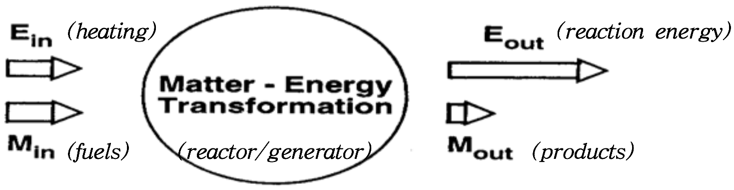
Chapter 1. Fundamentals of Fusion

Reading assignments: **Harms Chaps. 1, 2, 3, 7 Stacey Chap. 1,**

1. Origin of fusion energy

A. Fusion energy production

1) Mass-energy transformation



(e.g.) Hydro-electric process
Chemical reactions (combustion)
Fission process
Fusion process

Total energy conservation including rest mass energy :

$$E_{in} + M_{in} \rightarrow E_{out} + M_{out} \quad (1)$$

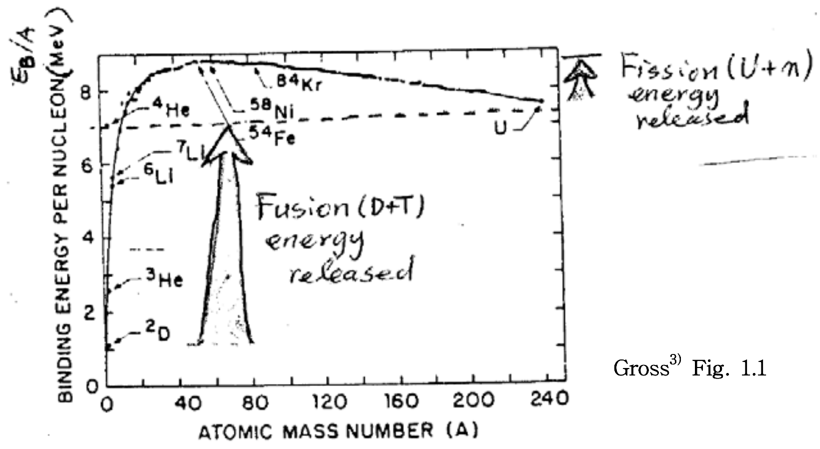
If $\Delta m = M_{out} - M_{in} < 0$, then we can get $E_{out} > E_{in}$.

2) Binding energy for an assembled nucleus

$$Zp + (A-Z)n \rightarrow {}^{A}_{Z}X + BE$$

where $BE \equiv [(Z\, m_p + (A-Z)\, m_n) - m_X]\, c^2 = -\Delta m\, c^2 \quad (2)$

$\Delta m < 0 \rightarrow$ Released energy (Exorthermic or Exoergic)



Gross[3] Fig. 1.1

3) Mass defect energy (or Q-value) of nuclear reaction

reactants products

a + b → d + e

$\Delta m = (m_d + m_e) - (m_a + m_b)$, $\Delta m < 0$: exothermic or exoergic

$\Delta m > 0$: endothermic or endoergic

$Q_{ab} = (-\Delta m)_{ab} c^2$ by Einstein's mass-energy relation (3)

Realization of energy production ($\Delta m < 0$) by nuclear ractions

$1 \leq A \lesssim 60$: fusion reaction

two light nuclei → new nuclei or n + Q

$60 \leq A$: fission reaction

heavy nucleus + n → two lighter nuclei + n_s + Q

For $E_a + E_b \ll Q_{ab}$

$$Q_{ab} \approx E_d + E_e = \frac{1}{2} m_d v_d^2 + \frac{1}{2} m_e v_e^2 \qquad (4)$$

By momentum conservation for reactions with center of mass at rest,

$$m_d v_d = m_e v_e$$

$$E_d = \left(\frac{m_e}{m_d + m_e} \right) Q_{ab}, \quad E_e = \left(\frac{m_d}{m_d + m_e} \right) Q_{ab} \qquad (5)$$

B. Interesting fusion fuels

1) 1st generation fuels

D-T : D + T → ^{4}He (3.5 *MeV*) + n (14.1 *MeV*)

Highest σ

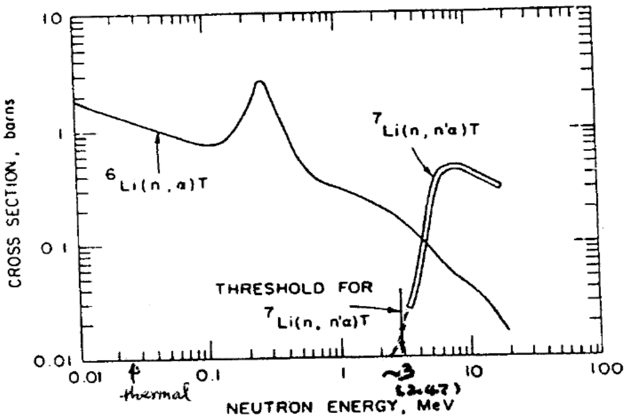
T-breeding is needed

Radioactivity of T (β emitter, 12.3 y), n activation and damage

T-breeding :

^{6}Li + n(thermal) → T + ^{4}He + 4.8 : *7.42% of natural Li*

^{7}Li + n(fast) → T + ^{4}He + n′ − 2.5 : *92.58% of natural Li*



Gross[3] Fig. 2.1 Reaction cross sections versus neutron energy for the ^{6}Li(n, α)T and ^{7}Li(n, n', α)T reactions used to manufacture tritium. One barn = 10^{-28} m^2.

2) Advanced fuels

(D-D)p : D + D → T (1) + p (3) : *Cheap, Limitless,*

(D-D)n : D + D → ^{3}He (0.81) + n (2.44) *Wide distribution*

D-^{3}He : D + ^{3}He → ^{4}He (3.6) + p (14.6) : *No n activation,*

↗ *Direct conversion,*

10^{-8} in regolith

3) Exotic fuels

p-^{6}Li : p + ^{6}Li → ^{4}He + ^{3}He + 4 : *Low activation, No n,*

p-^{11}B : p + ^{11}B → 3^{4}He + 8.7 *Direct conversion*

※ Semi-catalyzed D-D : (D-D) + (D-T)

5D → 2n + p + ^{3}He + ^{4}He + 24.9

※ Catalyzed D-D : (D-D) + (D-T) + (D-^{3}He)

6D → 2 (p + n + ^{4}He) + 43.1

No T-breeding, Higher T_{ig} *, Lower* P_{th} *than D-T*

(cf) Natural fusion in terrestrial conditions (Nucleosynthesis)

$p + p \rightarrow D + \beta^+ + \nu + 1.2\,MeV$

$p + D \rightarrow {}^3He + 5.5$

$.{}^3He + {}^3He \rightarrow {}^4He + 2p + 12.9$

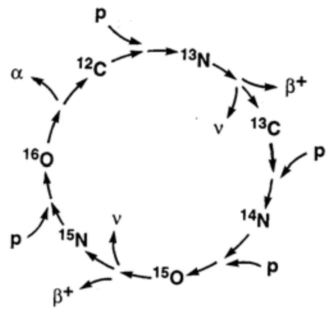
$.{}^3He + {}^4He \rightarrow {}^7Be + 1.6$

$.{}^7Be + \beta^- \rightarrow {}^7Li + 0.06$

. . . .

