Multi View Geometry (Spring '08)

Epipolar Geometry and the Fundamental Matrix

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Two View Geometry

- Epipolar geometry
- 3D reconstruction
- F-matrix computation
- Structure computation

Three questions

- **Correspondence geometry:**

✓ Given an image point **x** in the first image, how does this constrain the position of the corresponding point **x'** in the second image?

- **Camera geometry (motion):**

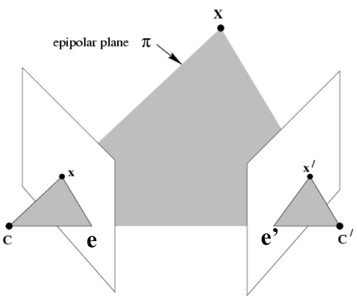
✓ Given a set of corresponding image points $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$, $i=1,\ldots,n$, what are the cameras **P** and **P'** for the two views?

- **Scene geometry (structure):**

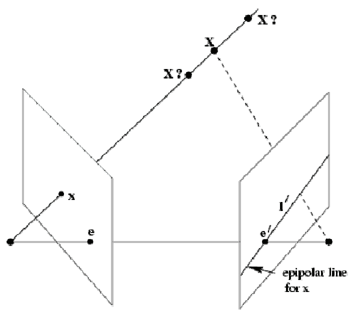
✓ Given corresponding image points $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ and cameras **P**, **P'**, what is the position of (their pre-image) **X** in space?

Epipolar geometry

- Point correspondence geometry



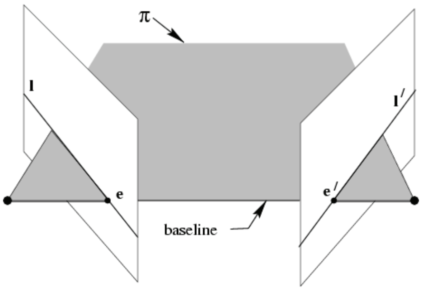
C,**C'**,**x**,**x'** and **X** are coplanar



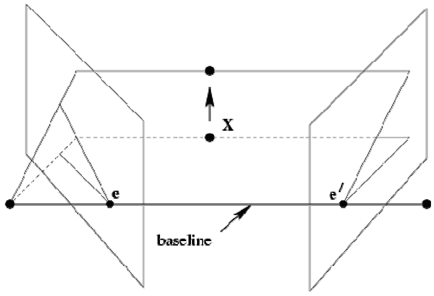
What if only **C**,**C'**,**x** are known?

- *Epipole* (**e**,**e'**):
  = intersection of baseline with image plane
  = projection of projection center in other image
  = vanishing point of camera motion direction
- *Epipolar plane* : plane containing baseline (1-D family)

Epipolar geometry



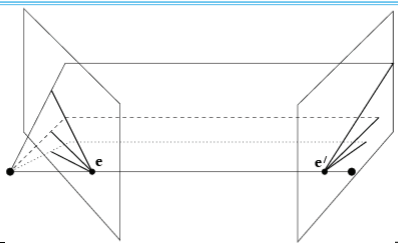
All points on π project on **l** and **l’**

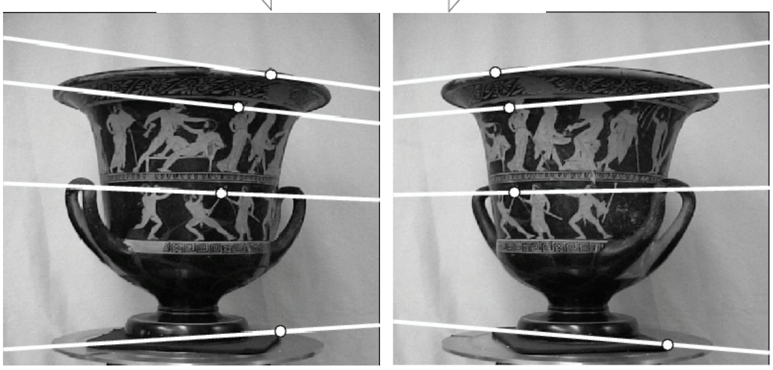


Family of planes π and lines **l** and **l’**
Intersection in **e** and **e’**

- *Epipolar line* : The intersection of an epipolar plane with the image plane. (always come in corresponding pairs)
- *Baseline* : Intersection line each image plane at the epipoles
- *Epipolar pencil* : Family of epipolar planes

