Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers
SECOND EDITION

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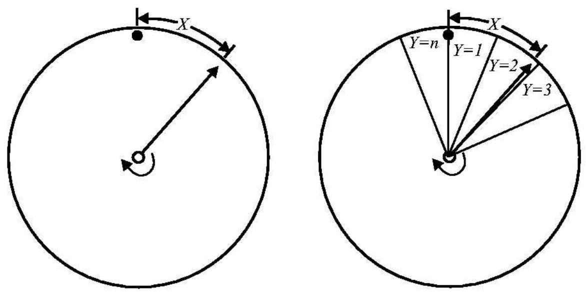
Definitions, Theorems, Proofs, Examples, Quizzes, Problems, Solutions

Chapter 3

Figure 3.1



The random pointer on disk of circumference 1.

Example 3.1 Problem

Suppose we have a wheel of circumference one meter and we mark a point on the perimeter at the top of the wheel. In the center of the wheel is a radial pointer that we spin. After spinning the pointer, we measure the distance, X meters, around the circumference of the wheel going clockwise from the marked point to the pointer position as shown in Figure 3.1. Clearly, $0 \leq X < 1$. Also, it is reasonable to believe that if the spin is hard enough, the pointer is just as likely to arrive at any part of the circle as at any other. For a given x, what is the probability $P[X = x]$?

Example 3.1 Solution

This problem is surprisingly difficult. However, given that we have developed methods for discrete random variables in Chapter 2, a reasonable approach is to find a discrete approximation to X. As shown on the right side of Figure 3.1, we can mark the perimeter with n equal-length arcs numbered 1 to n and let Y denote the number of the arc in which the pointer stops. Y is a discrete random variable with range $S_Y = \{1, 2, \ldots, n\}$. Since all parts of the wheel are equally likely, all arcs have the same probability. Thus the PMF of Y is

$$P_Y(y) = \begin{cases} 1/n & y = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

From the wheel on the right side of Figure 3.1, we can deduce that if $X = x$, then $Y = \lceil nx \rceil$, where the notation $\lceil a \rceil$ is defined as the smallest integer greater than or equal to a.

Example 3.1 Solution (continued)

Note that the event $\{X = x\} \subset \{Y = \lceil nx \rceil\}$, which implies that

$$P[X = x] \le P[Y = \lceil nx \rceil] = \frac{1}{n}.$$

At the limit,

$$P[X = x] \le \lim_{n\to\infty} P[Y = \lceil nx \rceil] = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Section 3.1

The Cumulative Distribution Function

Definition 3.1 *(CDF)*

The cumulative distribution function *(CDF) of random variable* X *is*

$$F_X(x) = P[X \leq x].$$

Theorem 3.1

For any random variable X,

(a) $F_X(-\infty) = 0$

(b) $F_X(\infty) = 1$

(c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

Definition 3.2 Continuous Random Variable

X *is a* continuous random variable *if the CDF* $F_X(x)$ *is a continuous function.*

Example 3.2 Problem

In the wheel-spinning experiment of Example 3.1, find the CDF of X.

Example 3.2 Solution

We begin by observing that any outcome $x \in S_X = [0, 1)$. This implies that $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x \geq 1$. To find the CDF for x between 0 and 1 we consider the event $\{X \leq x\}$ with x growing from 0 to 1. Each event corresponds to an arc on the circle in Figure 3.1. The arc is small when $x \approx 0$ and it includes nearly the whole circle when $x \approx 1$. $F_X(x) = P[X \leq x]$ is the probability that the pointer stops somewhere in the arc. This probability grows from 0 to 1 as the arc increases to include the whole circle. Given our assumption that the pointer has no preferred stopping places, it is reasonable to expect the probability to grow in proportion to the fraction of the circle occupied by the arc $X \leq x$. This fraction is simply x. To be more formal, we can refer to Figure 3.1 and note that with the circle divided into n arcs,

$$\{Y \leq \lceil nx \rceil - 1\} \subset \{X \leq x\} \subset \{Y \leq \lceil nx \rceil\}.$$

Therefore, the probabilities of the three events satisfy

$$F_Y(\lceil nx \rceil - 1) \leq F_X(x) \leq F_Y(\lceil nx \rceil).$$

[Continued]

Example 3.2 Solution (continued)

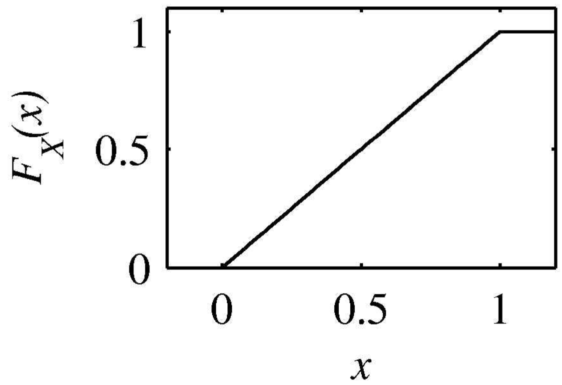
Note that Y is a discrete random variable with CDF

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ k/n & (k-1)/n < y \leq k/n, k = 1, 2, \ldots, n, \\ 1 & y > 1. \end{cases}$$

Thus for $x \in [0, 1)$ and for all n, we have

$$\frac{\lceil nx \rceil - 1}{n} \leq F_X(x) \leq \frac{\lceil nx \rceil}{n}.$$

In Problem 3.1.4, we ask the reader to verify that $\lim_{n\to\infty} \lceil nx \rceil / n = x$. This implies that as $n \to \infty$, both fractions approach x. The CDF of X is

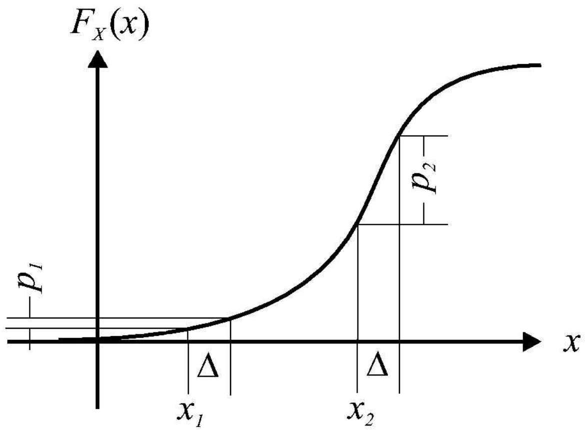


$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases}$$

Section 3.2

Probability Density Function

Figure 3.2



The graph of an arbitrary CDF $F_X(x)$.

Probability Density Function

Definition 3.3 (PDF)

The probability density function *(PDF) of a continuous random variable* X *is*

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

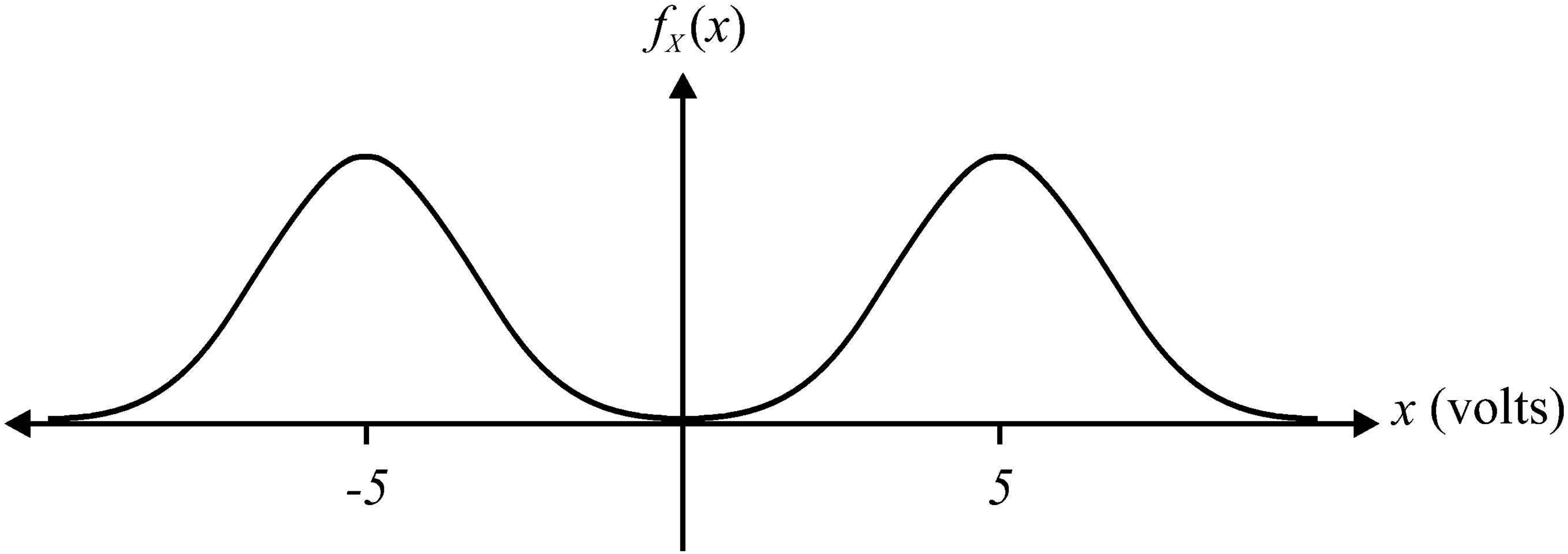
Example 3.3 Problem

Figure 3.3 depicts the PDF of a random variable X that describes the voltage at the receiver in a modem. What are probable values of X?

Example 3.3 Solution

Note that there are two places where the PDF has high values and that it is low elsewhere. The PDF indicates that the random variable is likely to be near -5 V (corresponding to the symbol 0 transmitted) and near $+5$ V (corresponding to a 1 transmitted). Values far from ± 5 V (due to strong distortion) are possible but much less likely.

Figure 3.3



The PDF of the modem receiver voltage X.

Theorem 3.2

For a continuous random variable X with PDF $f_X(x)$,

(a) $f_X(x) \geq 0$ for all x,

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$,

(c) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.

Proof: Theorem 3.2

The first statement is true because $F_X(x)$ is a nondecreasing function of x and therefore its derivative, $f_X(x)$, is nonnegative. The second fact follows directly from the definition of $f_X(x)$ and the fact that $F_X(-\infty) = 0$. The third statement follows from the second one and Theorem 3.1(b).

