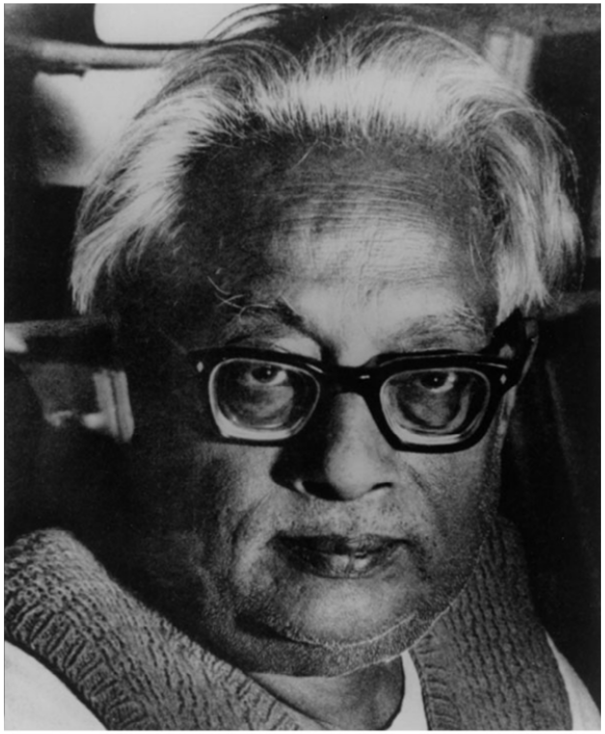
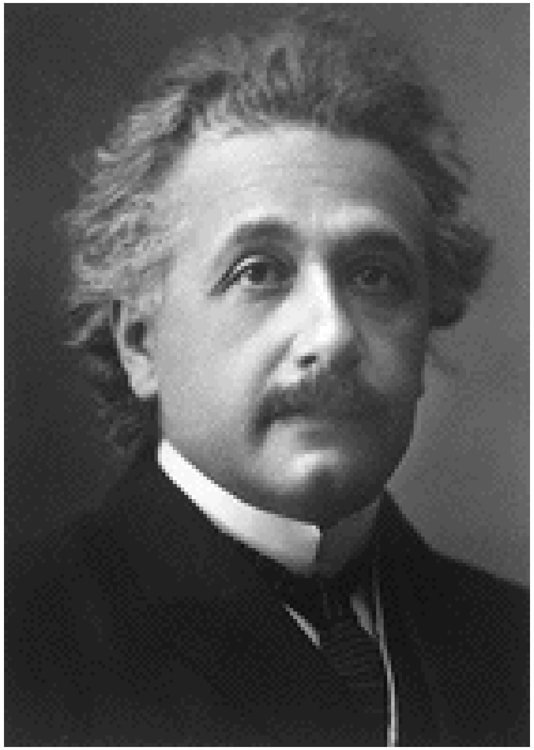
Bose and Einstein



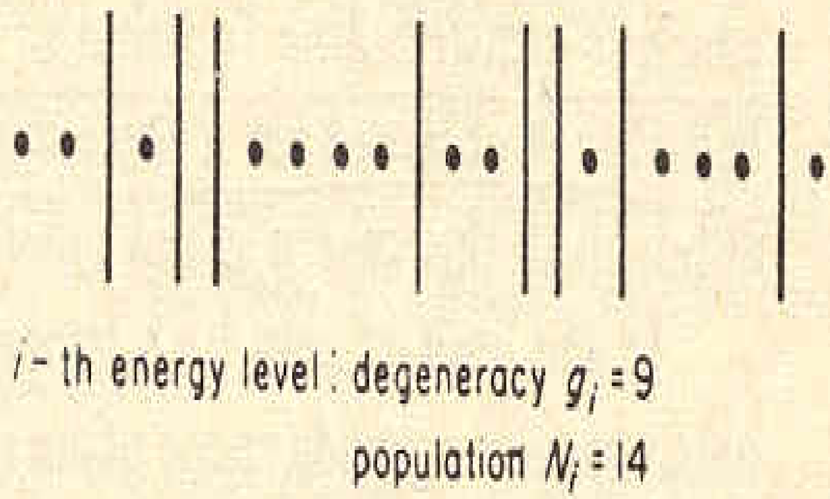
Satyendranath Bose
(1894-1974)



Albert Einstein
(1879-1955)

Bose-Einstein Distribution

FIGURE 5.13. A possible distribution of particles among quantum states in the *i*th energy level of a system to which the Pauli exclusion principle does not apply.