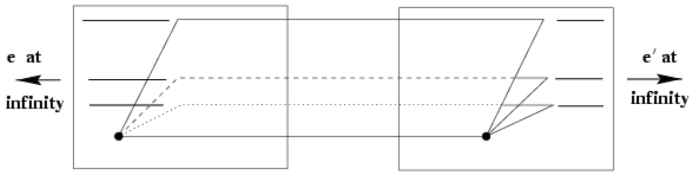
Epipolar geometry

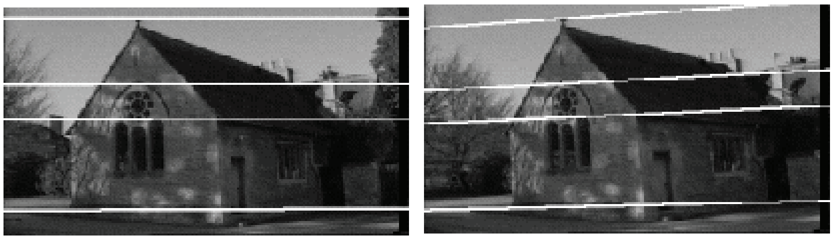




Epipolar geometry

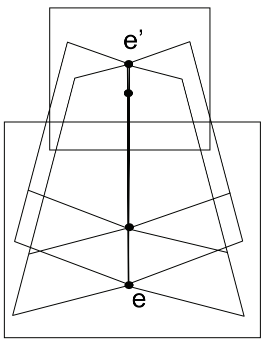
- Example: motion parallel with image plane

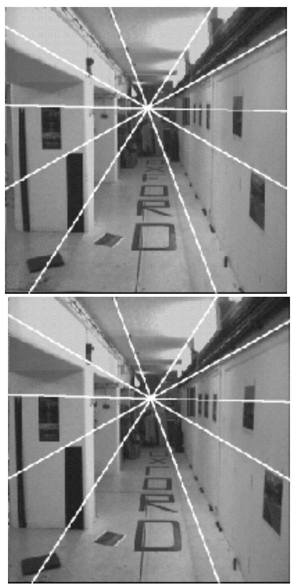




Epipolar geometry

- Example: forward motion





The fundamental matrix F

- Fundamental Matrix **F**:
- Algebraic representation of epipolar geometry

$$\mathbf{x} \mapsto \mathbf{l}'$$

- We will see that mapping is (singular) correlation (i.e. projective mapping from points to lines) represented by the fundamental matrix **F**

The fundamental matrix F

- Geometric derivation
  - ✓ Step 1 : Point transfer via a plane

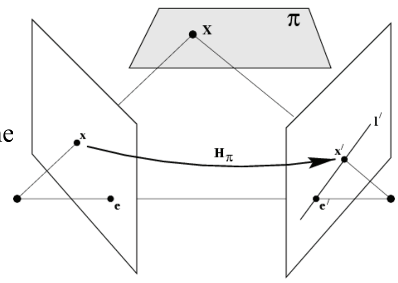
$$\mathbf{x}' = \mathbf{H}_\pi \mathbf{x}$$

  - ✓ Step 2 : Construction of the epipolar line

$$\mathbf{l}' = \mathbf{e}' \times \mathbf{x}' = [\mathbf{e}']_\times \mathbf{x}'$$

$$\mathbf{l}' = [\mathbf{e}']_\times \mathbf{H}_\pi \mathbf{x} = \mathbf{F}\mathbf{x}$$

$$\mathbf{F} = [\mathbf{e}']_\times \mathbf{H}_\pi$$



mapping from 2-D to 1-D family (rank 2)