Carbon cycle : $4p \rightarrow {}^4He + 2\beta^+ + 2\nu + 25.1\,MeV$

(closed fusion cycle)



Harms Fig. 1.3

C. *Promise of fusion*

1) Energy demand

Ancient - 1750 : 5 Q

1751 - 1850 : 2

1851 - 1950 : 4

1951 - 2000 : 15

* 1 Q = 10^{18} Btu
  ≃ 10^{21} J = 31.7 TW/yr
  ≃ 4x10^{10} t coal

* 2500 Cal/day ≃ 10^7 J/day
  ≃ 120 W to sustain life

* Energy consumption in 1975 ≃ 0.25 Q ≃ 8 TW
  1994 ≃ 11.7 TW (fossile fuels : 87 %)
  2050 ≃ 50 TW

2) Energy resources

Recoverable fossile fuels :

Coal	53 Q
Oil	12
Natural gas	13
Oil shale, Tar sand	3

Total 81 Q

Renewable resources --- Limited

Solar, Wind, Tide, Wave, Hydroelectric, Geothermal, Organic waste, Biomass

Fission fuels :

^{235}U 10 Q, ^{238}U, ^{232}Th 10^3 Q

Fusion fuels :

Li 10^3 Q (land deposit)

10^7 Q (sea water - 0.17 ppm of T)

D 10^{10} Q

* Inexhaustible large-energy options :

1) Solar energy 2) Fission breeders 3) Fusion

3) Advantages of fusion energy

- Essentially limitless, cheap, widely-distributed fuels
  D : 1/6500 (≃0.0153 %) of sea water (10^{14} tons)
  1 liter water ≃ 300 liter gasoline
- Lack of radioactivity and environmental problems
- Reduced danger of diversion of weapon-like materials
  (None of materials subject to the provisions of NPT)
- Inherent safety against destructive runaway (nuclear excursion)
- Multiple end uses
  (Direct energy conversion, Fusion-fission hybrid, RI product, Fuel synthesis, neutron source, waste treatment, etc.)

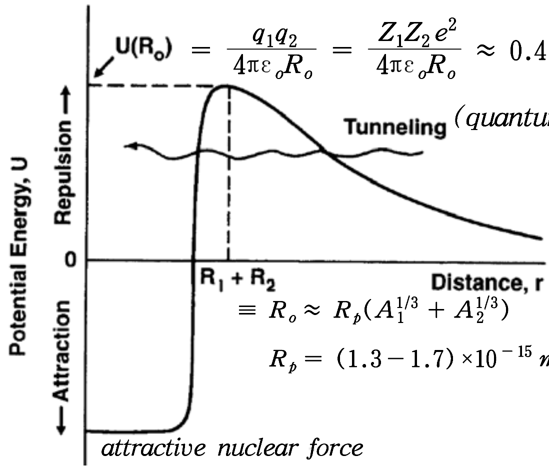
∗ Drawbacks :
Large unit size for ~ *GWe* → Large investment

2. Physical characterization of fusion reaction

A. Fusion reactions

1) Requirement for a fusion event between two colliding nuclei

$U(R_o) = \frac{q_1 q_2}{4\pi\varepsilon_o R_o} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o R_o} \approx 0.4\, Z_1 Z_2\, MeV$: *repulsive Coulomb force*



(*quantum mechanical effect*)

Harms Fig. 2.1

$\equiv R_o \approx R_p(A_1^{1/3} + A_2^{1/3})$

$R_p = (1.3 - 1.7) \times 10^{-15}\, m$ (*proton radius*)

attractive nuclear force

Kinetic energy of nuclei in center of mass system for a fusion event

$E_{cm} > U(R_o) \approx 0.4\ Z_1 Z_2$ [MeV] (6)

where

$E_{cm} = \frac{1}{2} m_r v^2$

$m_r = \frac{m_1 m_2}{m_1 + m_2}$: *reduced mass* (7)

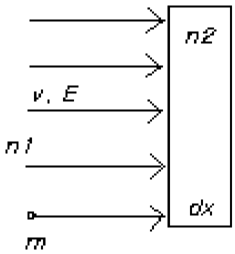
$v = |\mathbf{v}_1 - \mathbf{v}_2|$: *relative velocity*

※ Even where $E_{cm} < U(R_o)$, there are some fusion events due to barrier penetration by quantum mechanical tunneling effect.

$$\Pr obability(tunnneling) \propto \frac{1}{v} \exp\left(-\gamma \frac{q_1 q_2}{v}\right)$$

2) Fusion reaction cross sections

Beam-target collisions (Binary interactions)



For fixed target

$m = m_1, \quad v = v_1, \quad E = m_1 v_1^2/2$

For moving target

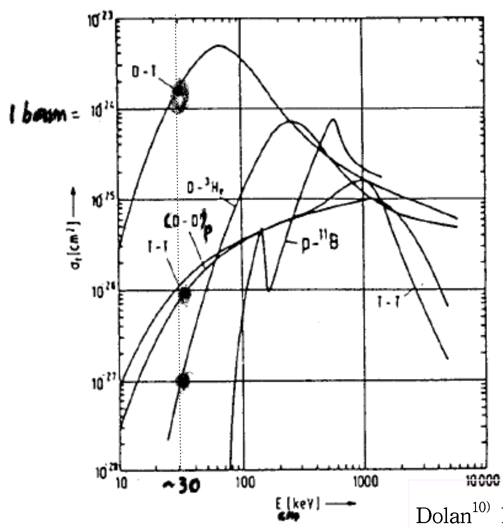
$m = m_r, \quad v = |\mathbf{v}_1 - \mathbf{v}_2|, \quad E = E_{cm}$

$$dn_1 = -\sigma_{12}(E)\, n_1 n_2\, dx \tag{8}$$

Fusion cross section for low energy $E_{cm} < U(R_o)$ by quantum mechanical tunneling process :