Bose-Einstein Distribution

$$N = N_1 + N_2 + \cdots + N_n$$

$$U = E_1 N_1 + E_2 N_2 + \cdots + E_n N_n$$

$$Q(N_1, N_2, \cdots, N_n) = \prod_{i=1}^{n} \frac{(N_i + g_i - 1)!}{N_i!(g_i - 1)!}$$

Bose-Einstein Distribution

Lagrange's multiplier method

$$\frac{\partial \ln Q}{\partial N_i} + \alpha \frac{\partial f}{\partial N_i} + \beta \frac{\partial h}{\partial N_i} = 0$$

Stirling's approximation

$$\ln\left(n!\right) \approx n \ln n - n \quad \text{for } n >> 1$$

$$\ln\left(N_i + g_i - 1\right) - \ln N_i + \alpha + \beta E_i = 0$$

$$f_{BE}\left(E_i\right) = \frac{N_i}{g_i} = \frac{1}{e^{-\alpha} e^{-\beta E_i} - 1} \qquad \beta = -\frac{1}{k_B T}$$

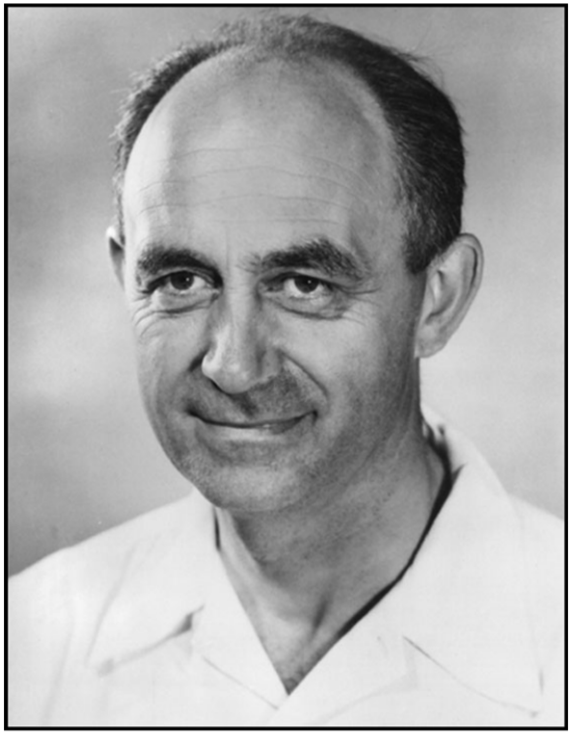
Bose-Einstein Distribution

$$f_{BE}(E) = \frac{1}{Be^{E/k_B T} - 1} \qquad \left(\frac{N}{V}\right)_{bosons} = \int_0^\infty g(E) f_{BE}(E)\, dE$$

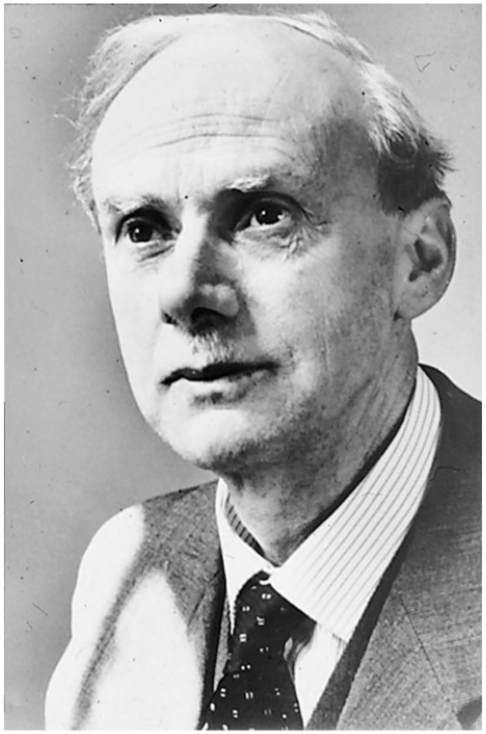
If there is no constraint of the conservation of partile number, then,

$$f_{BE}(E) = \frac{1}{e^{E/k_B T} - 1} \qquad \text{(for photons or phonons)}$$

Fermi and Dirac



Enrico Fermi
(1901-1954)