Theorem 3.3

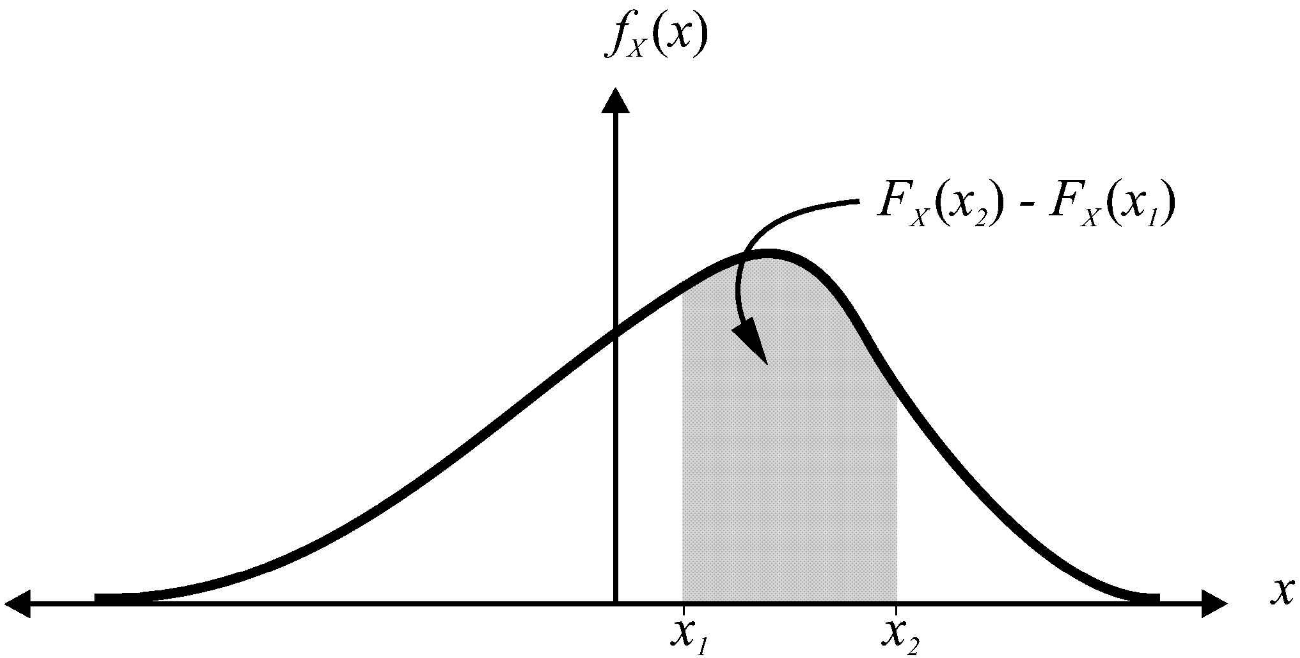
$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\, dx.$$

Proof: Theorem 3.3

From Theorem 3.2(b) and Theorem 3.1,

$$
\begin{aligned}
P[x_1 < X \leq x_2] &= P[X \leq x_2] - P[X \leq x_1] \\
&= F_X(x_2) - F_X(x_1) \\
&= \int_{x_1}^{x_2} f_X(x)\, dx.
\end{aligned}
$$

Figure 3.4



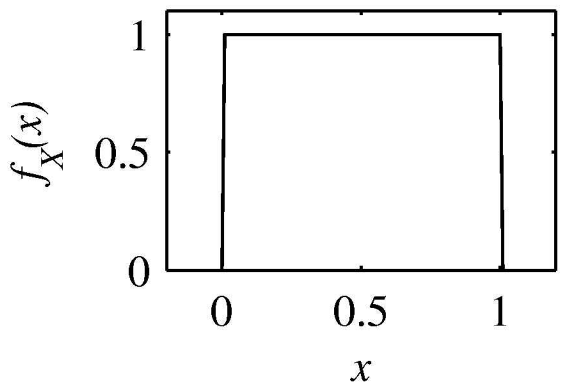
The PDF and CDF of X.

Example 3.4 Problem

For the experiment in Examples 3.1 and 3.2, find the PDF of X and the probability of the event $\{1/4 < X \leq 3/4\}$.

Example 3.4 Solution

Taking the derivative of the CDF in Equation (3.8), $f_X(x) = 0$, when $x < 0$ or $x \geq 1$. For x between 0 and 1 we have $f_X(x) = dF_X(x)/dx = 1$. Thus the PDF of X is



$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that the PDF is constant over the range of possible values of X reflects the fact that the pointer has no favorite stopping places on the circumference of the circle. To find the probability that X is between $1/4$ and $3/4$, we can use either Theorem 3.1 or Theorem 3.3. Thus

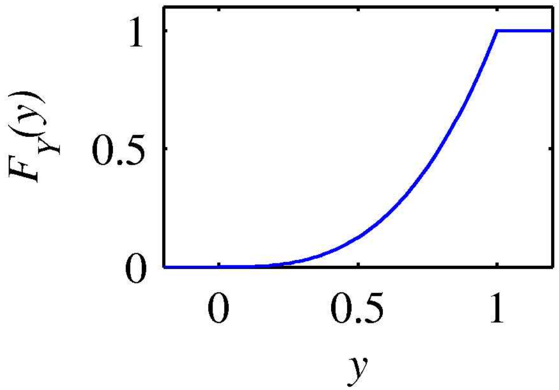
$$P[1/4 < X \leq 3/4] = F_X(3/4) - F_X(1/4) = 1/2,$$

and equivalently,

$$P[1/4 < X \leq 3/4] = \int_{1/4}^{3/4} f_X(x)\, dx = \int_{1/4}^{3/4} dx = 1/2.$$

Example 3.5 Problem

Consider an experiment that consists of spinning the pointer in Example 3.1 three times and observing Y meters, the maximum value of X in the three spins. In Example 5.8, we show that the CDF of Y is

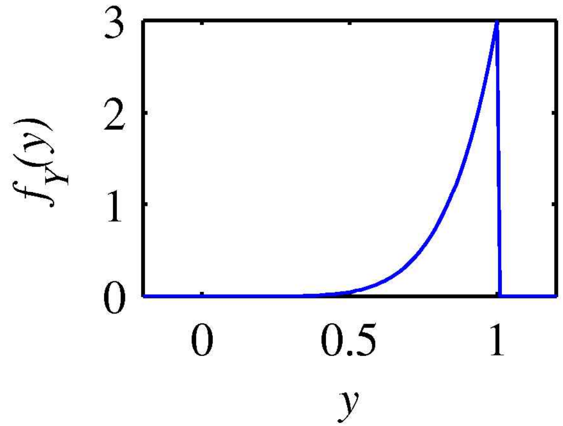


$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y^3 & 0 \le y \le 1, \\ 1 & y > 1. \end{cases}$$

Find the PDF of Y and the probability that Y is between $1/4$ and $3/4$.

Example 3.5 Solution

Applying Definition 3.3,



$$f_Y(y) = \begin{cases} df_Y(y)/dy = 3y^2 & 0 < y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the PDF has values between 0 and 3. Its integral between any pair of numbers is less than or equal to 1. The graph of $f_Y(y)$ shows that there is a higher probability of finding Y at the right side of the range of possible values than at the left side. This reflects the fact that the maximum of three spins produces higher numbers than individual spins. Either Theorem 3.1 or Theorem 3.3 can be used to calculate the probability of observing Y between $1/4$ and $3/4$:

$$P[1/4 < Y \le 3/4] = F_Y(3/4) - F_Y(1/4) = (3/4)^3 - (1/4)^3 = 13/32,$$

and equivalently,

$$P[1/4 < Y \le 3/4] = \int_{1/4}^{3/4} f_Y(y)\, dy = \int_{1/4}^{3/4} 3y^2\, dy = 13/32.$$

Quiz 3.2

Random variable X has probability density function

$$f_X(x) = \begin{cases} cxe^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the PDF and find the following:

(1) the constant c

(2) the CDF $F_X(x)$

(3) $P[0 \leq X \leq 4]$

(4) $P[-2 \leq X \leq 2]$

Section 3.3

Expected Values

Definition 3.4 Expected Value

The expected value *of a continuous random variable* X *is*

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Example 3.6 Problem

In Example 3.4, we found that the stopping point X of the spinning wheel experiment was a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1 & 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected stopping point $E[X]$ of the pointer.

Example 3.6 Solution

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x\, dx = 1/2 \text{ meter.}$$

With no preferred stopping points on the circle, the average stopping point of the pointer is exactly half way around the circle.

Example 3.7

In Example 3.5, find the expected value of the maximum stopping point Y of the three spins:

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 y(3y^2)\, dy = 3/4 \text{ meter.}$$

Example 3.8

Let X be a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = g(X) = 0$ if $X \leq 1/2$, and $W = g(X) = 1$ if $X > 1/2$. W is a discrete random variable with range $S_W = \{0, 1\}$.

Theorem 3.4

The expected value of a function, $g(X)$, of random variable X is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

Theorem 3.5

For any random variable X,

(a) $E[X - \mu_X] = 0$,

(b) $E[aX + b] = aE[X] + b$,

(c) $\mathrm{Var}[X] = E[X^2] - \mu_X^2$,

(d) $\mathrm{Var}[aX + b] = a^2 \mathrm{Var}[X]$.

Example 3.9 Problem

Find the variance and standard deviation of the pointer position in Example 3.1.

Example 3.9 Solution

To compute $\text{Var}[X]$, we use Theorem 3.5(c): $\text{Var}[X] = E[X^2] - \mu_X^2$. We calculate $E[X^2]$ directly from Theorem 3.4 with $g(X) = X^2$:

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2\, dx = 1/3.$$

In Example 3.6, we have $E[X] = 1/2$. Thus $\text{Var}[X] = 1/3 - (1/2)^2 = 1/12$, and the standard deviation is $\sigma_X = \sqrt{\text{Var}[X]} = 1/\sqrt{12} = 0.289$ meters.

Example 3.10 Problem

Find the variance and standard deviation of Y, the maximum pointer position after three spins, in Example 3.5.

Example 3.10 Solution

We proceed as in Example 3.9. We have $f_Y(y)$ from Example 3.5 and $E[Y] = 3/4$ from Example 3.7:

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^1 y^2 \left(3y^2\right) dy = 3/5.$$

Thus the variance is

$$\mathrm{Var}[Y] = 3/5 - (3/4)^2 = 3/80 \text{ m}^2,$$

and the standard deviation is $\sigma_Y = 0.194$ meters.