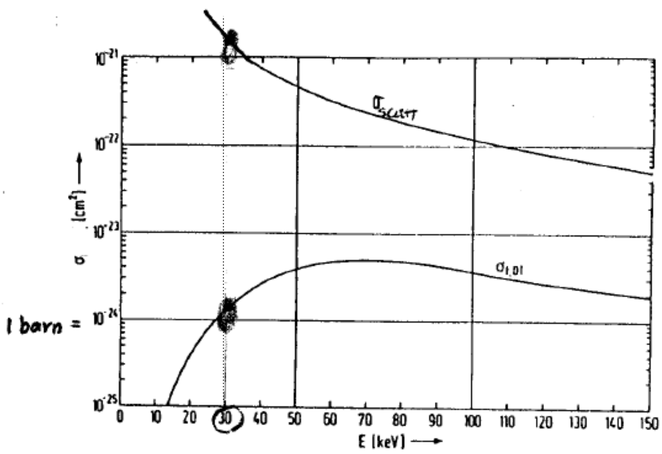
$$\sigma_{12}(E) = \frac{A}{E} e^{(-B/\sqrt{E})} \quad : \quad \textit{Gamow theory}\ (1938) \tag{9}$$

where A = const., $B = 2^{-1/2} \pi\, m_r^{1/2} Z_1 Z_2\, e^2/h\, \varepsilon_o$ *(cf) Gross TABLE 3.2*



Dolan[10] Fig. 2C1

Cross-section σ_f for various fusion reactions as a function of kinetic energy $E_{cm} = (m_r v^2)/2$ of the relative motion of the colliding nuclei.



Comparison of fusion cross-section $\sigma_{f,DT}$ of DT reaction with scattering cross-section σ_{scatt} for Coulomb collisions

$\sigma_{a-a} \simeq 10^{-16}$ cm^2

$\sigma_{e-a} \simeq 10^{-15}$

$\sigma_{exc} \simeq 10^{-16}$ (e-H)

$\sigma_{ion} \simeq 10^{-16}$ (e-H)

$\sigma_{cx} \simeq 10^{-15}$ (H^+-H)

Notes)

i) For $E \lesssim 100$ *keV*,

$\sigma_{D-T} > \sigma_{D-D} > \sigma_{D-{}^3He}$

ii) $\sigma_{fusion} \approx O(1\ barn)$, $\sigma_{scatt} \approx O(10^3\ barn)$ at $E \approx$ few 10 keV

iii) $\sigma_{scatt} \gg 100\ \sigma_{DT}$

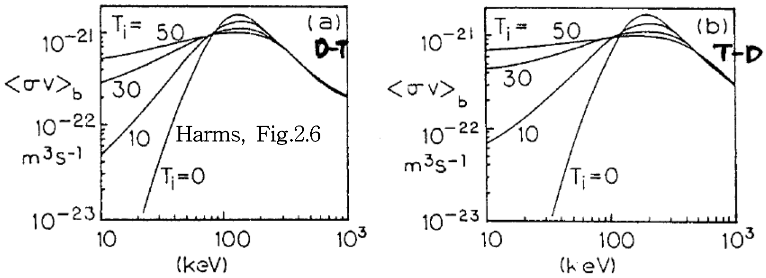
⇒ loss energy ≫ fusion energy

⇒ Fusion by beam-target collisions are not proper for practical energy-producing fusion reactors

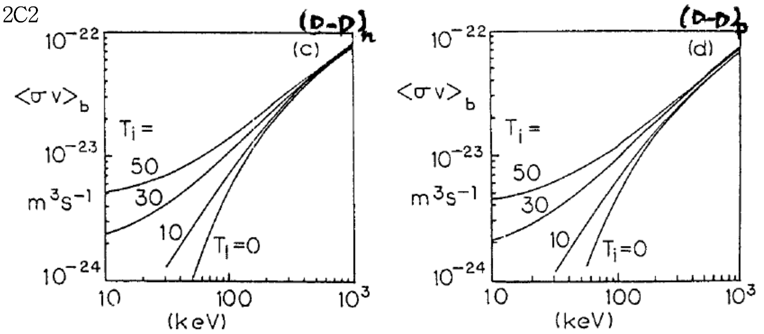
3) Fusion reaction rate parameter (or reactivity or $\sigma - v$ parameter)

$$\langle \sigma v \rangle_{12} \equiv \int_{v1} \int_{v2} f_1(\mathbf{v}_1)\, f_2(\mathbf{v}_2)\, \sigma_{12}(v) v \; d^3v_1 d^3v_2 \qquad (10)$$

For beam-plasma $\langle \sigma v \rangle_b$ as functions of beam energy in various plasma ion temperatures,



Dolan[10] Fig. 2C2

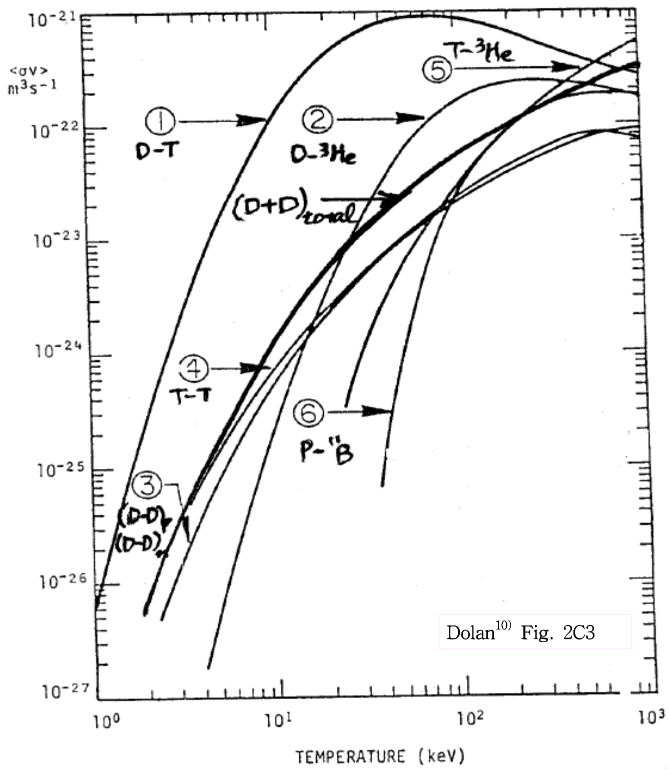


From calculations based on the theory (9) in (10) under two Maxwellian distributions of species 1 and 2 with T < 25 *keV*,

$$\langle \sigma v \rangle_{DT} \approx 3.7\times10^{-18}(kT)^{-2/3}\exp(-20/(kT)^{1/3}) \quad \mathrm{m^3/s} \qquad (10)_{DT}$$

$$\langle \sigma v \rangle_{DD} \approx 2.3\times10^{-20}(kT)^{-2/3}\exp(-18.8/(kT)^{1/3}) \qquad (10)_{DD}$$

in *keV*



(*cf*) Harms, Fig. 7.5

4) Fusion reaction rate (density)