Paul Adrien Maurice Dirac
(1902-1984)

Fermi-Dirac Distribution

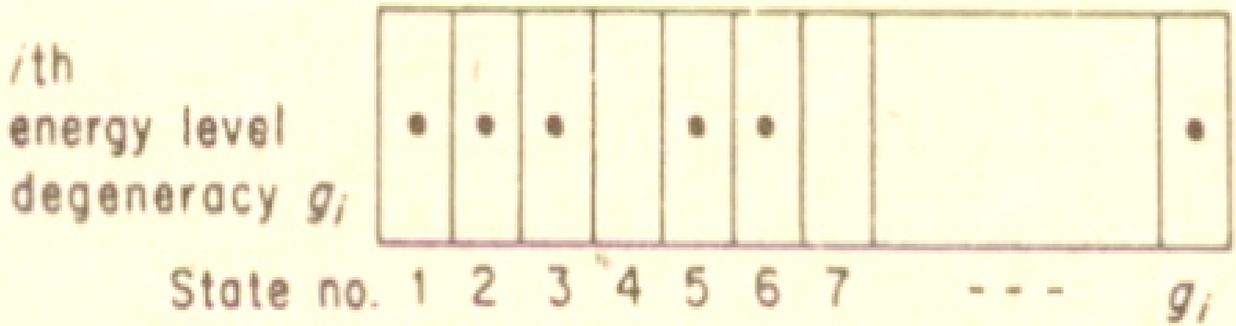


FIGURE 5.8. A possible distribution of particles among quantum states in the ith energy level of a system wherein the Pauli exclusion principle is applicable.

Fermi-Dirac Distribution

$$N = N_1 + N_2 + \cdots + N_n$$

$$U = E_1 N_1 + E_2 N_2 + \cdots + E_n N_n$$

$$Q(N_1, N_2, \cdots, N_n) = \prod_{i=1}^{n} \frac{g_i\,!}{N_i\,!\left(g_i - N_i\right)!}$$

Fermi-Dirac Distribution

$$-\ln N_i + \ln\left(g_i - N_i\right) + \alpha + \beta E_i = 0$$

$$f_{FD}\left(E_i\right) = \frac{N_i}{g_i} = \frac{1}{e^{-\alpha} e^{-\beta E_i} + 1} \qquad \beta = -\frac{1}{k_B T}$$