Section 3.4

Families of Continuous Random Variables

Definition 3.5 Uniform Random Variable

X is a uniform (a, b) random variable if the PDF of X is

$$f_X(x) = \begin{cases} 1/(b-a) & a \leq x < b, \\ 0 & \text{otherwise}, \end{cases}$$

where the two parameters are $b > a$.

Theorem 3.6

If X is a uniform (a, b) random variable,

(a) The CDF of X is

$$F_X(x) = \begin{cases} 0 & x \leq a, \\ (x-a)/(b-a) & a < x \leq b, \\ 1 & x > b. \end{cases}$$

(b) The expected value of X is $E[X] = (b+a)/2$.

(c) The variance of X is $\mathrm{Var}[X] = (b-a)^2/12$.

Example 3.11 Problem

The phase angle, Θ, of the signal at the input to a modem is uniformly distributed between 0 and 2π radians. Find the CDF, the expected value, and the variance of Θ.

Example 3.11 Solution

From the problem statement, we identify the parameters of the uniform (a, b) random variable as $a = 0$ and $b = 2\pi$. Therefore the PDF of Θ is

$$f_{\Theta}(\theta) = \begin{cases} 1/(2\pi) & 0 \leq \theta < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

The CDF is

$$F_{\Theta}(\theta) = \begin{cases} 0 & \theta \leq 0, \\ \theta/(2\pi) & 0 < x \leq 2\pi, \\ 1 & x > 2\pi. \end{cases}$$

The expected value is $E[\Theta] = b/2 = \pi$ radians, and the variance is $\text{Var}[\Theta] = (2\pi)^2/12 = \pi^2/3$ rad^2.

Theorem 3.7

Let X be a uniform (a, b) random variable, where a and b are both integers. Let $K = \lceil X \rceil$. Then K is a discrete uniform $(a + 1, b)$ random variable.

Proof: Theorem 3.7

Recall that for any x, $\lceil x \rceil$ is the smallest integer greater than or equal to x. It follows that the event $\{K = k\} = \{k - 1 < x \leq k\}$. Therefore,

$$P[K = k] = P_K(k) = \int_{k-1}^{k} P_X(x)\,dx = \begin{cases} 1/(b-a) & k = a+1, a+2, \ldots, b, \\ 0 & \text{otherwise.} \end{cases}$$

This expression for $P_K(k)$ conforms to Definition 2.9 of a discrete uniform $(a+1, b)$ PMF.

Definition 3.6 Exponential Random Variable

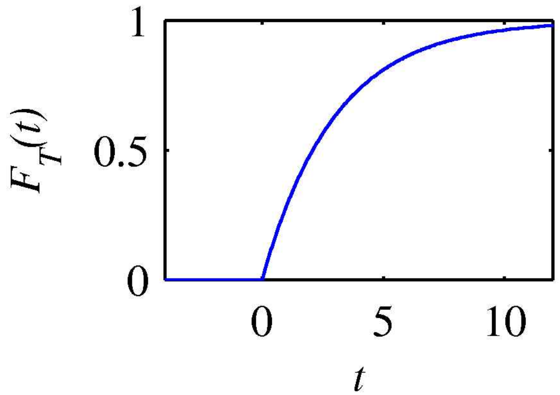
X *is an* exponential (λ) random variable *if the PDF of* X *is*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

where the parameter $\lambda > 0$.

Example 3.12 Problem

The probability that a telephone call lasts no more than t minutes is often modeled as an exponential CDF.

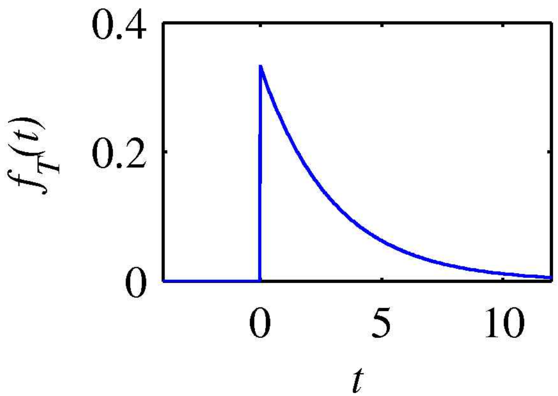


$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the PDF of the duration in minutes of a telephone conversation? What is the probability that a conversation will last between 2 and 4 minutes?

Example 3.12 Solution

We find the PDF of T by taking the derivative of the CDF:



$$f_T(t) = \frac{dF_T(t)}{dt} = \begin{cases} (1/3)e^{-t/3} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, observing Definition 3.6, we recognize that T is an exponential $(\lambda = 1/3)$ random variable. The probability that a call lasts between 2 and 4 minutes is

$$P[2 \leq T \leq 4] = F_4(4) - F_2(2) = e^{-2/3} - e^{-4/3} = 0.250.$$

Example 3.13 Problem

In Example 3.12, what is $E[T]$, the expected duration of a telephone call? What are the variance and standard deviation of T? What is the probability that a call duration is within ± 1 standard deviation of the expected call duration?

Example 3.13 Solution

Using the PDF $f_T(t)$ in Example 3.12, we calculate the expected duration of a call:

$$E[T] = \int_{-\infty}^{\infty} t f_T(t)\, dt = \int_0^{\infty} t \frac{1}{3} e^{-t/3}\, dt.$$

Integration by parts (Appendix B, Math Fact B.10) yields

$$E[T] = -te^{-t/3}\Big|_0^{\infty} + \int_0^{\infty} e^{-t/3}\, dt = 3 \text{ minutes}.$$

To calculate the variance, we begin with the second moment of T:

$$E\left[T^2\right] = \int_{-\infty}^{\infty} t^2 f_T(t)\, dt = \int_0^{\infty} t^2 \frac{1}{3} e^{-t/3}\, dt.$$

[Continued]

Example 3.13 Solution (continued)

Again integrating by parts, we have

$$E\left[T^2\right] = -t^2 e^{-t/3}\Big|_0^\infty + \int_0^\infty (2t)e^{-t/3}\,dt = 2\int_0^\infty te^{-t/3}\,dt.$$

With the knowledge that $E[T] = 3$, we observe that $\int_0^\infty te^{-t/3}\,dt = 3E[T] = 9$. Thus $E[T^2] = 6E[T] = 18$ and

$$\text{Var}\,[T] = E\left[T^2\right] - (E\,[T])^2 = 18 - 3^2 = 9.$$

The standard deviation is $\sigma_T = \sqrt{\text{Var}[T]} = 3$ minutes. The probability that the call duration is within 1 standard deviation of the expected value is

$$P\,[0 \le T \le 6] = F_T\,(6) - F_T\,(0) = 1 - e^{-2} = 0.865$$

Theorem 3.8

If X is an exponential (λ) random variable,

(a) $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$

(b) $E[X] = 1/\lambda$.

(c) $\mathrm{Var}[X] = 1/\lambda^2$.

Theorem 3.9

If X is an exponential (λ) random variable, then $K = \lceil X \rceil$ is a geometric (p) random variable with $p = 1 - e^{-\lambda}$.

Proof: Theorem 3.9

As in the proof of Theorem 3.7, the definition of K implies

$$P_K(k) = P[k-1 < X \leq k].$$

Referring to the CDF of X in Theorem 3.8, we observe

$$P_K(k) = F_x(k) - F_x(k-1) = e^{-\lambda(k-1)} - e^{-\lambda k} = (e^{-\lambda})^{k-1}(1-e^{-\lambda}).$$

If we let $p = 1 - e^{-\lambda}$, we have $P_K(k) = p(1-p)^{k-1}$, which conforms to Definition 2.6 of a geometric (p) random variable with $p = 1 - e^{-\lambda}$.

Example 3.14 Problem

Phone company A charges \$0.15 per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone Company B also charges \$0.15 per minute. However, Phone Company B calculates its charge based on the exact duration of a call. If T, the duration of a call in minutes, is an exponential $(\lambda = 1/3)$ random variable, what are the expected revenues per call $E[R_A]$ and $E[R_B]$ for companies A and B?

Example 3.14 Solution

Because T is an exponential random variable, we have in Theorem 3.8 (and in Example 3.13), $E[T] = 1/\lambda = 3$ minutes per call. Therefore, for phone company B, which charges for the exact duration of a call,

$$E[R_B] = 0.15E[T] = \$0.45 \text{ per call.}$$

Company A, by contrast, collects \$0.15$\lceil T \rceil$ for a call of duration T minutes. Theorem 3.9 states that $K = \lceil T \rceil$ is a geometric random variable with parameter $p = 1 - e^{-1/3}$. Therefore, the expected revenue for Company A is

$$E[R_A] = 0.15E[K] = 0.15/p = (0.15)(3.53) = \$0.529 \text{ per call.}$$

Definition 3.7 Erlang Random Variable

X *is an* Erlang (n, λ) *random variable if the PDF of* X *is*

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

where the parameter $\lambda > 0$*, and the parameter* $n \geq 1$ *is an integer.*

Theorem 3.10

If X is an Erlang (n, λ) random variable, then

$$E[X] = \frac{n}{\lambda}, \qquad \text{Var}[X] = \frac{n}{\lambda^2}.$$

Theorem 3.11

Let K_α denote a Poisson (α) random variable. For any $x > 0$, the CDF of an Erlang (n, λ) random variable X satisfies

$$F_X(x) = 1 - F_{K_{\lambda x}}(n-1) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}.$$

Gamma Random Variable

The pdf of the gamma random variable has two parameters, $\alpha > 0$ and $\lambda > 0$, and is given by

$$f_x(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \quad 0 < x < \infty,$$

where $\Gamma(z)$ is the gamma function, which is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty x^{z-1} e^{-x} dx \quad z > 0.$$

The gamma function has the following properties:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \;\; \Gamma(z+1) = z\Gamma(z) \text{ for } z > 0, \text{ and}$$