※Cross products : If $\mathbf{a} = (a_1, a_2, a_3)^T$ is a 3 vector, then one defines a corresponding skew-symmetric matrix $[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Matrix $[\mathbf{a}]_\times$ is singular, and **a** is its null-vector (right or left)

The fundamental matrix F

- Algebraic derivation
  - ✓ ray back-projected from **x** by **P**

$$\mathbf{X}(\lambda) = \mathbf{P}^{+}\mathbf{x} + \lambda\mathbf{C}, \quad \mathbf{P}^{+} = \mathbf{P}^{T}(\mathbf{P}\mathbf{P}^{T})^{-1} \qquad (\mathbf{P}^{+}\mathbf{P} = \mathbf{I})$$

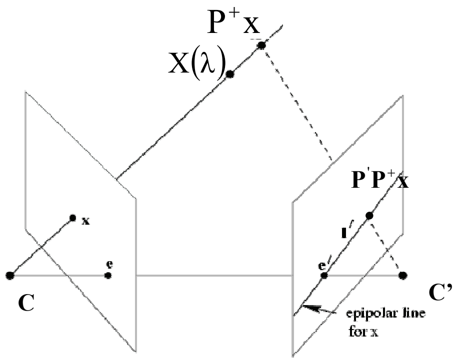
  - ✓ consider two points

$$\mathbf{P}^{+}\mathbf{x}(\lambda = 0), \quad \mathbf{C}(\lambda = \infty)$$

  - ✓ The epipolar line by **P'**

$$\begin{aligned} \mathbf{l}' &= (\mathbf{P}'\mathbf{C})\times(\mathbf{P}'\mathbf{P}^{+}\mathbf{x}) \\ &= [\mathbf{e}']_{\times}(\mathbf{P}'\mathbf{P}^{+})\mathbf{x} \\ &= \mathbf{F}\mathbf{x} \end{aligned}$$

$$\therefore \mathbf{F} = [\mathbf{e}']_{\times}\mathbf{P}'\mathbf{P}^{+}$$



The fundamental matrix F

- Example
- ✓ Suppose the camera matrices are those of a calibrated stereo rig with the world origin at the first camera

$$\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}] \qquad \mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}].$$

then

$$\mathtt{P}^{+} = \begin{bmatrix} \mathtt{K}^{-1} \\ \mathbf{0}^{\top} \end{bmatrix} \qquad \mathbf{C} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

and

$$\begin{aligned} \mathtt{F} &= [\mathtt{P}'\mathbf{C}]_{\times}\mathtt{P}'\mathtt{P}^{+} \\ &= [\mathtt{K}'\mathbf{t}]_{\times}\mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\top}\mathtt{R}[\mathtt{R}^{\top}\mathbf{t}]_{\times}\mathtt{K}^{-1} = \mathtt{K}'^{-\top}\mathtt{R}\mathtt{K}^{\top}[\mathtt{K}\mathtt{R}^{\top}\mathbf{t}]_{\times} \end{aligned}$$

where the various forms follow equation

$$[t\mathbf{e}']_{\times}(\mathbf{P}'\mathbf{P}^{+})\mathbf{x} = \mathbf{F}\mathbf{x}[\mathbf{e}']_{\times}$$

- ✓ Note that the epipoles are

$$\mathbf{e} = \mathtt{P}\begin{pmatrix} -\mathtt{R}^{\top}\mathbf{t} \\ 1 \end{pmatrix} = \mathtt{K}\mathtt{R}^{\top}\mathbf{t} \qquad \mathbf{e}' = \mathtt{P}'\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathtt{K}'\mathbf{t}.$$

$$\mathbf{F} = [\mathbf{e}']_{\times}\mathbf{H}_{\infty} \qquad (\mathbf{H}_{\infty} = \mathbf{K}'\mathbf{R}\mathbf{K}^{-1})$$