Differential reaction rate per unit volume by Eq. (8) :

$$dR_{12} \equiv \frac{d}{dt}(-dn) = dn_1 dn_2 \sigma_{12}(v)\, v \qquad (11)$$

where $dn_1 = n_1 f_1(\boldsymbol{v}_1)\, d^3 v_1, \quad dn_2 = n_2 f_2(\boldsymbol{v}_2)\, d^3 v_2$

$f_1,\ f_2$ = normalized distribution functions

Total reaction rate :

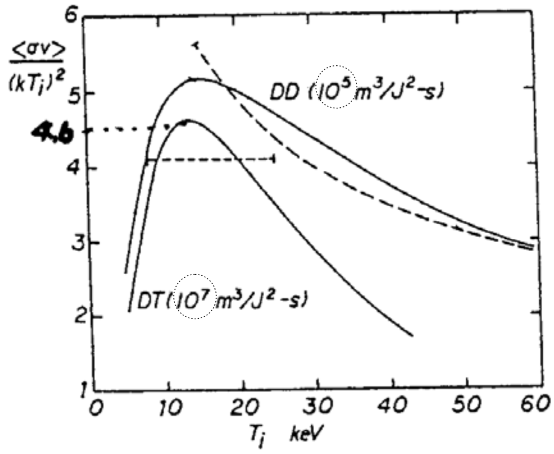
$$R_{12} = n_1 n_2 \langle \sigma v \rangle_{12} \qquad (12)$$

where the fusion reaction rate parameter is

$$\langle \sigma v \rangle_{12} \equiv \int_{v_1} \int_{v_2} \sigma_{12}(v)\, v\, f_1(\boldsymbol{v}_1) f_2(\boldsymbol{v}_2)\, d^3 v_1 d^3 v_2$$

Notes) i) $R_{12} = n_1 n_2 \langle \sigma v \rangle_{12}$, $p = nkT$

$\Rightarrow R_{12} \propto \langle \sigma v \rangle_{12} / (kT)^2$ at a given p



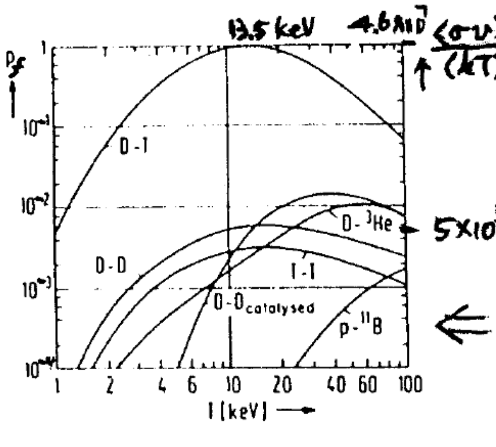
Approximation formula :

$\langle \sigma v \rangle_{DT} / (kT)^2 \approx 4.1\times10^7$ m³/J²s for $8 \lesssim kT \lesssim 25$ *keV*

$\langle \sigma v \rangle_{DD} / (kT)^2 \approx 4.0\times10^5\,(30\ keV/T)^{1/2}$ for $15 \lesssim kT \lesssim 70$ *keV*

Dolan[10] Fig. 2D1

ii) Fusion power density : $P_f = R_{12} Q_{12} \propto \dfrac{\langle \sigma v \rangle}{(kT)^2}$



<σv>/(kT)² (m³/J²·s)

Fusion power densities as a function of temperature. Values are normalized to the maximum obtained for DT reaction at T = 13.5 keV.

iii) For a given pressure

$(R_{12})_{max}$ occurs at kT ≈ 13 *keV* for DT

15 for DD

60 for D³He

iv) At given T

$R_{DT} \approx 100\ R_{DD}$

v) At given T and p (constant plasma ion density n)

$$R_{DT} = \frac{n^2}{4}\langle \sigma v \rangle_{DT} \quad \text{for 50\%-50\% DT}$$

$$R_{DD} = \frac{n^2}{2}\langle \sigma v \rangle_{DD} \quad \text{for DD}$$

⇒ $R_{DT} \approx 50\ R_{DD}$

⇒ D-T reaction has less severe conditions of T and P than D-D reaction

⇒ **1st generation reactor = D-T reactor !**

5) Fusion power density

= *Rate of energy density released by fusion reactions*

$$P_{12} = R_{12}E_f = n_1 n_2 \langle \sigma v \rangle_{12} Q_{12} \quad W/\mathrm{m}^3 \tag{13}$$

$$P_{DD} = \frac{n_D^2}{2}\langle \sigma v \rangle_{DD} \times 3.65 \quad MeV$$

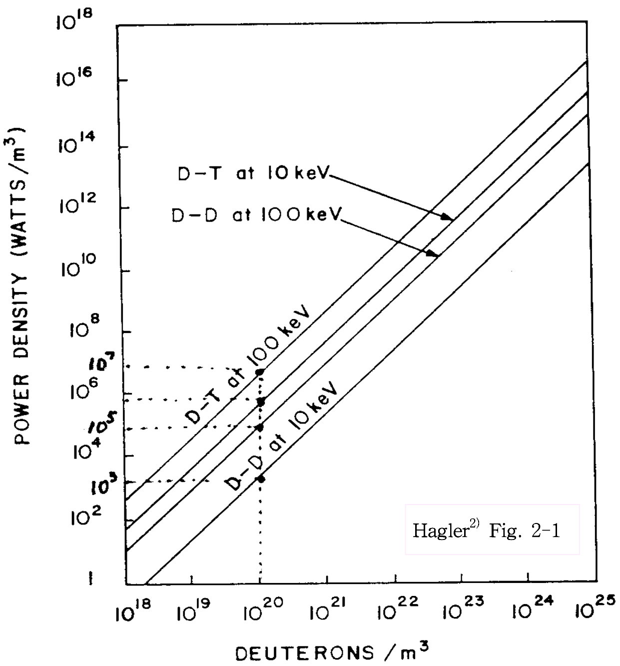
$\approx 3\times10^{-37}\ n_D^2$ W/m^3 for 10 keV