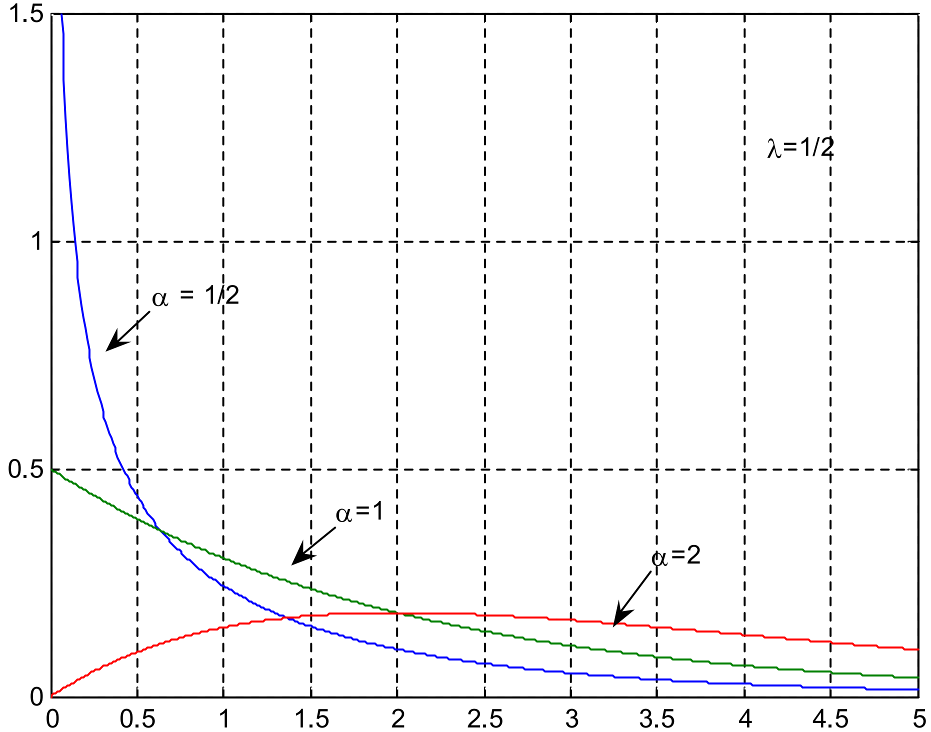
$$\Gamma(m+1) = m! \text{ for } m \text{ a nonnegative integer.}$$

Gamma Random Variable (continued)

Special cases:

$\alpha = 1$:	exponential r. v.
$\lambda = \frac{1}{2}$ and $\alpha = \frac{k}{2}$ (k a positive integer):	chi-square r. v.
$\alpha = m$ (m a positive integer):	*m*-Erlang r. v.

Gamma Random Variable (continued)



Gamma pdfs for $\alpha = 1/2$, $\alpha = 1$, and $\alpha = 2$ and $\lambda = 1/2$

Section 3.5

Gaussian Random Variables

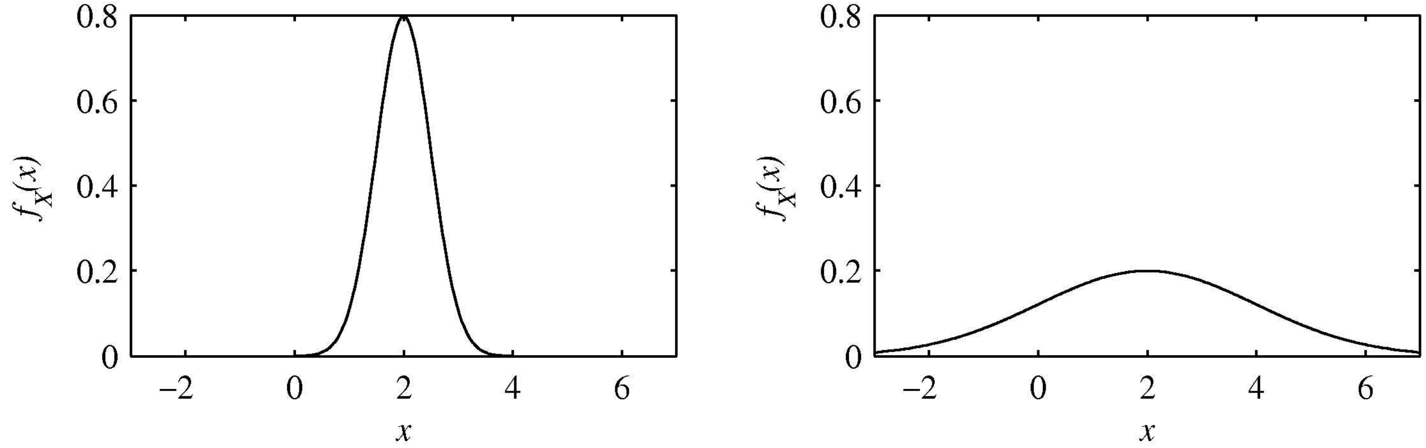
Definition 3.8 Gaussian Random Variable

X is a Gaussian (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ can be any real number and the parameter $\sigma > 0$.

Figure 3.5



(a) $\mu = 2, \sigma = 1/2$ **(b)** $\mu = 2, \sigma = 2$

Two examples of a Gaussian random variable X with expected value μ and standard deviation σ.

Theorem 3.12

If X is a Gaussian (μ, σ) random variable,

$$E[X] = \mu \qquad \mathrm{Var}[X] = \sigma^2.$$

Theorem 3.13

If X is Gaussian (μ, σ), $Y = aX + b$ is Gaussian $(a\mu + b, a\sigma)$.

Definition 3.9 Standard Normal Random Variable

The standard normal random variable Z *is the Gaussian* $(0, 1)$ *random variable.*

Definition 3.10 Standard Normal CDF

The CDF of the standard normal random variable Z is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

Theorem 3.14

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$$P[a < X \le b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Define $erf\,(x) = \dfrac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$, then

$$erf\,(z) = 2\Phi\left(\sqrt{2}\,z\right) - 1 \text{ or } \Phi\left(z\right) = \frac{1}{2}\left[1 + erf\,(\frac{z}{\sqrt{2}})\right]$$

$$F_X(x) = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right];\ P\left[a < X \le b\right] = \frac{1}{2}\left[erf\left(\frac{b-\mu}{\sqrt{2}\sigma}\right) - erf\left(\frac{a-\mu}{\sqrt{2}\sigma}\right)\right]$$

Example 3.15 Problem

Suppose your score on a test is $x = 46$, a sample value of the Gaussian $(61, 10)$ random variable. Express your test score as a sample value of the standard normal random variable, Z.

Example 3.15 Solution

Equation (3.54) indicates that $z = (46 - 61)/10 = -1.5$. Therefore your score is 1.5 *standard deviations less than the expected value*.

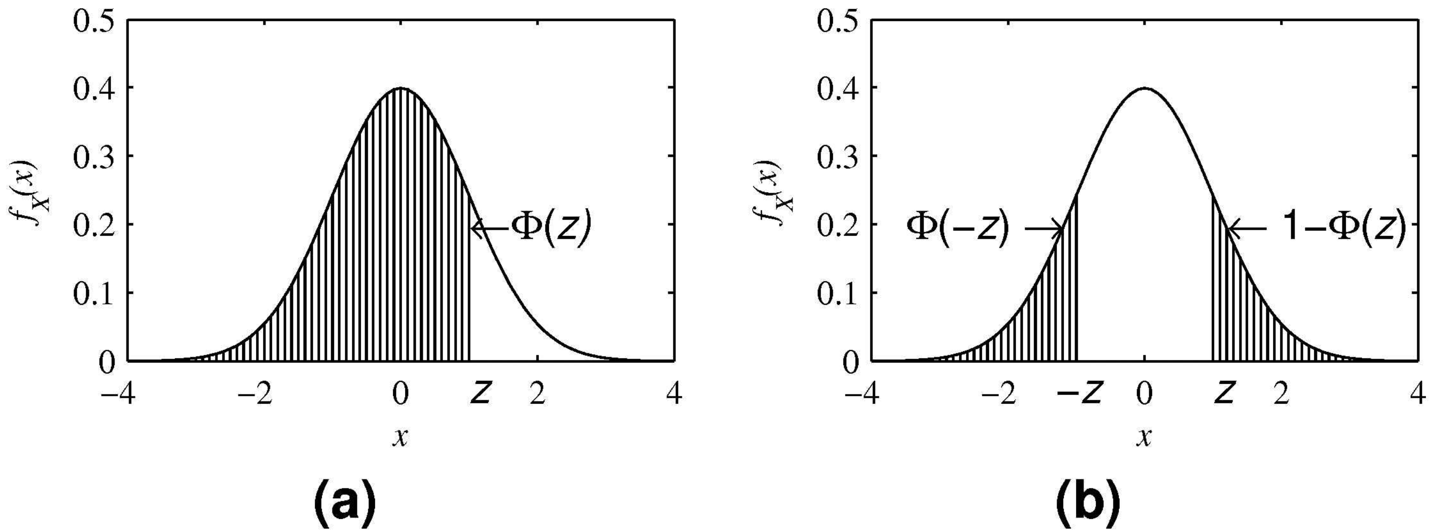
Theorem 3.15

$$\Phi(-z) = 1 - \Phi(z).$$

or

$$erf\,(\text{-}x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\,dt = -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2}\,dt = -erf\,(x)$$

Figure 3.6



Symmetry properties of th Gaussian $(0, 1)$ PDF.

Example 3.16 Problem

If X is the Gaussian $(61, 10)$ random variable, what is $P[X \leq 46]$?

Example 3.16 Solution

Applying Theorem 3.14, Theorem 3.15 and the result of Example 3.15, we have

$$P[X \leq 46] = F_X(46) = \Phi(-1.5) = 1 - \Phi(1.5) = 1 - 0.933 = 0.067.$$

This suggests that if your test score is 1.5 standard deviations below the expected value, you are in the lowest 6.7% of the population of test takers.

Using the error function,

$$P[X \leq 46] = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)\right] = \frac{1}{2}\left[1 + erf\left(\frac{46-61}{\sqrt{2}\cdot 10}\right)\right]$$

$$= \frac{1}{2}\left[1 - erf(1.0607)\right] = 0.0668.$$

Example 3.17 Problem

If X is a Gaussian random variable with $\mu = 61$ and $\sigma = 10$, what is $P[51 < X \leq 71]$?