Thus

$$\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\top}\mathtt{R}[\mathtt{R}^{\top}\mathbf{t}]_{\times}\mathtt{K}^{-1} = \mathtt{K}'^{-\top}\mathtt{R}\mathtt{K}^{\top}[\mathbf{e}]_{\times}.$$

The fundamental matrix F

- Correspondence condition:

  ✓ The fundamental matrix satisfies the condition that for any pair of corresponding points $\mathbf{x} \leftrightarrow \mathbf{x}'$ in the two images

$$\mathbf{x}'^{\mathrm{T}} \mathbf{F}\mathbf{x} = 0 \quad \left(\mathbf{x}'^{\mathrm{T}} \mathbf{l}' = 0\right)$$

  ✓ $\mathbf{F}$ is the unique 3x3 rank 2 matrix that satisfies $\mathbf{x}'^{\mathrm{T}}\mathbf{F}\mathbf{x}=0$ for all $\mathbf{x} \leftrightarrow \mathbf{x}'$

The fundamental matrix F

- F is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence**: If $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points, then $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0$.
- **Epipolar lines**:
  - ◇ $\mathbf{l}' = \mathrm{F}\mathbf{x}$ is the epipolar line corresponding to $\mathbf{x}$.
  - ◇ $\mathbf{l} = \mathrm{F}^{\top}\mathbf{x}'$ is the epipolar line corresponding to $\mathbf{x}'$.
- **Epipoles**:
  - ◇ $\mathrm{F}\mathbf{e} = \mathbf{0}$.
  - ◇ $\mathrm{F}^{\top}\mathbf{e}' = \mathbf{0}$.
- **Computation from camera matrices** P, P':
  - ◇ General cameras,
    $\mathrm{F} = [\mathbf{e}']_{\times}\mathrm{P}'\mathrm{P}^{+}$, where P^{+} is the pseudo-inverse of P, and $\mathbf{e}' = \mathrm{P}'\mathbf{C}$, with $\mathrm{P}\mathbf{C} = \mathbf{0}$.
  - ◇ Canonical cameras, $\mathrm{P} = [\mathrm{I} \mid \mathbf{0}]$, $\mathrm{P}' = [\mathrm{M} \mid \mathbf{m}]$,
    $\mathrm{F} = [\mathbf{e}']_{\times}\mathrm{M} = \mathrm{M}^{-\top}[\mathbf{e}]_{\times}$, where $\mathbf{e}' = \mathbf{m}$ and $\mathbf{e} = \mathrm{M}^{-1}\mathbf{m}$.
  - ◇ Cameras not at infinity $\mathrm{P} = \mathrm{K}[\mathrm{I} \mid \mathbf{0}]$, $\mathrm{P}' = \mathrm{K}'[\mathrm{R} \mid \mathbf{t}]$,
    $\mathrm{F} = \mathrm{K}'^{-\top}[\mathbf{t}]_{\times}\mathrm{R}\mathrm{K}^{-1} = [\mathrm{K}'\mathbf{t}]_{\times}\mathrm{K}'\mathrm{R}\mathrm{K}^{-1} = \mathrm{K}'^{-\top}\mathrm{R}\mathrm{K}^{\top}[\mathrm{K}\mathrm{R}^{\top}\mathbf{t}]_{\times}$.

The fundamental matrix F

- Properties of fundamental matrix

(i) Transpose : If $\mathbf{F}=(\mathbf{P},\mathbf{P}')$, then $\mathbf{F}^T=(\mathbf{P}',\mathbf{P})$

(ii) Epipolar line : For any point $\mathbf{x}$, $\mathbf{x}'$ in the first and second image

$$\mathbf{l}' = \mathbf{F}\mathbf{x}, \quad \mathbf{l} = \mathbf{F}^T\mathbf{x}'$$

(iii) The epipole : The epipolar line $\mathbf{l}'=\mathbf{F}\mathbf{x}$ contains the epipole $\mathbf{e}'$.