$$f_{FD}\left(E\right) = \frac{1}{e^{\left(E - E_F\right)/k_B T} + 1}$$

E_F : Fermi energy, Fermi level, chemical potential

$$\left(\frac{N}{V}\right)_{fermions} = \int_0^\infty g(E) f_{FD}(E) dE$$

Fermi-Dirac Distribution

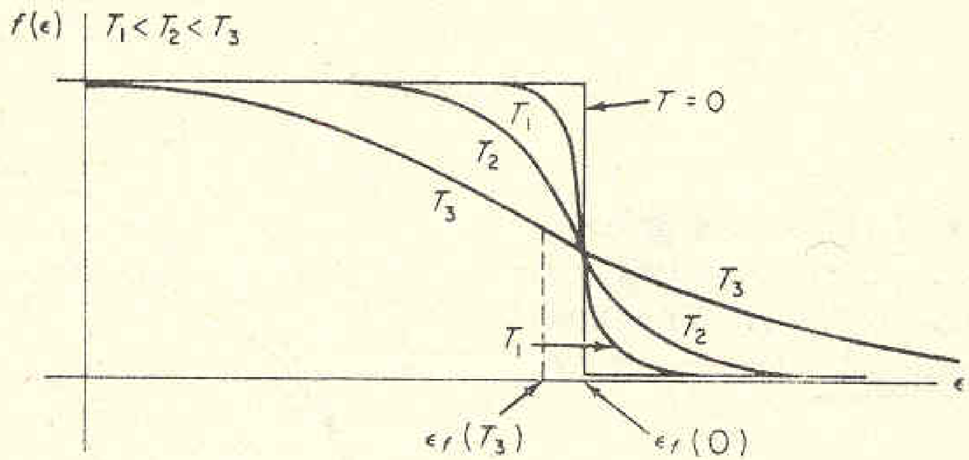
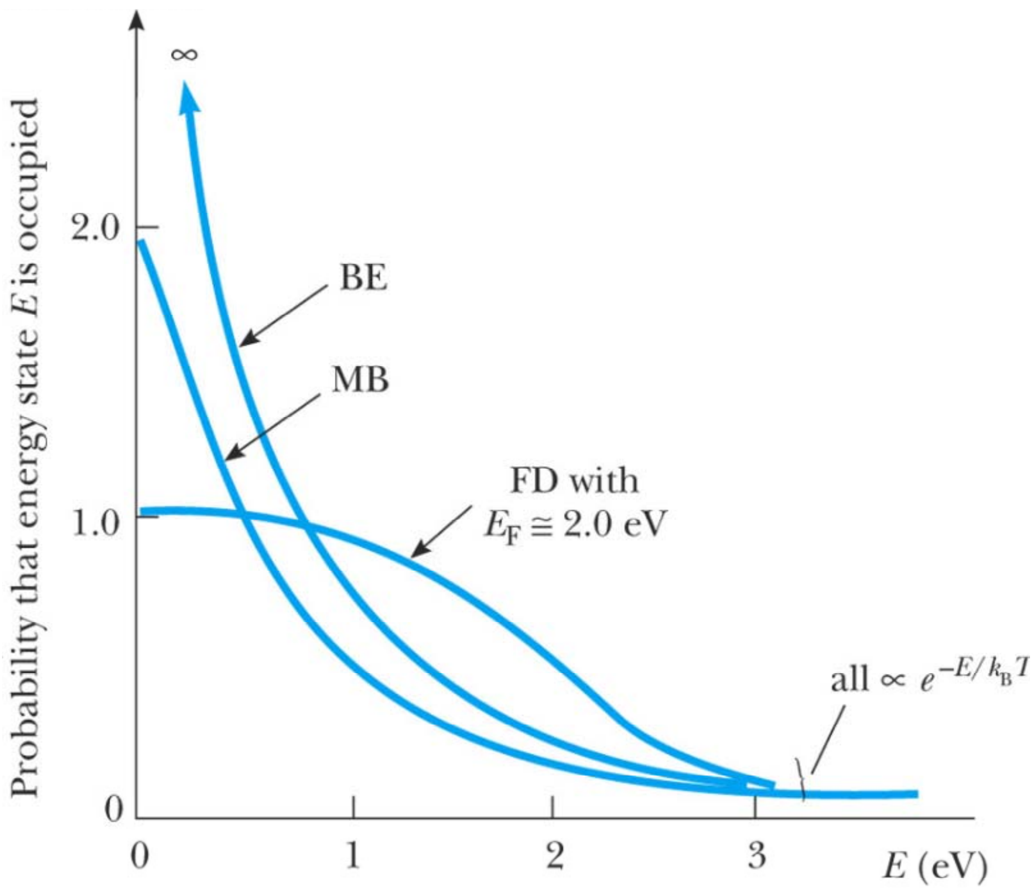


FIGURE 5.10. Schematic representation of the Fermi distribution function for four different temperatures. Note the variation of the Fermi energy with temperature. The temperature dependence of the Fermi energy depicted here is typical of a three-dimensional free-electron gas, but the actual variation in any particular system will depend critically upon the density of states function (or level degeneracies) for that system.