Example 3.17 Solution

Applying Equation (3.54), $Z = (X - 61)/10$ and the event $\{51 < X \leq 71\}$ corresponds to the event $\{-1 < Z \leq 1\}$. The probability of this event is

$$P[-1 < Z \leq 1] = \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1 = 0.683.$$

Using the error function,

$$P[-1 < Z \leq 1] = \frac{1}{2}\left[erf\left(\frac{1}{\sqrt{2}}\right) - erf\left(-\frac{1}{\sqrt{2}}\right)\right] = erf\left(\frac{1}{\sqrt{2}}\right) = 0.6827.$$

Standard Normal

Definition 3.11 Complementary CDF

The standard normal complementary CDF *is*

$$Q(z) = P[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-u^2/2}\,du = 1 - \Phi(z).$$

Example 3.18 Problem

In an optical fiber transmission system, the probability of a binary error is $Q(\sqrt{\gamma/2})$, where γ is the signal-to-noise ratio. What is the minimum value of γ that produces a binary error rate not exceeding 10^{-6}?

Example 3.18 Solution

Referring to Table 3.1, we find that $Q(z) < 10^{-6}$ when $z \geq 4.75$. Therefore, if $\sqrt{\gamma/2} \geq 4.75$, or $\gamma \geq 45$, the probability of error is less than 10^{-6}.

Section 3.6

Delta Functions, Mixed Random Variables

Definition 3.12 Unit Impulse (Delta) Function

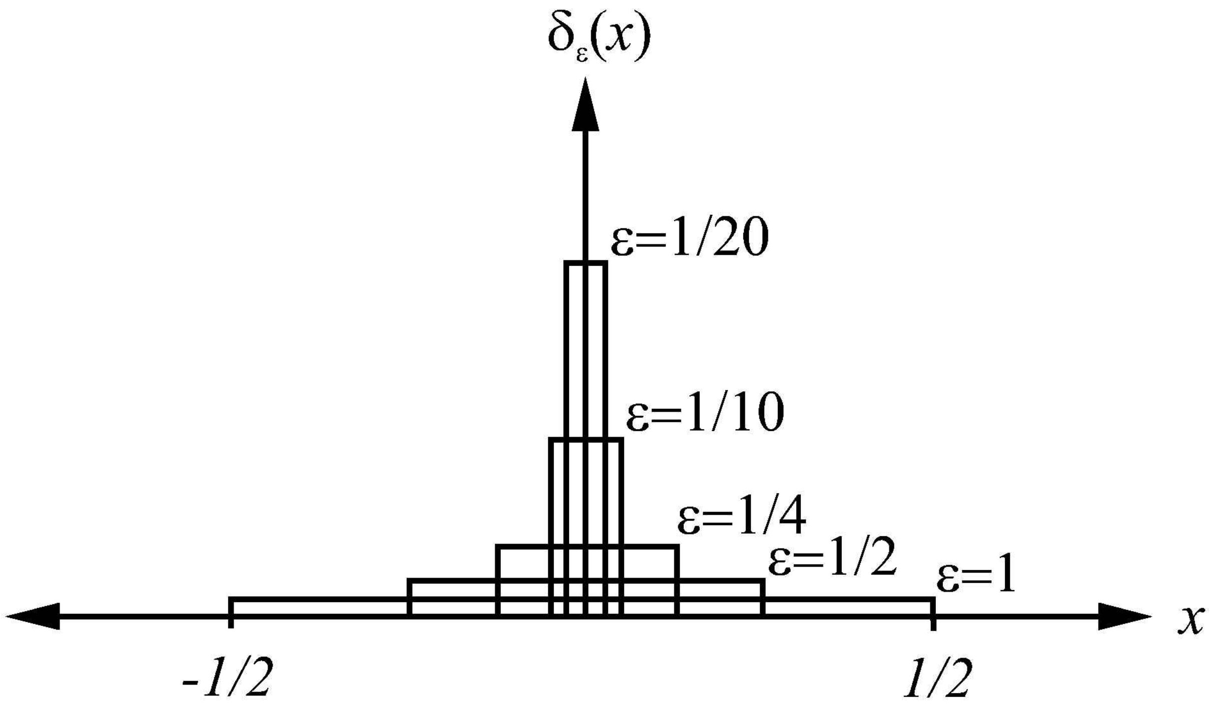
Let

$$d_\epsilon(x) = \begin{cases} 1/\epsilon & -\epsilon/2 \le x \le \epsilon/2, \\ 0 & \textit{otherwise}. \end{cases}$$

The unit impulse function *is*

$$\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(x).$$

Figure 3.7



As $\epsilon \to 0$, $d_\epsilon(x)$ approaches the delta function $\delta(x)$. For each ϵ, the area under the curve of $d_\epsilon(x)$ equals 1.

Theorem 3.16

For any continuous function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)\, dx = g(x_0).$$

Definition 3.13 Unit Step Function

The unit step function *is*

$$u(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Theorem 3.17

$$\int_{-\infty}^{x} \delta(v)\, dv = u(x).$$

Example 3.19

Suppose Y takes on the values $1, 2, 3$ with equal probability. The PMF and the corresponding CDF of Y are

$$P_Y(y) = \begin{cases} 1/3 & y = 1, 2, 3, \\ 0 & \text{otherwise}, \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 1, \\ 1/3 & 1 \le y < 2, \\ 2/3 & 2 \le y < 3, \\ 1 & y \ge 3. \end{cases}$$

Using the unit step function $u(y)$, we can write $F_Y(y)$ more compactly as

$$F_Y(y) = \frac{1}{3}u(y-1) + \frac{1}{3}u(y-2) + \frac{1}{3}u(y-3).$$

The PDF of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{3}\delta(y-1) + \frac{1}{3}\delta(y-2) + \frac{1}{3}\delta(y-3).$$

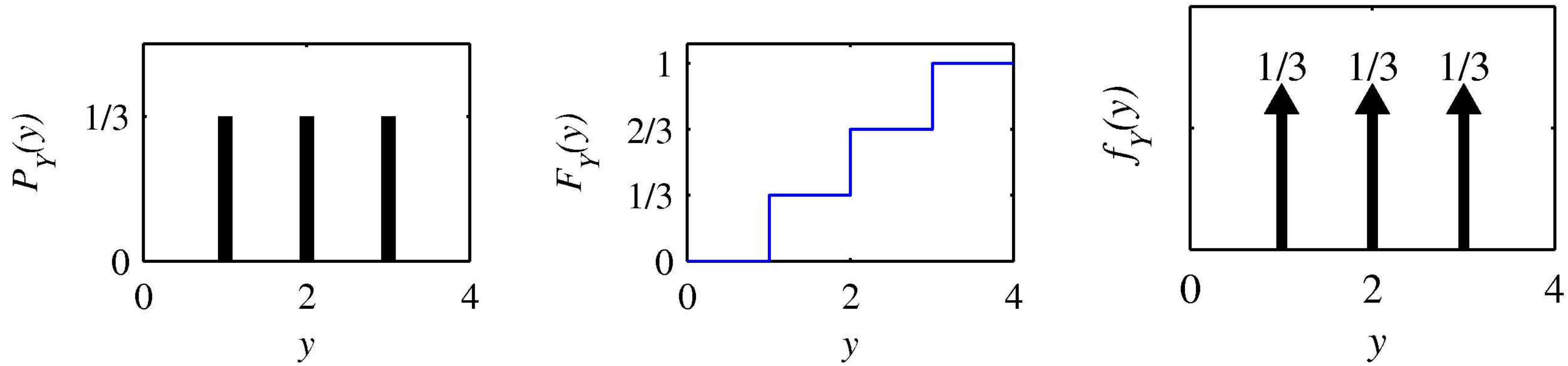
[Continued]

Example 3.19 (continued)

We see that the discrete random variable Y can be represented graphically either by a PMF $P_Y(y)$ with bars at $y = 1, 2, 3$, by a CDF with jumps at $y = 1, 2, 3$, or by a PDF $f_Y(y)$ with impulses at $y = 1, 2, 3$. These three representations are shown in Figure 3.8. The expected value of Y can be calculated either by summing over the PMF $P_Y(y)$ or integrating over the PDF $f_Y(y)$. Using the PDF, we have

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dy \\
&= \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-1)\, dy + \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-2)\, dy + \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-3)\, dy \\
&= 1/3 + 2/3 + 1 = 2.
\end{aligned}
$$

Figure 3.8



The PMF, CDF, and PDF of the mixed random variable Y.

Example 3.20

For the random variable Y of Example 3.19,

$$F_Y\left(2^-\right) = 1/3, \qquad F_Y\left(2^+\right) = 2/3.$$

Theorem 3.18

For a random variable X, we have the following equivalent statements:

(a) $P[X = x_0] = q$

(b) $P_X(x_0) = q$

(c) $F_X(x_0^+) - F_X(x_0^-) = q$

(d) $f_X(x_0) = q\delta(0)$

Definition 3.14 *Mixed Random Variable*

X *is a* mixed *random variable if and only if* $f_X(x)$ *contains both impulses and nonzero, finite values.*

Example 3.21 Problem

Observe someone dialing a telephone and record the duration of the call. In a simple model of the experiment, $1/3$ of the calls never begin either because no one answers or the line is busy. The duration of these calls is 0 minutes. Otherwise, with probability $2/3$, a call duration is uniformly distributed between 0 and 3 minutes. Let Y denote the call duration. Find the CDF $F_Y(y)$, the PDF $f_Y(y)$, and the expected value $E[Y]$.