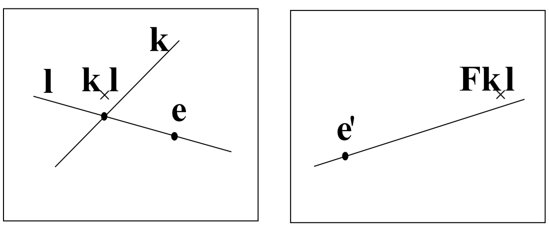
$$\mathbf{e}'^T(\mathbf{F}\mathbf{x}) = (\mathbf{e}'^T\mathbf{F})\mathbf{x} = 0$$

(iv) **F** has 7 DOF: 3x3 -1 (scaling) – 1(rank2) =7.

(v) **F** is a correlation : projetive mapping from **x** to a line $\mathbf{l}'=\mathbf{F}\mathbf{x}$,
however it is not a proper correlation (not invertible).

The Epipolar line homography

- Suppose $\mathbf{l}$,$\mathbf{l}'$ epipolar lines, $\mathbf{k}$ any line not through $\mathbf{e}$
- Then $\mathbf{l}'=\mathbf{F}[\mathbf{k}]_\times\mathbf{l}$ and symmetrically $\mathbf{l}=\mathbf{F}^T[\mathbf{k}']_\times\mathbf{l}'$



(pick $\mathbf{k}=\mathbf{e}$, since $\mathbf{e}^T\mathbf{e}\neq 0$)

$$\mathbf{l}' = \mathbf{F}[\mathbf{e}]_\times\mathbf{l} \qquad \mathbf{l} = \mathbf{F}^T[\mathbf{e}']_\times\mathbf{l}'$$

Special motions

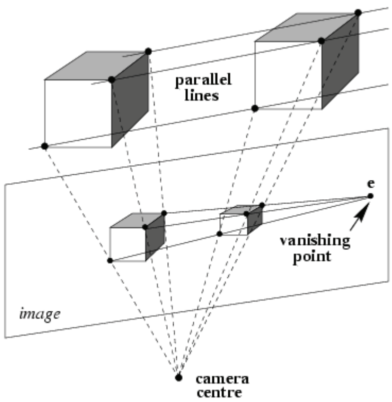
- A special motion arises from a particular relationship between the translation direction **t** and the direction of the rotation axis **a**.

- Pure translation **:** No rotation
- Pure planar motion **: t** is orthogonal to **a .**

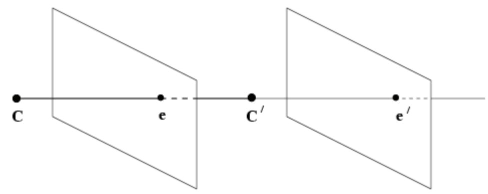
※ The 'pure' indicates that there is no change in the internal parameters.

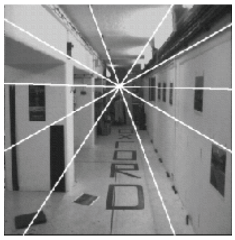
Pure Translation

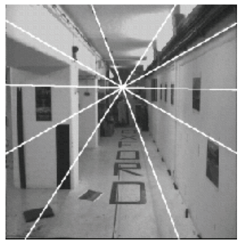
- The camera is stationary, and the world undergoes a translation – **t** .
- Vanishing point **v** in the direction of **t** is the epipole and parallel lines are the epipolar lines.



