mean the upper bound, for every choice of a finite system e of non-overlapping intervals Δ contained in $0 \leqslant t \leqslant 1$, of the sum

$$\sum_e \Psi(|\Delta f|),$$

† L. C. Young, 1, § 8 [*cf.* in particular, 259, (8.2*a*)].

‡ L. C. Young, 2.

where Δf denotes the difference of the values of $f(t)$ at the ends of Δ. The classical total variation is the case $\Psi(u) = u$; the total p-th power variation is the case $\Psi(u) = u^p$; and so on. The function $\Psi(u)$ is generally supposed increasing from $\Psi(0) = 0$ to $\Psi(\infty) = \infty$ with u, but in our case only the values $u \leqslant 2M$ are relevant and we may suppose $\Psi(u)$ suitably modified for $u > 2M$.

The general lines of the proof of (7.3) have already been indicated. We may, therefore, limit ourselves to the case

$$\Psi(u) = u\{\log [3M/u]\}^{-q}$$

where $q > 1$.

Let us denote by E the finite set of points of the form $t = x \pm u$, where† $u = 2^{-r}x/3 > 1/(2n)$, together with those of the form $t = 2^{-s} > 4x/3$ for which, moreover, $t > 1/n$ (r and s are non-negative integers).

This being so, let e denote any finite set of non-overlapping intervals in $0 \leqslant t \leqslant 1$. We have to show that

$$\sum_e \Psi(|\Delta g_n|) < K, \tag{7.4}$$

where K is independent of e, n, x. We denote by e_1 the set of the intervals of e which contains at least one point of E. Clearly, in view of the remarks already made, it follows from §5 that

$$\sum_{e_1} |\Delta g_n| < K,$$

and therefore that

$$\sum_{e_1} \Psi(|\Delta g_n|) < K. \tag{7.5}$$

The remaining intervals of e can be ranged in groups e_2 such that each group e_2 consists of intervals between the same pair of points t', t'' of E. For each such group e_2 we have

$$\begin{aligned}\sum_{e_2} \Psi(|\Delta g_n|) &\leqslant [\log(3M/\max|\Delta g_n|)]^{-q} \sum_{e_2} |\Delta g_n| \\ &\leqslant [\log(3M/\operatorname{Osc} g_n)]^{-q} \int_{t'}^{t''} |g_n'(t)|\,dt \\ &\leqslant K[\log(3M/\operatorname{Osc} g_n)]^{-q}.\end{aligned}$$

Now we have seen that the oscillations of $g_n(t)$ in the partial intervals determined by the consecutive pairs of points t', t'' of E are majorized by

† The symbol u has now no connection with the variable occurring in $\Psi(u)$.

the terms of three geometric series $\sum_m K 2^{-m}$. It follows that the groups of terms

$$\sum_{e_2} \Psi(|\Delta g_m|)$$

are majorized by single terms of three series $\sum_m K m^{-q}$. Hence, adding the contributions of the various groups e_2 to that of e_1 which is estimated in (7.5), we obtain (7.4) as required.

[*Note.*—If we write $g_n(t)$ as a function of t, x in the form $g_n(t, x)$ say, it is natural to ask whether the p-th power, and other, variations in x for constant t are subject to inequalities at all similar to those established here for the corresponding variations in t for constant x. The analogy with trigonometric Fourier series rather suggests this, but there appear to be serious formal difficulties in the way.

One type of conclusion which could be drawn from such results (if true) concerns the existence of a bound for certain power variations of the functions $x^{\frac{1}{2}} s_n(x)$ in the case in which $x^{\frac{1}{2}} f(x)$ is of a corresponding bounded power variation. This follows from an inequality for the "Stieltjes Faltung" which has been given in a recent paper† in these *Proceedings*.]

8. *Extension of the convergence criteria.* We now come to our main theorem which asserts the following:

(8.1) *In* (6.1) *the term "function of bounded variation" may, without affecting the validity of the theorem, be interpreted to mean either* (*a*) *"function of bounded p-th power variation", or* (*b*) *"function of bounded Φ-variation" where, for the relevant*‡ u,

(8.2) $$\Phi(u) = \exp(-u^{-c}), \quad 0 < c < \tfrac{1}{2}.$$

Proof of (8.1). Denoting by $\mu = \mu(\delta; x)$ the oscillation of the function $t^{\frac{1}{2}} f(t)$ in $x-\delta \leqslant t < x$ or in $x < t \leqslant x+\delta$ (whichever is the larger), we observe that, by the results proved in § 6, it is sufficient to show that the integrals

(8.3) $$\int_{x-\delta}^{x} \{t^{\frac{1}{2}} f(t) - x^{\frac{1}{2}} f(x-0)\}\, dg_n(t) \quad \text{and} \quad \int_{x}^{x+\delta} \{t^{\frac{1}{2}} f(t) - x^{\frac{1}{2}} f(x+0)\}\, dg_n(t)$$

(in which the functions concerned are supposed constant for $t \leqslant 0$ and $t \geqslant 1$ if necessary) are less in absolute value than an expression of the form

† L. C. Young, 3.