Example 3.21 Solution

Let A denote the event that the phone was answered. Since $Y \geq 0$, we know that for $y < 0$, $F_Y(y) = 0$. Similarly, we know that for $y > 3$, $F_Y(y) = 1$. For $0 \leq y \leq 3$, we apply the law of total probability to write

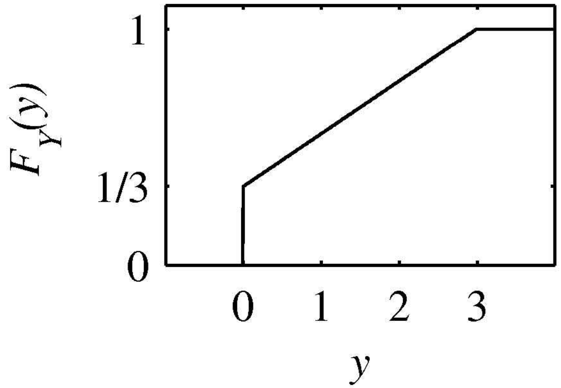
$$F_Y(y) = P[Y \leq y] = P\left[Y \leq y|A^c\right] P\left[A^c\right] + P[Y \leq y|A] P[A].$$

When A^c occurs, $Y = 0$, so that for $0 \leq y \leq 3$, $P[Y \leq y|A^c] = 1$. When A occurs, the call duration is uniformly distributed over $[0, 3]$, so that for $0 \leq y \leq 3$, $P[Y \leq y|A] = y/3$. So, for $0 \leq y \leq 3$,

$$F_Y(y) = (1/3)(1) + (2/3)(y/3) = 1/3 + 2y/9.$$

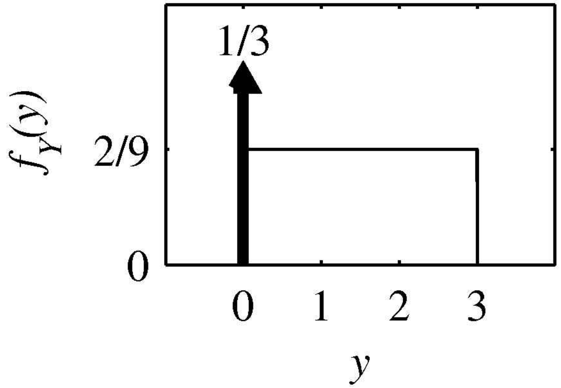
Finally, the complete CDF of Y is [Continued]

Example 3.21 Solution (continued)



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ 1/3 + 2y/9 & 0 \le y < 3, \\ 1 & y \ge 3. \end{cases}$$

Consequently, the corresponding PDF $f_Y(y)$ is



$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

For the mixed random variable Y, it is easiest to calculate $E[Y]$ using the PDF:

$$E[Y] = \int_{-\infty}^{\infty} y \frac{1}{3}\delta(y)\,dy + \int_0^3 \frac{2}{9} y\,dy = 0 + \frac{2}{9}\frac{y^2}{2}\bigg|_0^3 = 1.$$

Quiz 3.6

The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/4 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

Sketch the CDF and find the following:

(1) $P[X \le 1]$

(2) $P[X < 1]$

(3) $P[X = 1]$

(4) the PDF $f_X(x)$

Section 3.7

Probability Models of Derived Random Variables

Probability Models of Derived Random Variables

Consider

$$y = g(x)$$

where $y = g(x)$ is a real function of *x* and x is a r. v. with $F_x(x)$ and $f_x(x)$.

Express $F_Y(y)$ in terms of $F_X(x)$ and $f_Y(y)$ in terms of $f_X(x)$.

By definition

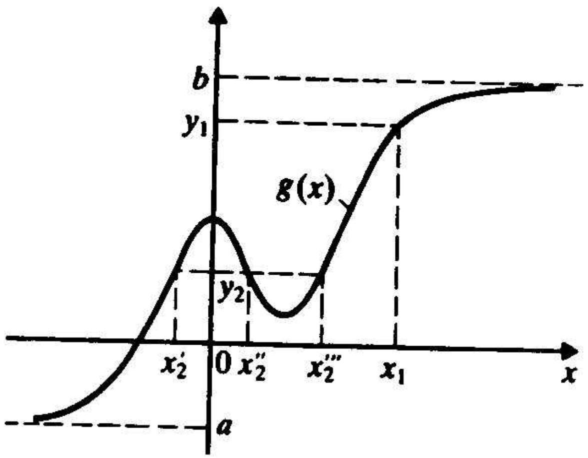
$$F_y(y) = P[y \le y] = P[g(x) \le y] = P\left[\, x \in R_y \,\right]$$

where $R_y = \{x : g(x) < y\}$.

Distribution Function of g(x)

We shall express the distribution function $F_y(y)$ of the random variables $y = g(x)$ in terms of the distribution function $F_x(x)$ of the random variable X and the function $g(x)$.

1. Continuous and Smooth Function, g(x)



Consider the function $g(x)$ in the above figure where $g(x)$ is between a and b for any x.

If $y \geq b$, then $g(x) \leq y$ for every x, hence $P[\mathrm{y} \leq y] = 1$;
if $y < a$, then there is no x such that $g(x) \leq y$, hence $P[\mathrm{y} \leq y] = 0$. Thus,

$$F_Y(y) = \begin{cases} 1 & y \geq b \\ 0 & y < a \end{cases}$$

with x_1 and $y_1 = g(x_1)$ as shown, we observe that $g(x) \leq y_1$ for $x \leq x_1$. Hence,

$$F_Y(y_1) = P[\,\mathrm{x} \leq x_1\,] = F_{\mathrm{x}}(x_1).$$

We finally note that $g(x) \leq y_2$ if $x \leq x_2'$ or $x_2'' \leq x \leq x_2'''$. Hence,

$$F_{\mathrm{y}}(y_2) = P[\,\mathrm{x} \leq x_2'] + P[\,x_2'' \leq \mathrm{x} \leq x_2''' \,\}$$
$$= F_{\mathrm{x}}(x_2') + F_{\mathrm{x}}(x_2''') - F_{\mathrm{x}}(x_2'').$$

Example ($\mathrm{Y} = aX + b$)

To find $F_{\mathrm{Y}}(y)$, we must find the values of x such that $ax + b \le y$.

(a) If $a > 0$, $ax + b \le y$ for $x \le \dfrac{(y-b)}{a}$. (Fig. 3.7-2(a)) Hence,

$$F_Y(y) = P[\mathrm{x} \le \frac{y-b}{a}] = F_X(\frac{y-b}{a}), \text{ a>0.}$$

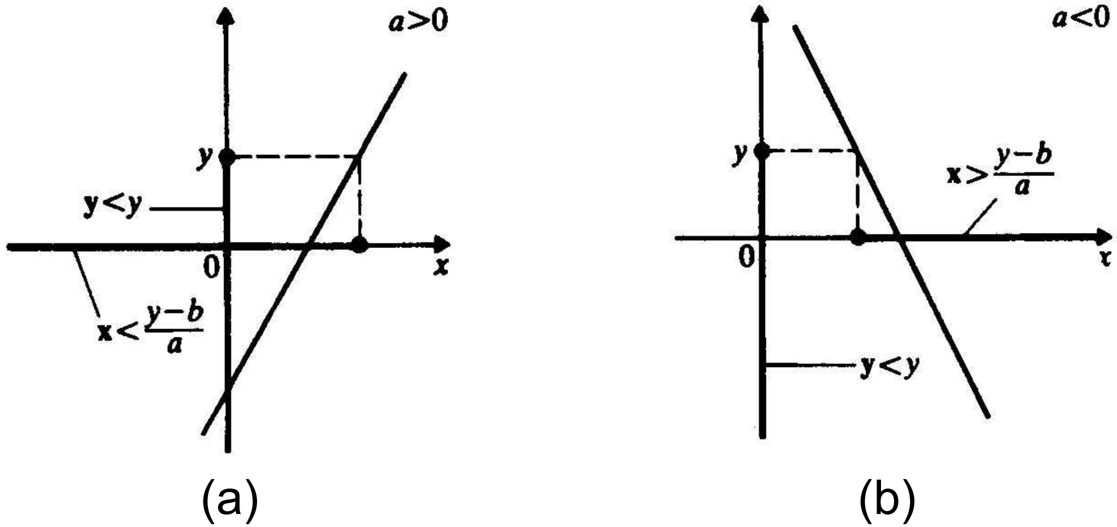


Figure 3.7-2. $\mathrm{y} = a\mathrm{x} + b$

(b) If $a < 0$, $ax + b \leq y$ for $x > \frac{(y-b)}{a}$. (Fig. 3-2(b)) Hence,

$$F_y(y) = P[\mathrm{x} \geq \frac{y-b}{a}] = 1 - F_x(\frac{y-b}{a}) \text{ , a<0.}$$

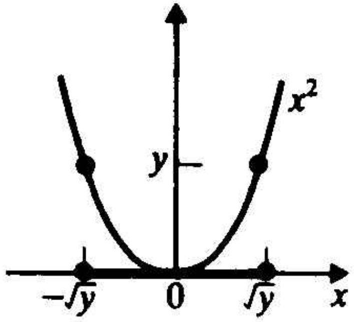
Example ($Y = X^2$)

(1) When $y \geq 0$

$$x^2 \leq y \qquad \text{for} \qquad -\sqrt{y} \leq x \leq \sqrt{y}$$

$$F_Y(y) = P\left\{ -\sqrt{y} \leq x \leq \sqrt{y} \right\}$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$



(a) $y = x^2$

(2) When $y < 0$,

$$F_y(y) = P[\phi] = 0$$

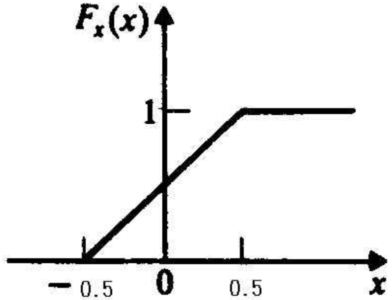
Let $\quad f_X(x) = \begin{cases} 1 & -\tfrac{1}{2} \leq x \leq \tfrac{1}{2} \\ 0 & \text{else} \end{cases}$

then, $F_X(x) = \dfrac{1}{2} + x, \qquad |x| < \dfrac{1}{2}$

and

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

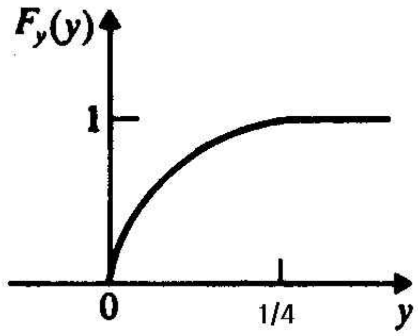
$$= \sqrt{y} + \sqrt{y} = 2\sqrt{y} \qquad \textit{for} \quad 0 \leq y \leq \tfrac{1}{4}.$$



(b) $F_x(x)$

In summary,

$$F_{\mathrm{y}}(y)=\begin{cases}1 & y>1/4\\ 2\sqrt{y} & 0\le y\le 1/4\\ 0 & y<0.\end{cases}$$



(c) $F_{\mathrm{y}}(y)$

2. Continuous but not Smooth Function, **g(x)**

Suppose the function $g(x)$ is constant in an interval (x_0, x_1) such that

$$g(x) = y_1, \quad x_0 < x \le x_1.$$

In this case

$$P[\mathrm{y} = y_1] = P[x_0 < \mathrm{x} \le \mathrm{x}_1] = F_{\mathrm{x}}(x_1) - F_{\mathrm{x}}(x_0).$$

Hence $F_{\mathrm{y}}(y)$ is discontinuous at $\mathrm{y} = y_1$ and its discontinuity equals $F_{\mathrm{x}}(x_1) - F_{\mathrm{x}}(x_0)$.