Pure Translation







Pure translation

- Example

✓ Two cameras $\mathbf{P} = \mathbf{K}[\mathbf{I} \mid 0]$ and $\mathbf{P}' = \mathbf{K}[\mathbf{I} \mid \mathbf{t}]$, then

$$\mathtt{F} = [\mathtt{e}']_\times \mathtt{K}\mathtt{K}^{-1} = [\mathtt{e}']_\times$$

✓ If the camera translation is parallel to the x-axis, then $\mathbf{e}' = (1,0,0)^{\mathrm{T}}$ so,

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}_\times$$

$$\mathbf{x}'^{\mathrm{T}} \mathbf{F}\mathbf{x} = 0 \Leftrightarrow y = y'$$

✓ If x is normalized as $\mathbf{x}=(x,y,1)^T$ then from

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \mathbf{K}[\mathbf{I} \mid \mathbf{0}]\mathbf{X} \Rightarrow (X,Y,Z)^T = Z\mathbf{K}^{-1}\mathbf{x} \qquad \mathbf{F} = [\mathbf{e}']_\times$$

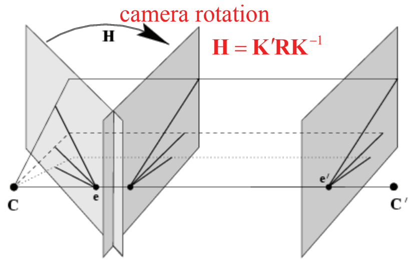
skew symmetric

$$\mathbf{x}' = \mathbf{P}'\mathbf{X} = \mathbf{K}[\mathbf{I} \mid \mathbf{t}]\mathbf{X} \Rightarrow \mathbf{x}' = \mathbf{x} + \mathbf{K}\mathbf{t}/Z$$

✓ If pure translation, $\mathbf{F}$ only 2DOF, $\mathbf{x}^{\mathrm{T}}[\mathbf{e}']_\times \mathbf{x} = 0$ ⇨ auto-epipolar

General Motion

- Given two arbitrary cameras with rotation and different calibration matrices



- The **F** matrix between the corrected first and the second image is $\hat{\mathbf{F}} = [\mathbf{e}']_\times$, satisfying $\mathbf{x}'^T\hat{\mathbf{F}}\hat{\mathbf{x}} = 0$, where $\hat{\mathbf{x}} = \mathbf{H}\mathbf{x}$

- Thus, $\mathbf{x}'^T [\mathbf{e}']_\times \mathbf{H}\mathbf{x} = 0 \quad \Rightarrow \mathbf{F} = [\mathbf{e}']_\times \mathbf{H}$

General Motion

- Example

  ✓ Consider two cameras $\mathbf{P} = \mathbf{K}[\mathbf{I} \mid 0]$, $\mathbf{P}' = \mathbf{K}'[\mathbf{R} \mid \mathbf{t}]$

$$\mathbf{F} = [\mathbf{e}']_\times \mathbf{H}_\infty \qquad \mathbf{H}_\infty = \mathbf{K}'\mathbf{R}\mathbf{K}^{-1}$$

infinity homography

  ✓ If x is normalized as $\mathbf{x}=(x,y,1)^T$ then

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \mathbf{K}[\mathbf{I} \mid \mathbf{0}]\mathbf{X} \Rightarrow (X,Y,Z)^T = Z\mathbf{K}^{-1}\mathbf{x}$$

$$\mathbf{x}' = \mathbf{P}'\mathbf{X} = \mathbf{K}'[\mathbf{R} \mid \mathbf{t}]\mathbf{X} \Rightarrow \mathbf{x}' = \mathbf{K}'\mathbf{R}\mathbf{K}^{-1}\mathbf{x} + \mathbf{K}'\mathbf{t}/Z$$

Projective transformation and invariance

- Derivation based purely on projective concepts

$$\hat{\mathbf{x}} = \mathbf{Hx},\ \hat{\mathbf{x}}' = \mathbf{H}'\mathbf{x}' \Rightarrow \hat{\mathbf{F}} = \mathbf{H}'^{-T}\mathbf{F}\mathbf{H}^{-1}$$

- **F** is invariant to transformations of projective 3-space

$$\mathbf{x} = \mathbf{PX} = (\mathbf{PH})(\mathbf{H}^{-1}\mathbf{X}) = \hat{\mathbf{P}}\hat{\mathbf{X}}$$
$$\mathbf{x}' = \mathbf{P}'\mathbf{X} = (\mathbf{P}'\mathbf{H})(\mathbf{H}^{-1}\mathbf{X}) = \hat{\mathbf{P}}'\hat{\mathbf{X}}$$