‡ We suppose that Φ increases to ∞ with u. This implies that the function concerned is bounded. But once we know the function to be bounded, the large values of u are irrelevant.

$K\,\epsilon(\mu)$, where $\epsilon(\mu)$ is a bounded function of μ (for the relevant values of μ) and tends to 0 as $\mu \to 0$. It is sufficient to prove this in the case of the former of the two integrals (8.3), since the proof for the second integral proceeds on symmetrical lines.

Now it is easy to see that for any $p_1 > p$ in case (*a*), or for any c_1 such that $c < c_1 < \frac{1}{2}$ in case (*b*) and for the corresponding function $\Phi_1(u)$ obtained by replacing c by c_1 in the definition of $\Phi(u)$, (8.2), we have

$$(8.4) \qquad (a) \quad V_{p_1}(F) < \epsilon(\mu) \quad \text{or} \quad (b) \quad V_{\Phi_1}(F) < \epsilon(\mu),$$

where $F(t)$ denotes the function defined for $x-\delta \leqslant t \leqslant x$ and equal to $t^{\frac{1}{2}} f(t)$ except perhaps† at $t = x$, where it has the value $x^{\frac{1}{2}} f(x-0)$.

Moreover we have

$$(8.5) \qquad (a) \quad V_{q_1}(g_n) < K \quad \text{and} \quad (b) \quad V_{\Psi_1}(g_n) < K,$$

when we choose q_1 so that $1 < q_1 < p_1/(p_1-1)$ and

$$\Psi_1(u) = u/[\log(3M/u)]^{1/(2c_1)}.$$

This being so, we apply the fundamental inequalities for Stieltjes integrals of the author's papers in the *Acta Mathematica* and in the *Mathematische Annalen*‡. We obtain for the absolute value of the integral

$$\int_{x-\delta}^{x} \{F(t) - F(x)\}\, dg_n(t),$$

which is equal to the first integral (8.3), the majorant§

$$\epsilon(\mu) . K$$

in case (*a*) since $(1/p_1) + (1/q_1) > 1$, in view of (8.4) (*a*) and (8.5) (*a*); and the majorant||

$$(8.6) \qquad K \sum_m \phi_1\left(\frac{\epsilon(\mu)}{m}\right) \psi_1\left(\frac{K}{m}\right)$$

in case (*b*), where $\phi_1(u)$, $\psi_1(u)$ denote the inverse functions of $\Phi_1(u)$, $\Psi_1(u)$, in view of (8.4) (*b*) and (8.5) (*b*). This completes the proof in case (*a*), while in case (*b*) it is now only necessary to verify that the series

$$(8.7) \qquad \sum_m \phi_1\left(\frac{1}{m}\right) \psi_1\left(\frac{1}{m}\right)$$

† We suppose the function $t^{\frac{1}{2}} f(t)$ continued for $t < 0$ and for $t > 1$ if necessary, by defining it to be zero.

‡ L. C. Young, **1**, **2**.

§ L. C. Young, **1**, 266 (10.9).

|| L. C. Young, **2**, 597, Theorem (5.1)(*i*).

converges, since, if (8.7) converges, (8.6) split up at $m = N$ is at most $KN\phi_1[\epsilon(\mu)] + \epsilon_N$, where $\epsilon_N \to 0$ independently of μ as $N \to \infty$, so that (8.6) is at most another $\epsilon(\mu)$. This completes the proof since the convergence of (8.7) is an immediate consequence of the following relations, valid for sufficiently small $\epsilon > 0$,

$$\phi_1(u) = [\log(1/u)]^{-1/c_1}, \quad \psi_1(u) < Ku[\log(K/u)]^{(1+\epsilon)/(2c_1)}$$

and $$\phi_1(u)\psi_1(u) < Ku[\log(K/u)]^{-(1-\epsilon)(2c_1)} < Ku[\log(K/u)]^{-(1+\epsilon)}.$$

As a particular case of the theorem just proved, we may mention that *if the function* $F(x) = \sqrt{x}f(x)$ *satisfies uniformly in* x *the Lipschitz condition*

$$|F(x+h) - F(x)| < K[\log(1/h)]^{-(2+\epsilon)},$$

where $\epsilon > 0$, *then it is uniformly in* x *the sum of its orthogonal and normalized Fourier-Bessel series, i.e. the limit of* $x^{\frac{1}{2}}s_n(x)$.

Actually the analogy with ordinary Fourier series at once suggests that the exponent $-(2+\epsilon)$ in this last result may be replaced by -1.

The present writer is unable to state definitely whether this extension is possible and whether it has been made. What is certain is that the problem of a corresponding extension of the range of c in (8.1) (b) is unsolved.

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