Comparison



Probability that energy state E is occupied
∞
2.0
1.0
0
BE
MB
FD with
$E_F \cong 2.0$ eV
all $\propto e^{-E/k_B T}$
0
1
2
3
E (eV)

Comparison and Bose-Einstein Condensation

Distinguishable particles

$$\psi_I = \psi_a(1)\psi_b(2)$$

$$\psi_{II} = \psi_a(2)\psi_b(1)$$

Bosons

$$\psi_B = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1)\right]$$

Fermions

$$\psi_F = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right]$$

For $b = a$,

Distinguishable particles

$$\psi_M = \psi_a(1)\psi_a(2)$$

Bosons

$$\psi_B = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_a(2) + \psi_a(2)\psi_a(1)\right] = \sqrt{2}\psi_M$$

Fermions

$$\psi_F = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_a(2) - \psi_a(2)\psi_a(1)\right] = 0$$

Comparison

	막스웰-볼츠만 (Maxwell-Boltzmann)	보즈-아인슈타인 (Bose-Einstein)	페르미-디락 (Fermi-Dirac)
적용되는 계	동일, 구별 가능 입자	동일, 구별 불가능 입자이며 배타 원리를 따르지 않는다.	동일, 구별 불가능 입자이며 배타 원리를 따른다.
입자의 범주	고전적	보존	페르미온
입자의 성질	임의의 스핀, 입자들이 충분히 떨어져 있어서 파동 방정식의 중첩이 없음.	스핀 $0, 1, 2, \ldots$; 파동 방정식은 입자의 교환에 대해 대칭임.	스핀 $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$; wave 파동 방정식은 입자의 교환에 대해 비대칭임.
예	기체 분자	공동내의 광자; 고체내의 포논; 저온에서의 액체 헬륨	금속내의 자유전자; 원자가 붕괴하는 별 (백색 왜성)에서의 전자
분포함수 (온도 T일 때 에너지 ε 상태를 점유할 수 있는 입자의 개수)	$f_{\mathrm{MB}}(\varepsilon) = Ae^{-\varepsilon/kT}$	$f_{\mathrm{BE}}(\varepsilon) = \dfrac{1}{e^{\alpha}e^{\varepsilon/kT} - 1}$	$f_{\mathrm{FD}}(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\varepsilon_{\mathrm{F}})/kT} + 1}$
분포의 성질	상태 당 점유할 수 있는 입자수가 무제한	상태 당 점유할 수 있는 입자수가 무제한; 낮은 에너지에서는 f_{MB}보다 많은 입자; 높은 에너지에서는 f_{MB}에 접근	상태상 단 1개 이상의 입자를 점유 할 수 없음; 낮은 에너지에서는 f_{MB}보다 적은 입자; 높은 에너지에서는 f_{MB}에 접근

Cavity Modes of Light

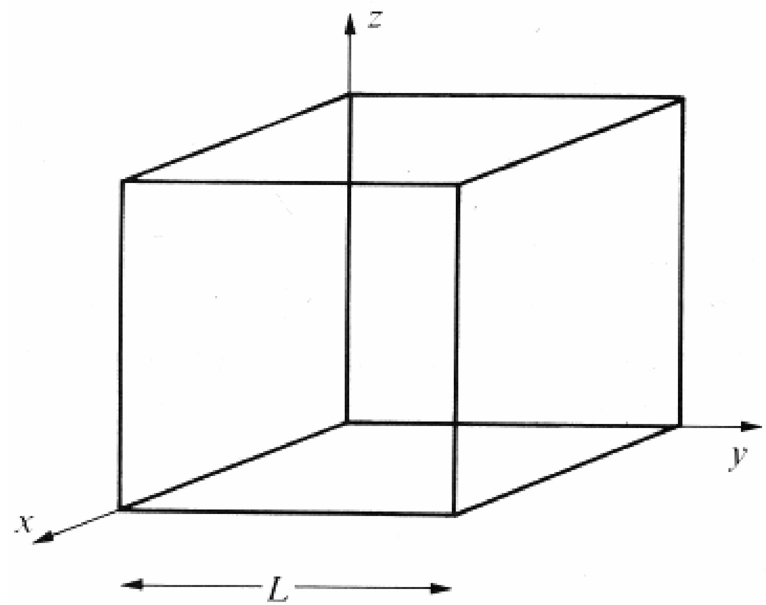
$$\nabla^2 \mathbf{E}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0 \qquad k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} = \frac{\omega}{c}$$