$(\mathbf{P},\mathbf{P}') \mapsto \mathbf{F}$ unique

$\mathbf{F} \mapsto (\mathbf{P},\mathbf{P}')$ not unique

$\mathbf{F} = [\mathbf{e}']_\times \mathbf{P}'\mathbf{P}^+$

canonical form

$$\mathbf{P} = [\mathbf{I} \mid \mathbf{0}] \qquad \mathbf{P}' = [\mathbf{M} \mid \mathbf{m}] \qquad \mathbf{F} = [\mathbf{m}]_\times \mathbf{M}$$

Projective ambiguity of cameras given F

- Show that if **F** is same for (**P**,**P'**) and $(\tilde{\mathbf{P}},\tilde{\mathbf{P}}')$, there exists a projective transformation **H** so that $\tilde{\mathbf{P}} = \mathbf{PH}$ and $\tilde{\mathbf{P}}' = \mathbf{P}'\mathbf{H}$

$$\mathbf{P} = [\mathbf{I}|\mathbf{0}]\ \ \mathbf{P}' = [\mathbf{A} \mid \mathbf{a}],\ \ \tilde{\mathbf{P}} = [\mathbf{I} \mid \mathbf{0}]\ \ \tilde{\mathbf{P}}' = [\tilde{\mathbf{A}} \mid \tilde{\mathbf{a}}]$$

$$\mathbf{F} = [\mathbf{a}]_\times \mathbf{A} = [\tilde{\mathbf{a}}]_\times \tilde{\mathbf{A}}$$

<u>lemma:</u> $\tilde{\mathbf{a}} = \mathrm{k}\mathbf{a},\ \ \tilde{\mathbf{A}} = k^{-1}(\mathbf{A} + \mathbf{a}\mathbf{v}^{\mathrm{T}})$

$$\mathbf{aF} = \mathbf{a}[\mathbf{a}]_\times \mathbf{A} = 0 = \tilde{\mathbf{a}}\mathbf{F} \overset{\text{rank 2}}{\Longrightarrow} \tilde{\mathbf{a}} = \mathrm{k}\mathbf{a}$$

$$[\mathbf{a}]_\times \mathbf{A} = [\tilde{\mathbf{a}}]_\times \tilde{\mathbf{A}} \Rightarrow [\mathbf{a}]_\times (\mathrm{k}\tilde{\mathbf{A}} - \mathbf{A}) = 0 \Rightarrow (\mathrm{k}\tilde{\mathbf{A}} - \mathbf{A}) = \mathbf{a}\mathbf{v}^{\mathrm{T}}$$

$$\mathbf{H} = \begin{bmatrix} k^{-1}\mathbf{I} & \mathbf{0} \\ k^{-1}\mathbf{v}^{\mathrm{T}} & k \end{bmatrix} \quad \mathbf{P}'\mathbf{H} = [\mathbf{A} \mid \mathbf{a}]\begin{bmatrix} k^{-1}\mathbf{I} & \mathbf{0} \\ k^{-1}\mathbf{v}^{\mathrm{T}} & k \end{bmatrix} = [k^{-1}(\mathbf{A} - \mathbf{a}\mathbf{v}^{\mathrm{T}}) \mid k\mathbf{a}] = \tilde{\mathbf{P}}'$$

(22-15=7, ok)

Canonical Cameras given F

- **F determines the camera pairs up to a 3D projective transformation. What is the specific forms then?**
- **F** matrix corresponds to **P**,**P'** *iff* **P'^{T}FP** is skew-symmetric

$$\left(\mathbf{X}^T\mathbf{P'}^T\,\mathbf{FPX} = \mathbf{x}'^T\mathbf{Fx} = 0, \forall \mathbf{X}\right)$$