$10^{-35}\ n_D^2$ for 100

$$P_{DT} = \frac{n_D n_T}{4}\langle \sigma v \rangle_{DT} \times 17.6 \quad MeV$$

$\approx 6\times10^{-35}\ n_D n_T$ W/m^3 for 10 keV

$5\times10^{-34}\ n_D n_T$ for 100



Hagler[2] Fig. 2-1

Note) Expected optimum operation density

Design value in fission reactor core : 20 ~ 60 MW/m^3

For a fusion reactor of $L \approx 3$ m & $P_{th} = 10^6\ kW$,

$P_{DT} = P_{th}/L^3 \approx 40\ MW/m^3$

⇒ $\mathbf{n_D \approx 10^{20} \sim 10^{21}\ m^{-3}}$ **for 10 *keV***

B. Coulomb scattering

1) Collisional processes in laboratory and center-of-mass(c.m.) reference frames

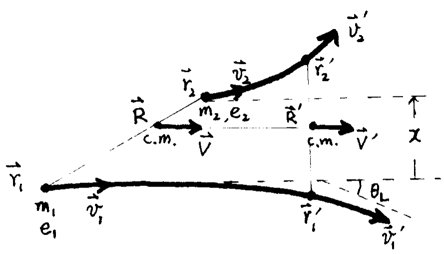
Type of interactions between charged particles in plasmas :

Weak, long-range, collective Coulomb interactions of many particles ($\lambda_D < L$)

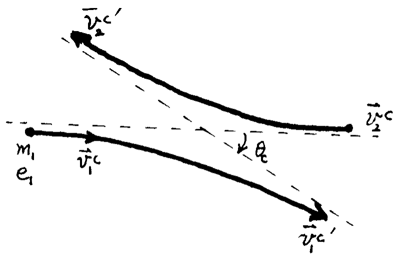
∴ *Accumulation of small-angle scattering (Coulomb collisions)*

⇒ *Large-angle deflection (Collective behavior)*

Consider an elastic Coulomb collision between a test particle (1) and a field particle (2)



Lab. reference frame



c.m. reference frame

Conservation laws :

$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = m_1\boldsymbol{v}_1{}' + m_2\boldsymbol{v}_2{}' \quad : \text{ momentum}$$

$$m_1\boldsymbol{v}_1^2 + m_2\boldsymbol{v}_2^2 = m_1\boldsymbol{v}_1{}'^2 + m_2\boldsymbol{v}_2{}'^2 \quad : \text{ energy}$$

⇒ $\boldsymbol{V} = \boldsymbol{V}'$: *c.m. velocity is unchanged by collisions*

$|\boldsymbol{v}| = |\boldsymbol{v}'|$: *Magnitude of relative velocity is unchanged by collisions, although its direction is changed.*

where

$$\boldsymbol{V} \equiv \frac{d\boldsymbol{R}}{dt} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2} \quad : \text{ c.m. velocity}$$

$$\boldsymbol{v} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2 \quad : \text{ relative velocity}$$

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2 \quad : \text{ relative position}$$

Equations of motions :

In lab. frame,

$$\boldsymbol{F}_{12} \equiv m_1\frac{d^2\boldsymbol{r}_1}{dt^2} = \frac{q_1q_2(\boldsymbol{r}_1 - \boldsymbol{r}_2)}{4\pi\varepsilon_o|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}$$

$$\boldsymbol{F}_{12} \equiv m_2\frac{d^2\boldsymbol{r}_2}{dt^2} = \frac{q_1q_2(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{4\pi\varepsilon_o|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}$$

In c.m. frame,

$$(m_1 + m_2)\frac{d^2R}{dt^2} = 0 \quad : \text{ constant c.m. motion}$$

$$m_r \frac{d^2 \boldsymbol{r}}{dt^2} = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \equiv C\frac{\boldsymbol{r}}{r^3} \quad : \textit{ relative motion} \qquad (14)$$

where $m_r \equiv \dfrac{m_1 m_2}{m_1 + m_2}$ = reduced mass

Note) (14) describes motion of a single fictitious particle of (reduced) mass m_r, (relative) velocity **v** which is changed only direction, but not in magnitude.

Relationship between Θ_L(measured in lab.) & Θ_c(calculated in c.m.) :

$$\cot\Theta_L = \frac{m_1}{m_2}\csc\Theta_c + \cot\Theta_c \qquad (15)$$

Notes) $m_1 << m_2$: $\Theta_L \approx \Theta_c$

$m_1 \approx m_2$: $\Theta_L \approx \Theta_c/2$

$m_1 >> m_2$: $\Theta_L \approx (m_2/m_1)\Theta_c$

Transferred energy from 1 to 2 in a collision :

$$\frac{\triangle E}{E_o} = \frac{4m_1 m_2}{(m_1+m_2)^2}\sin^2\frac{\Theta_c}{2} \qquad (16)$$

where $\triangle E = \frac{1}{2} m_1(v_1^2 - v_1'^2) = E_o(1 - \frac{v_1'^2}{v_1^2})$

Notes) $m_1 << m_2$: $\triangle E << E_o$

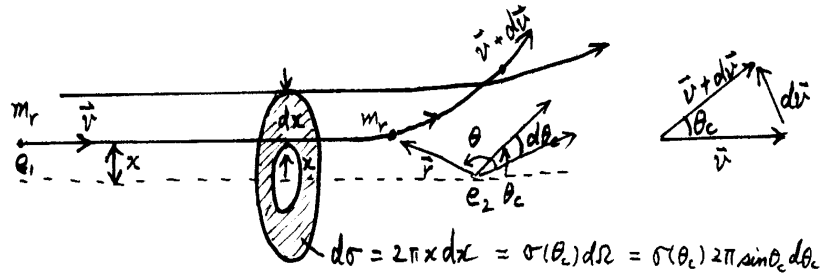
$m_1 \approx m_2$: $\triangle E \approx E_o$

$m_1 >> m_2$: $\triangle E << E_o$

2) Coulomb scattering cross section

(or Rutherford scattering cross section)