$$E_x(\mathbf{r},t) = E_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z)$$

$$E_y(\mathbf{r},t) = E_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z)$$

$$E_z(\mathbf{r},t) = E_z(t)\sin(k_x x)\sin(k_y y)\cos(k_z z)$$

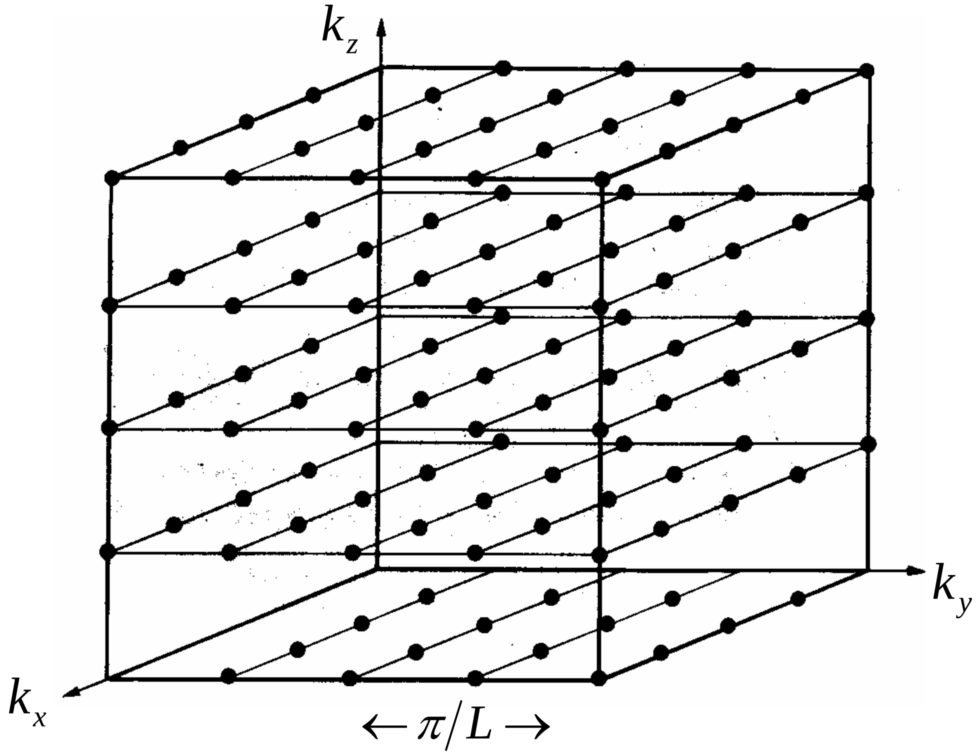


Cavity Modes of Light

$$k_x = \pi n_x / L, \quad k_y = \pi n_y / L, \quad k_z = \pi n_z / L$$

$$n_x, n_y, n_z = 0, 1, 2, 3, \ldots$$

$$\mathbf{k} \cdot \mathbf{E}(t) = 0$$



Mode Density

$$\frac{\frac{1}{8}\left(4\pi k^2 dk\right)}{\left(\pi / L\right)^3}\times 2$$

$$G(k)\,dk = k^2\,dk/\pi^2$$

$$\nu = \frac{ck}{2\pi}$$

$$G(\nu)\,d\nu = G(k)\,dk$$

$$G(\nu) = G(k)\frac{dk}{d\nu} = \frac{k^2}{\pi^2}\frac{2\pi}{c} = \frac{8\pi\nu^2}{c^3}$$

Rayleigh-Jeans Formula vs. Planck Radiation Law

Rayleigh-Jeans Formula

$$u(\nu)\,d\nu = \bar{E}G(\nu)\,d\nu = k_B T G(\nu)\,d\nu = \frac{8\pi\nu^2 k_B T d\nu}{c^3}$$

Planck's Law with Bose-Einstein

$$\bar{E} = h\nu \times \frac{1}{e^{h\nu/k_B T} - 1} = \frac{h\nu}{e^{h\nu/k_B T} - 1}$$

$$u(\nu)\,d\nu = \bar{E}G(\nu)\,d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/k_B T} - 1}$$