Example (Continuous but not Smooth Function)

Consider the function shown in Fig. 3.7-4 (center-level clipper)

$$g(x) = \begin{cases} x - c & \text{for } x > c \\ 0 & \text{for } -c \le x \le c \\ x + c & \text{for } x < -c \end{cases}$$

In this case, $F_{\mathrm{Y}}(y)$ is discontinuous for $y = 0$ and its discontinuity equals $F_{\mathrm{x}}(c) - F_{\mathrm{x}}(-c)$.

Furthermore, if $y \ge 0$ then $P[\mathrm{y} \le y] = P[\mathrm{x} \le y + c] = F_{\mathrm{x}}(y + c)$;

if $y < 0$ then $P[\mathrm{y} \le y] = P[\mathrm{x} \le y - c] = F_{\mathrm{x}}(y - c)$.

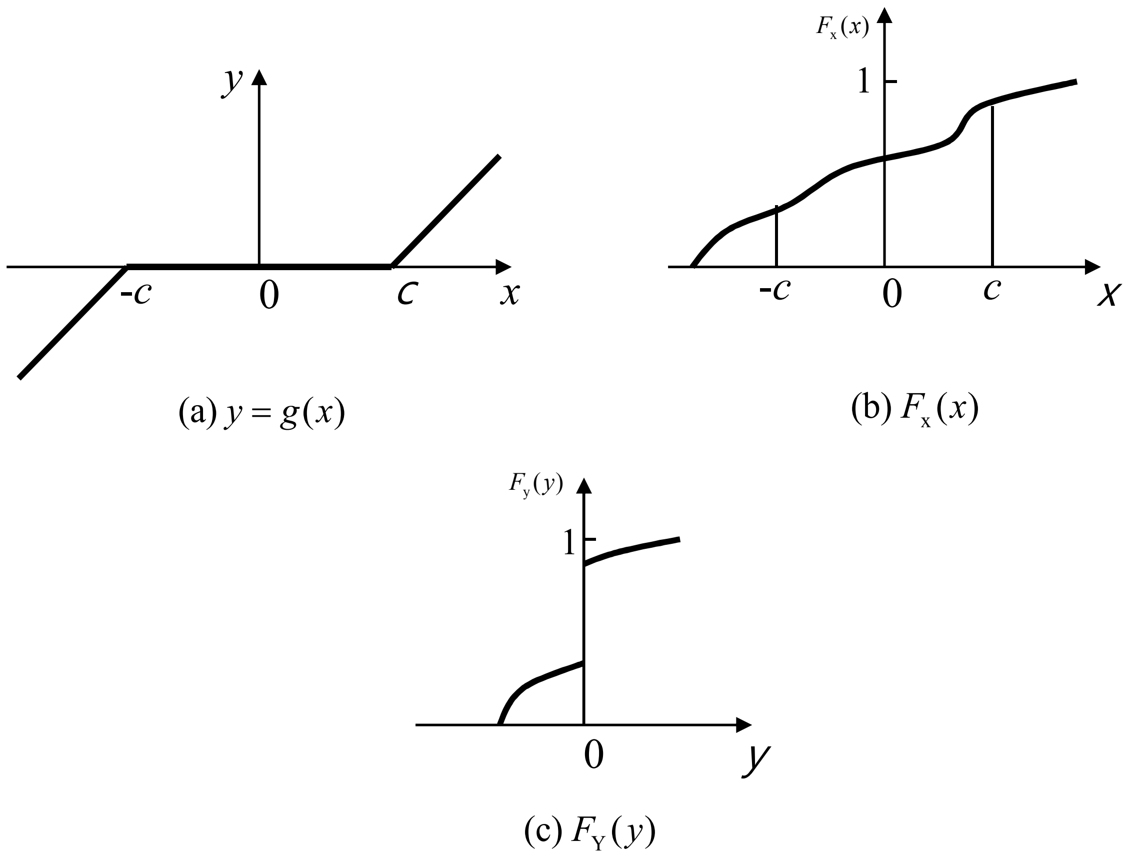


Fig. A Continuous But Not Smooth Function and Its Distribution Function

3. Discontinuous, Staircase Function, **g(x)**

Assume that $g(x)$ is a discontinuous, staircase function

$$g(x) = g(x_i) = y_i \text{ for } x_{i-1} < x < x_i.$$

In this case, the random variable $\mathrm{y} = g(\mathrm{x})$ is of discrete type taking the values y_i with

$$P[\mathrm{y} = y_i] = P[x_{i-1} < \mathrm{x} \le \mathrm{x_i}] = F_\mathrm{x}(x_i) - F_\mathrm{x}(x_{i-1}).$$

Example (Hard Limiter Function)

If

$$g(x) = \begin{cases} 1 \ \text{ for } x > 0 \\ -1 \ \text{ for } x \le 0, \end{cases}$$

then, y takes the values ± 1 with

$$P[\mathrm{y} = -1] = P[\mathrm{x} \le 0] = F_\mathrm{x}(0)$$

$$P[\mathrm{y} = 1] = P[\mathrm{x} > 0] = 1 - F_\mathrm{x}(0).$$

Hence, $F_\mathrm{y}(y)$ is a staircase function as in Fig. 3.7-5.

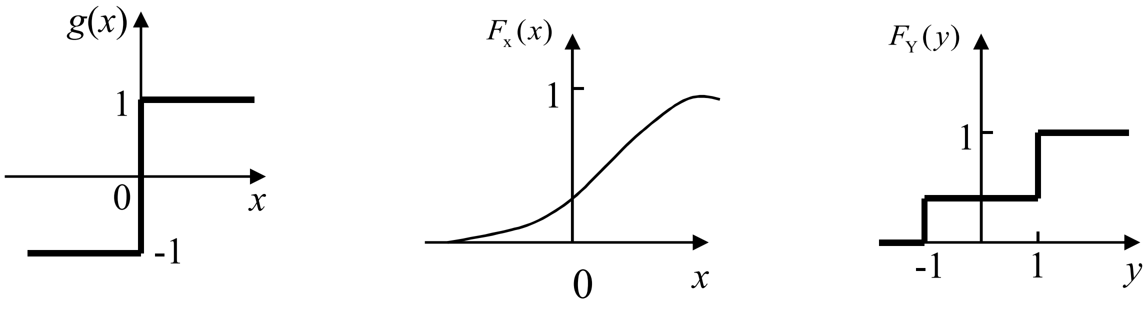


Figure 3.7-5. The Hard Limiter Function and Its Distribution Function

Determination of $f_{\mathrm{y}}(y)$

We wish to determine the density of $\mathrm{Y} = g(X)$ in terms of the density of X.
Suppose, first, that the set R of the y-axis is not in the range of the function $g(x)$.
In other words, $g(x)$ is not a point of R for any x. In this case, the probability that $g(x)$ is in R equals 0. Hence, $f_{\mathrm{y}}(y) = 0$ for $y \in R$. It suffices, therefore, to consider the values of y such that for some x, $g(x) = y$.

Fundamental Theorem

We are given a continuous r.v. X with pdf $f_X(x)$ and a differentiable function $g(x)$ of the real variable x. What is the pdf of $Y = g(X)$?

Theorem: Let $x_1, x_2, \cdots, x_n, \cdots$ be the real values satisfying $y = g(x)$. Then,

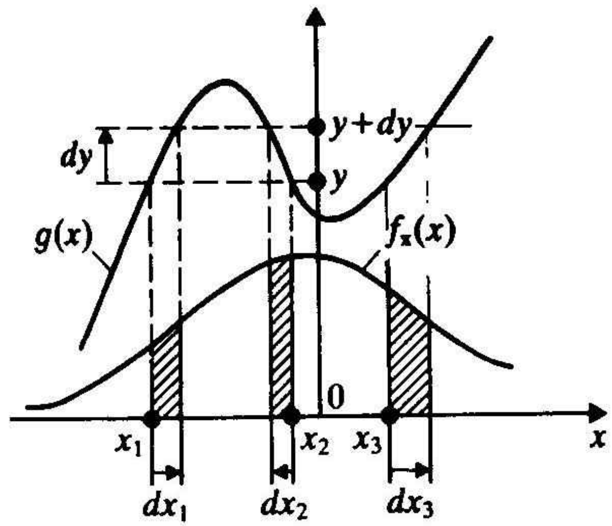
$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|} + \cdots$$

where $g'(x_i) = \left. \dfrac{dg(x)}{dx} \right|_{x = x_i}$.

(Proof)

For simplicity, we prove the case that $y = g(x)$ has three roots, $x_1, x_2,$ and x_3.

$$
\begin{aligned}
f_Y(y)\,dy = P\{\ y < Y \le y + dy\ \} \\
&= P\{\ x_1 < X \le x_1 + dx_1\ \} \\
&\quad + P\{\ x_2 + dx_2 < X \le x_2 \} \\
&\quad + P\{\ x_3 < X \le x_3 + dx_3\ \} \\
&= f_X(x_1)|dx_1| + f_X(x_2)|dx_2| + f_X(x_3)|dx_3| \\
&\qquad \text{for small } dx_i \\
&= \frac{f_X(x_1)}{|g'(x_1)|}dy + \frac{f_X(x_2)}{|g'(x_2)|}dy + \frac{f_X(x_3)}{|g'(x_3)|}dy
\end{aligned}
$$



where

$$
g'(x_1) = \left.\frac{dy}{dx}\right|_{x=x_1}.
$$

Example ($Y = aX + b$ and Its Density Function)

Given

$$f_X(x) = N\left(m , \sigma^2 \right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}$$

$$Y = aX + b$$

find $f_Y(y)$.