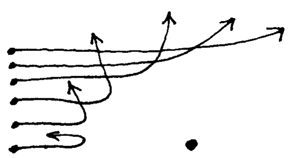
From Eq. (14), binary Coulomb collision can be treated as the scattering of a particle m_r with $\boldsymbol{v}$ by a fixed center of force *(Rutherford-type scattering)*



Solution of Eq. (14) : Harms Fig. 3.2

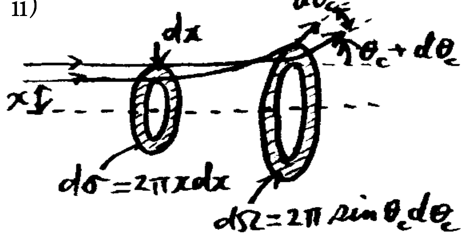
$$\tan\left(\frac{\Theta_c}{2}\right) = \left(\frac{q_1 q_2}{4\pi\varepsilon_o}\right)\frac{1}{m_r v^2 x} = \frac{C}{m_r v^2}\frac{1}{x} \equiv \frac{K}{x} \qquad (17)$$

Notes) i)



$\Theta_c \uparrow$ as $x \downarrow$

ii)



Since particles passing through a geometrical annular area $dA \equiv d\sigma_s = 2\pi x\, dx$ will be scattered into a solid angle $d\Omega = 2\pi \sin\Theta_c d\Theta_c$, we can expect that $d\sigma_s \propto d\Omega$.

From this relation, the ***differential scattering cross section*** for scattering into (Θ_c, $\Theta_c + d\Theta_c$) is written as:

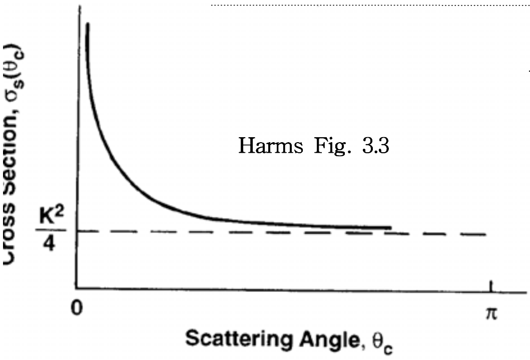
$$d\sigma_s = \frac{d\sigma_s}{d\Omega} d\Omega \equiv \sigma_s'(\Theta_c) d\Omega = \sigma_s'(\Theta_c) 2\pi \sin\Theta_c d\Theta_c = 2\pi x\, dx \quad (18)$$

Then, **Coulomb cross section** ***(Rutherford scattering cross section)*** can be found as

$$\sigma_s'(\Theta_c) \equiv \frac{d\sigma_s}{d\Omega} = \frac{x\,dx}{\sin\Theta_c d\Theta_c}$$

$$\overset{(18)}{=} \frac{1}{4}\left(\frac{C}{m_r v^2}\right)^2 \frac{1}{\sin^4(\Theta_c/2)} = \frac{K^2}{4} \frac{1}{\sin^4(\Theta_c/2)}$$

$$\overset{(17)}{=} \left(\frac{q_1 q_2}{16\pi\varepsilon_o}\right)^2 \frac{1}{\left(\frac{m_r v^2}{2}\right)^2 \sin^4\frac{\Theta_c}{2}} \quad (19)$$



Harms Fig. 3.3

Notes)

For 'head-on' collisions ($\Theta_c \to \pi$, $x \to 0$),
$\sigma'_s(\Theta_c) \to K^2/4$

For 'glancing' collisions($\Theta_c \to 0$, $x \to \infty$),
$\sigma'_s(\Theta_c) \to \infty$: mostly small-angle scattering by long-range Coul. force

Total scattering cross section :

$$\sigma_s = \int_0^\pi \sigma'_s(\Theta_c)\, d\Omega = 2\pi \int_0^\pi \left[\frac{K^2}{4\sin^4(\Theta_c/2)}\right] \sin\Theta_c d\Theta_c$$

$$= \pi K^2 \int_0^\pi \frac{\cos(\Theta_c/2)}{\sin^3(\Theta_c/2)}\, d\Theta_c = 2\pi K^2 \int_{\sin(\Theta_{min}/2)}^{\sin(\pi/2)} \frac{d(\sin\Theta)}{\sin^3\Theta}$$

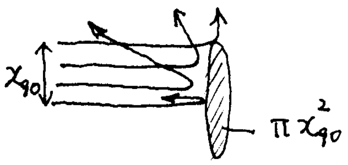
$$= \pi K^2 \left\{\left[\sin\left(\frac{\Theta_{min}}{2}\right)\right]^{-2} - 1\right\} \quad (20)$$

where Θ_{min} follows from the inversion of Eq.(17) for $x = \lambda_D$:

$$\Theta_{min} = 2\tan^{-1}\left(\frac{K}{\lambda_D}\right) \quad (21)$$

3) Effective Coulomb scattering cross section

In a single encounter, a large-angle ($\Theta_c > 90^o$) cross-section is



$$\sigma_s(\Theta_c > 90^o) = \pi x_{90}^2$$

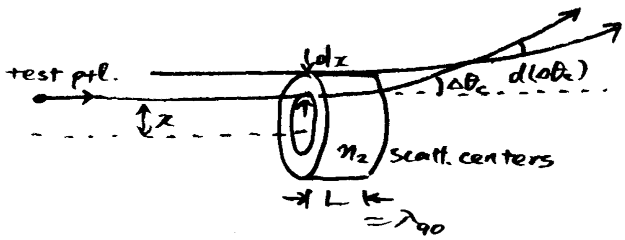
$$= \frac{(q_1 q_2)^2}{\pi(4\varepsilon_o m_r v^2)^2} \approx \frac{1.6\times10^{-24}}{E^2(keV)}\ m^2 \quad (25)$$

(17) for $\Theta_c/2 = 45°$

Effective large-angle deflection = Cumulative multiple small-angle deflections
due to long-range collective interactions
with charged particles in Debye sphere
≠ A single large-angle deflection
due to a close encounter (in λ_D)
by a binary Coulomb collision

Deflection angle for a single small-angle scattering :

(17) $\Rightarrow \quad \Delta\Theta_c \approx \dfrac{2q_1q_2}{4\pi\varepsilon_o m_r v^2 x}$