- **F** matrix, **S** any skew-symmetric matrix, then

$$\mathbf{P} = [\mathbf{I} \mid \mathbf{0}] \quad \mathbf{P}' = [\mathbf{SF} \mid \mathbf{e}'] \quad \text{(fund.matrix=F)}$$

$$\left([\mathbf{SF} \mid \mathbf{e}']^T \mathbf{F}[\mathbf{I} \mid \mathbf{0}] = \begin{bmatrix} \mathbf{F}^T\mathbf{S}^T\mathbf{F} & \mathbf{0} \\ \mathbf{e}'^T\,\mathbf{F} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{F}^T\mathbf{S}^T\mathbf{F} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}\right)$$

skew symmetric

Possible choice:

$$\mathbf{P} = [\mathbf{I} \mid \mathbf{0}] \quad \mathbf{P}' = [[\mathbf{e}']_\times \mathbf{F} \mid \mathbf{e}']$$

General canonical representation:

$$\mathbf{P} = [\mathbf{I} \mid \mathbf{0}] \quad \mathbf{P}' = [[\mathbf{e}']_\times \mathbf{F} + \mathbf{e}'\mathbf{v}^T \mid \lambda \mathbf{e}']$$

The Essential matrix

- Normalized camera (known **K**)

P with K=I

$$\mathbf{x} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\mathbf{X} \Rightarrow \hat{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x} = [\mathbf{R} \mid \mathbf{t}]\mathbf{X}$$

- Fundamental matrix for calibrated cameras (remove **K**)
- Fundamental matrix with the normalized image coordinates
- The fundamental matrix of a pair of normalized cameras

$$\mathbf{P} = [\mathbf{I} \mid 0] \quad \mathbf{P}' = [\mathbf{R} \mid \mathbf{t}] \quad \Rightarrow \mathbf{E} = [\mathbf{t}]_\times \mathbf{R} = \mathbf{R}[\mathbf{R}^T\mathbf{t}]_\times$$

- And

$$\hat{\mathbf{x}}'^T\,\mathbf{E}\hat{\mathbf{x}} = 0 \Rightarrow \mathbf{x}'^T\mathbf{K}'^{-T}\mathbf{E}\mathbf{K}^{-1}\mathbf{x} = 0 \quad \left(\hat{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x};\ \hat{\mathbf{x}}' = \mathbf{K}^{-1}\mathbf{x}'\right)$$

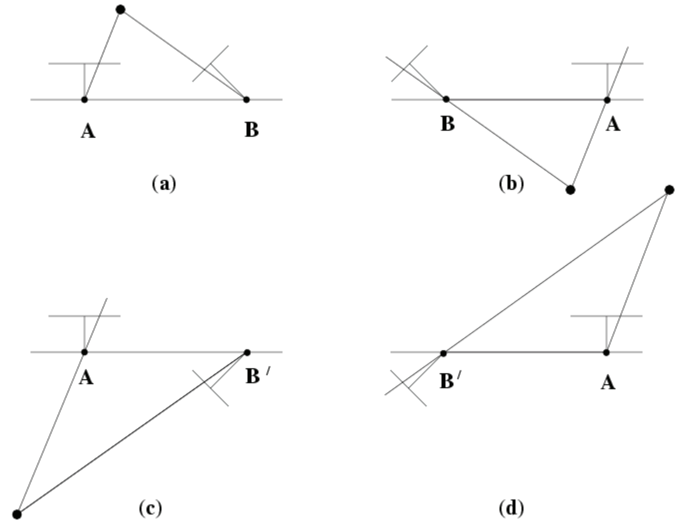
$$\therefore \mathbf{E} = \mathbf{K}'^T\,\mathbf{FK}$$

5 DOF (3 for **R**; 2 for **t** up to scale)

- **E** is essential matrix if and only if two singular values are equal (and third=0)

$$\mathbf{E} = \mathbf{U}\mathrm{diag}(1,1,0)\mathbf{V}^T$$

Four possible reconstructions from E



(only one solution where points is in front of both cameras)