By calculating a mean square deflection of test particle traveling in plasma with scattering center density n_2

$$\overline{(\Delta\Theta_c)^2} = \int_{\Delta\Theta_{min}}^{\Delta\Theta_{max}} (\Delta\Theta_c)^2 f(\Delta\Theta_c)\, d(\Delta\Theta_c)$$

where $\Delta\Theta_{max} = q_1q_2 / 2\pi\varepsilon_o m_r v^2 x_{min}$

$\Delta\Theta_{min} = q_1q_2 / 2\pi\varepsilon_o m_r v^2 x_{max}$

$f(\Delta\Theta_c)\, d(\Delta\Theta_c) = n_2 L\, 2\pi x\, dx$: # of collisions resulting in $\Delta\Theta_c$

Here, $x_{max} = \lambda_D = \left(\dfrac{\varepsilon_o k T_2}{n_2 q_2^2}\right)^{1/2}$: Debye length of 2

$x_{min} = x_{90} = |\, q_1q_2 \,| / 4\pi\varepsilon_o m_r v^2 = |\, q_1q_2 \,| / 12\pi\varepsilon_o kT$,

the **effective Coulomb scattering cross section** can be found :

$$\sigma_{s,\,eff} = \frac{1}{n_2\lambda_s} = \frac{(q_1q_2)^2 \ln\Lambda}{2\pi(\varepsilon_o m_r v^2)^2} \qquad (26)$$

where $\ln\Lambda \equiv \ln\left(\dfrac{x_{max}}{x_{min}}\right)$: Coulomb logarithm

Notes)

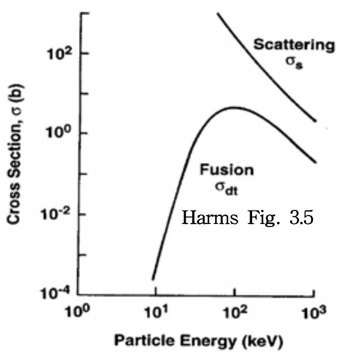
i) $\sigma_{s,\,eff} \approx \dfrac{2.6\times 10^{-22}}{E^2(keV)}\ m^2$ for $Z=1$ & $\ln\Lambda = 20$

$\approx 10^{-9} z^2 \ln\Lambda / T^2(K)$

$\sigma_{s.eff} \gg \sigma_{DT} \quad \Rightarrow \quad \sigma_{scatt} = 10^3 \sim 10^4\, \sigma_{fus}$

(e.g.) $\sigma_{s,\,eff} \approx 10^4$ *barn* $(H^+ + H^+$ *at* 10 *keV*$)$

$\sigma_{DT} \approx 10^{-1}$ *barn* (*at* ~25 *keV*)



Harms Fig. 3.5

ii) $\dfrac{Multi-coll.}{Single\ coll.} = \dfrac{\sigma_{s,\,eff}}{\sigma_s(\Theta_c > 90^\circ)} = 8 \ln\Lambda \approx O(10^2)$

iii) For high T, $\sigma_{90} \propto v^{-4} \propto T^{-2}$: collisionless plasma

(e.g.) n = $10^{20}\ m^{-3}$, T = 100 *keV* $\Rightarrow$ $\lambda_s \approx 10^6\ m$, $\nu_s \approx v/\lambda_s$ = 10 ~ 100/s

4) Characteristic scattering (or confinement) times

a) Mean collision (or scattering) time

= Momentum relaxation (transfer) time

= Particle confinement time

$$\frac{dP}{dt} = -\frac{P}{\tau_s}$$

$$\tau_s = \frac{\lambda_s}{v}$$

In c.m. reference frame,

$$\tau_{s,12} = \frac{\lambda_{s,12}}{v} = \frac{1}{n_2 \sigma_{s,eff} v}$$

$$= \frac{2\pi\varepsilon_o^2 m_r^2 v^3}{n_2 (q_1 q_2)^2 \ln\Lambda} = \frac{2\pi\sqrt{m_r}\varepsilon_o^2 (3kT_1)^{3/2}}{n_2 (q_1 q_2)^2 \ln\Lambda} \qquad (27)$$

where

$$\Theta_L \approx \begin{cases} \Theta_c & \text{for } m_1 \ll m_2 \\ \Theta_c/2 & m_1 = m_2 \\ (m_2/m_1)\Theta_c & m_1 \gg m_2 \end{cases} \quad \Rightarrow \quad (\tau_s)_L \approx \begin{cases} (\tau_s)_c \\ 2(\tau_s)_c \\ (m_1/m_2)(\tau_s)_c \end{cases}$$

$$\tau_c^{ei} \equiv (\tau_{s,ei})_L = 6\sqrt{3}\pi \frac{\varepsilon_o^2 m_e^{1/2} (kT_e)^{3/2}}{Z n_e e^4 \ln\Lambda} \qquad (28)$$

$$\approx 2\times10^5 \frac{T_e^{3/2}(K)}{n_e \ln\Lambda} \approx 10^{16} \frac{T_e^{3/2}(keV)}{n_e \ln\Lambda} \approx 2\times10^{11} \frac{T_e^{3/2}(eV)}{n_e \ln\Lambda}$$

$$\nu_{ei} = (\tau_c^{ei})^{-1} \approx 5\times10^{-6} \frac{n_e \ln\Lambda}{T_e^{3/2}(k)}$$: mean collision time between e & i

$$\tau_c^{ee} : \tau_c^{ei} : \tau_c^{ii} : \tau_c^{ie} = 1 : \frac{1}{Z}\left(\frac{T_i}{T_e}\right)^{3/2} : \frac{1}{Z^3}\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_i}{T_e}\right)^{3/2} : \frac{1}{Z^2}\frac{m_i}{m_e}$$

For $T_i = T_e = 10\ keV$, $Z = 1$, $n_e = 10^{20}\ m^{-3}$,

$$\tau_c^{ee} : \tau_c^{ei} : \tau_c^{ii} : \tau_c^{ie} \approx 1 : 1 : \sqrt{\frac{m_i}{m_e}} : \frac{m_i}{m_e} \qquad (29)$$

$$\approx 10^{-4} : 10^{-4} : 10^{-2} : 1 \quad (sec)$$

b) Energy confinement time

= Energy relaxation (transfer) time

= Temperature relaxation time

= Equilibration time

$$\frac{dT}{dt} = \frac{T_2 - T_1}{\tau_{eq}}$$

$\tau_{eq} \approx \tau_E^{12}$ = (# of collisions required to transfer all energy)

× (mean collision time)

$$= \frac{1}{(\triangle E/E_o)_s} \tau_c^{12}$$

where energy transferred in a large-angle scattering in c.m. system is

$$\left(\frac{\triangle E}{E_o}\right)_s = \frac{2m_1m_2}{(m_1+m_2)^2} \approx 2m_1/m_2 \quad \text{for } m_1 << m_2$$

$$1/2 \quad \text{for } m_1 = m_2$$

$$2m_2/m_1 \quad \text{for } m_1 >> m_2$$

Then,

$$\tau_E^{ee} \approx \tau_c^{ee} \approx \tau_c^{ei}, \qquad \tau_E^{ii} \approx \tau_c^{ii} \approx \sqrt{\frac{m_i}{m_e}}\tau_c^{ee},$$

$$\tau_E^{ei} \approx \frac{m_i}{m_e}\tau_c^{ei}, \qquad \tau_E^{ie} \approx \frac{m_i}{m_e}\tau_c^{ei}$$

$$\therefore \quad \tau_E^{ee} : \tau_E^{ii} : \tau_E^{ei} : \tau_E^{ie} \approx 1 : \sqrt{\frac{m_i}{m_e}} : \frac{m_i}{m_e} : \frac{m_i}{m_e} \tag{30}$$

$$\approx 10^{-4} : 10^{-2} : 1 : 1 \quad (sec)$$

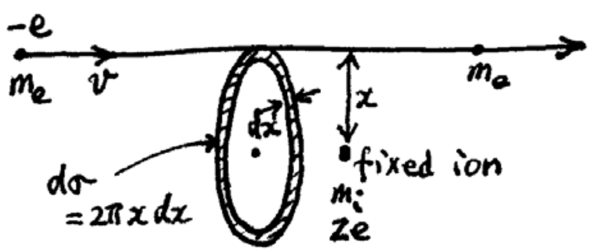
C. Radiation power losses

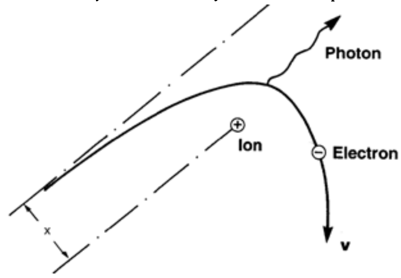
1) Bremsstrahlung (Braking radiation or Free-free radiation)

Mainly due to electron acceleration or deceleration in e-i collisions since radiation fields($P_{rad} = E_{rad} \times H_{rad}$) exactly cancel in e-e or i-i collisions due to oppositely accelerated particles.

Assume small-angle Coulomb scattering.

$P_{before\ coll} - P_{after\ coll} = P_{photon}$





Harms Fig. 3.6

$$|\boldsymbol{F}| = m_e\left|\frac{d\boldsymbol{v}}{dt}\right| \approx \frac{Ze^2}{4\pi\varepsilon_o x^2} \quad \Rightarrow \quad \bar{a} = \left|\frac{d\boldsymbol{v}}{dt}\right| \approx \frac{Ze^2}{4\pi\varepsilon_o m_e x^2}$$

$\triangle t_{coll} = \frac{x}{v} \ll 1$, otherwise continuous spectrum

Power radiated from a **single** e-i collision :

$$P_{rad} = \frac{1}{6\pi\varepsilon_o c^3} q_e^2 \left|\frac{d\boldsymbol{v}}{dt}\right|^2 = \frac{Z^2e^6}{96\pi^3\varepsilon_o^2 c^3 m_e^2 x^4}$$

\# of collisions per unit volume of n_e with n_i in $d\sigma$ during t_{coll} :

$$dN = n_e n_i d\sigma\, v\, t_{coll} = n_e n_i 2\pi x^2 dx$$

Total bremsstrahlung power density at all impact parameters x :

$$P_{br} = \int_{x_{min}}^{x_{max}} P_{rad} dN = \frac{Z^2e^6 n_e n_i}{48\pi^2\varepsilon_o^3 c^3 m_e^2} \int_{\hbar/m_e v}^{\lambda_D \to \infty} \frac{dx}{x^2}$$

$v = \sqrt{8T_e/\pi m_e}$ $\qquad$ $x_{min} = \lambda_{deBrogile} = \hbar/\triangle p$ since $\triangle x \triangle p = \hbar$

$$= \frac{Z^2e^6 n_e n_i v}{24\pi\varepsilon_o^3 c^2 m_e h} = A_{br} n_e n_i Z^2 (kT_e)^{1/2} \quad W/m^3 \tag{35}$$

where $A_{br} \approx 4.8\times 10^{-37}$ in keV unit for kT_e

1.6×10^{-38} in eV unit for kT_e

Notes) In fusion plasma, soft X-ray range of brems. can penetrates thru plasma $(\omega_{pe} \ll \omega_X)$ $\Rightarrow$ Optically thin (transparent)

Serious energy losses

2) Cyclotron radiation

$\bar{a} = v^2/r_L \approx r_{Le}\omega_{ce}^2 = v_{\perp e}eB/m_e \Rightarrow P_{rad} \Rightarrow P_{cyc} = n_e P_{rad}$

$$P_{cyc} = 6.2\times 10^{-17} B^2 n_e [kT_e(keV)] \quad W/m^3 \tag{36}$$

$\approx 1.5\ MW/m^3 > P_{br}$

Notes)

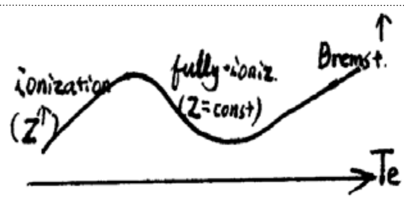
Optically thick(opaque) to fundamental frequency (IR, MW)

Losses at harmonics $\approx 10^{-2} \sim 10^{-3}$ P_{cyc} $\rightarrow$ reflectors

3) Line (Excitation+Recombination) radiations from impurities

(bound-bound, free-bound)

$$P_{imp} = 5.4\times 10^{-37} n_e n_z \left(Z^2 (kT_e)^{1/2} + \frac{3.8\times 10^{-2} Z^4}{(kT_e)^{1/2}} + \frac{8.6\times 10^{-4} Z^6}{(kT_e)^{3/2}} \right) \quad W/m^3 \tag{37}$$



Homework :

Harms Chap.1 Problems 1.3, 1.5, 1.8

Chap.2 2.2

Chap.3 3.3, 3.4, 3.5, 3.7

Chap.7 7.1, 7.4, 7.6