(Solution)

Apply the fundamental theorem and we obtain

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} \text{ where } x_1 = \frac{y-b}{a}.$$

Considering the following equation

$$g'(x_1) = \frac{d}{dx}(ax+b)\Big|_{x_1} = a$$

.

we get,

$$
\begin{aligned}
f_Y(y) &= \frac{f_X\left(\dfrac{y-b}{a}\right)}{|a|} \\
&= \frac{1}{\sqrt{2\pi}\sigma|a|} e^{-(\frac{y-b}{a}-m)^2 \Big/ 2\sigma^2} \\
&= \frac{1}{\sqrt{2\pi}\,|a\sigma|} e^{-(y-b-a\ m)^2 \Big/ 2(a\sigma)^2}.
\end{aligned}
$$

Note that f_Y is the Gaussian with mean $(b+am)$ and standard deviation $|a\sigma|$, which illustrates that a linear transformation of Gaussian r.v. is also Gaussian.

Example ($y = ax^2$ and Its Density Function)

Given

$$Y = a\,X^2\,, \qquad a > 0$$

$$X = N\,(\,0\,, \sigma_X{}^2\,)$$

Find $f_Y(y)$.

(Answer)

Solving $y = ax^2$ for x gives

$$x_1 = \sqrt{\frac{y}{a}},\, x_2 = -\sqrt{\frac{y}{a}}.$$

And

$$|\,g'(x_1)\,| = |\,2a\,x_1\,| = \left|2\sqrt{ay}\right| = 2\sqrt{ay}$$

$$|\,g'(x_2)\,| = |\,2a\,x_2\,| = \left|-2\sqrt{ay}\right| = 2\sqrt{ay}.$$

Now

$$
\begin{aligned}
f_Y(y) &= \frac{f_X(x_1)}{\left|g'(x_1)\right|} + \frac{f_X(x_2)}{\left|g'(x_2)\right|} \\
&= \frac{1}{2\sqrt{ay}}\left[f_X(\sqrt{\tfrac{y}{a}}) + f_X(-\sqrt{\tfrac{y}{a}}) \right], \quad y > 0 \\
&= \frac{1}{2\sqrt{ay}} \cdot \frac{1}{\sqrt{2\pi}\sigma_X}\left[e^{-\left(\sqrt{y/a}\right)^2 / 2\sigma_X{}^2} + e^{-\left(-\sqrt{y/a}\right)^2 / 2\sigma_X{}^2} \right] \\
&= \frac{1}{\sqrt{\pi} \cdot \sqrt{2a\sigma_X{}^2} \cdot \sqrt{y}} e^{-\frac{y}{2a\sigma_X{}^2}} U(y). \qquad (1)
\end{aligned}
$$

Gamma density function is defined by

$$
f_V(v) = \frac{\lambda^\alpha v^{\alpha-1} e^{-\lambda v}}{\Gamma(\alpha)} \; U(v). \qquad (2)
$$

Comparing Eqs. (1) and (2), we can tell that $f_Y(y)$ is the Gamma density function

with $\lambda = \dfrac{1}{2a\sigma_X{}^2}$ and $\alpha = \dfrac{1}{2}$.

Note: 1) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

2) For $a=1$ and $\sigma_X{}^2=1$, with $\lambda=\frac{1}{2}$ and $\alpha=\frac{1}{2}$,

$$f_Y(y) \;=\; \frac{(y)^{-1/2}\,e^{-y/2}}{2^{1/2}\,\Gamma\left(1/2\right)}.$$

This is the chi - square density with 1 degree of freedom.

Section 3.8

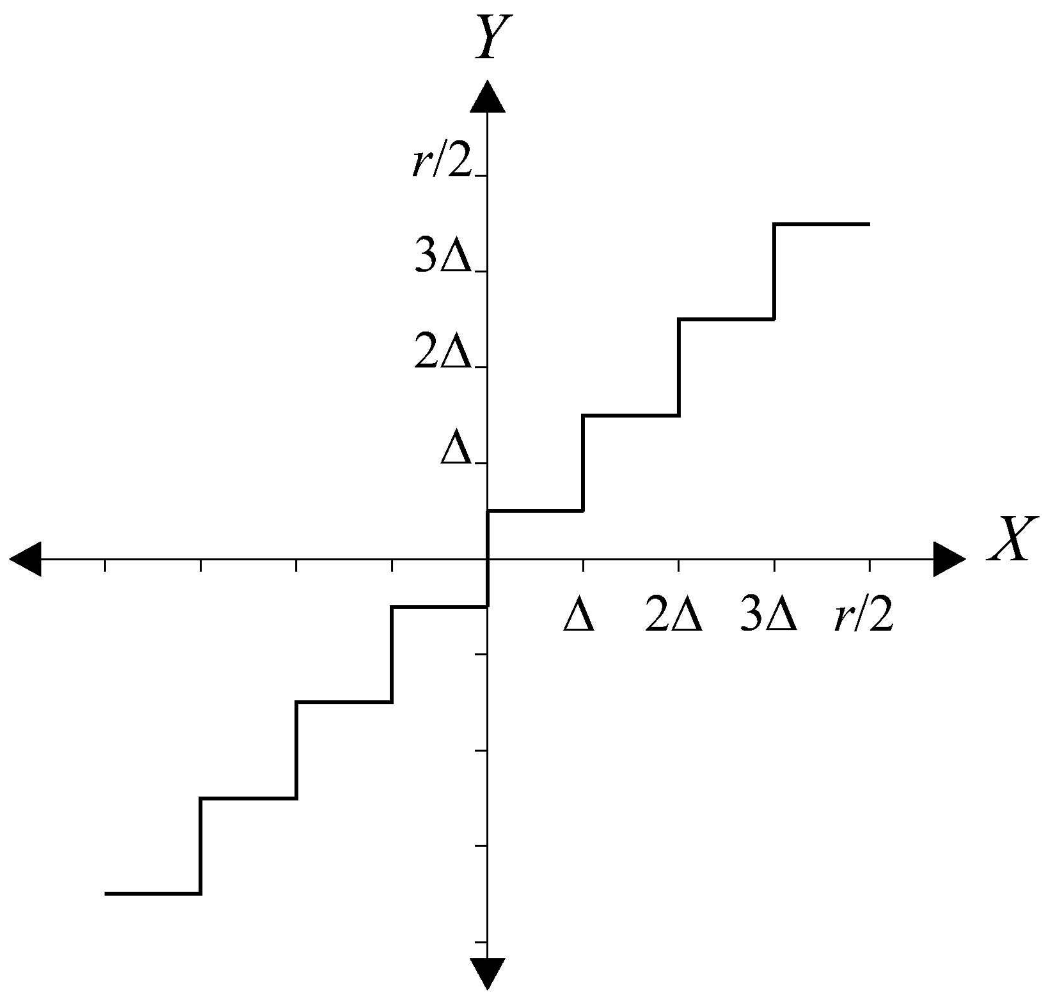
Conditioning a Continuous Random Variable

Definition 3.15 Conditional PDF given an Event

For a random variable X *with PDF* $f_X(x)$ *and an event* $B \subset S_X$ *with* $P[B] > 0$*, the* conditional PDF of X given B *is*

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]} & x \in B, \\ 0 & \textit{otherwise.} \end{cases}$$

Figure 3.9



Example 3.30 Problem

For the wheel-spinning experiment of Example 3.1, find the conditional PDF of the pointer position for spins in which the pointer stops on the left side of the circle. What are the conditional expected value and the conditional standard deviation?

Example 3.30 Solution

Let L denote the left side of the circle. In terms of the stopping position, $L = [1/2, 1)$. Recalling from Example 3.4 that the pointer position X has a uniform PDF over $[0, 1)$,

$$P[L] = \int_{1/2}^{1} f_X(x)\, dx = \int_{1/2}^{1} dx = 1/2.$$

Therefore,

$$f_{X|L}(x) = \begin{cases} 2 & 1/2 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.31 Problem

The uniform $(-r/2, r/2)$ random variable X is processed by a b-bit uniform quantizer to produce the quantized output Y. Random variable X is rounded to the nearest quantizer level. With a b-bit quantizer, there are $n = 2^b$ quantization levels. The quantization step size is $\Delta = r/n$, and Y takes on values in the set

$$Q_Y = \{y_i = \Delta/2 + i\,\Delta | i = -n/2, -n/2+1, \ldots, n/2-1\}.$$

This relationship is shown for $b = 3$ in Figure 3.9. Given the event B_i that $Y = y_i$, find the conditional PDF of X given B_i.

Example 3.31 Solution

In terms of X, we observe that $B_i = \{i\Delta \leq X < (i+1)\Delta\}$. Thus,

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} f_X(x)\, dx = \frac{\Delta}{r} = \frac{1}{n}.$$

By Definition 3.15,

$$f_{X|B_i}(x) = \begin{cases} \dfrac{f_X(x)}{P[B_i]} & x \in B_i, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1/\Delta & i\Delta \leq x < (i+1)\Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Given B_i, the conditional PDF of X is uniform over the ith quantization interval.

Theorem 3.23

Given an event space $\{B_i\}$ and the conditional PDFs $f_{X|B_i}(x)$,

$$f_X(x) = \sum_i f_{X|B_i}(x)\, P[B_i].$$

Example 3.32 Problem

Continuing Example 3.3, when symbol "0" is transmitted (event B_0), X is the Gaussian $(-5, 2)$ random variable. When symbol "1" is transmitted (event B_1), X is the Gaussian $(5, 2)$ random variable. Given that symbols "0" and "1" are equally likely to be sent, what is the PDF of X?

Example 3.32 Solution

The problem statement implies that $P[B_0] = P[B_1] = 1/2$ and

$$f_{X|B_0}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x+5)^2/8}, \qquad f_{X|B_1}(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x-5)^2/8}.$$

By Theorem 3.23,

$$\begin{aligned} f_X(x) &= f_{X|B_0}(x)\, P\left[B_0\right] + f_{X|B_1}(x)\, P\left[B_1\right] \\ &= \frac{1}{4\sqrt{2\pi}} \left(e^{-(x+5)^2/8} + e^{-(x-5)^2/8} \right). \end{aligned}$$

Problem 3.9.2 asks the reader to graph $f_X(x)$ to show its similarity to Figure 3.3.

Conditional Expected Value

Definition 3.16 Given an Event

If $\{x \in B\}$, *the conditional expected value of* X *is*

$$E[X|B] = \int_{-\infty}^{\infty} x f_{X|B}(x)\, dx.$$

Example 3.33 Problem

Continuing the wheel spinning of Example 3.30, find the conditional expected value and the conditional standard deviation of the pointer position X given the event L that the pointer stops on the left side of the circle.

Example 3.33 Solution

The conditional expected value and the conditional variance are

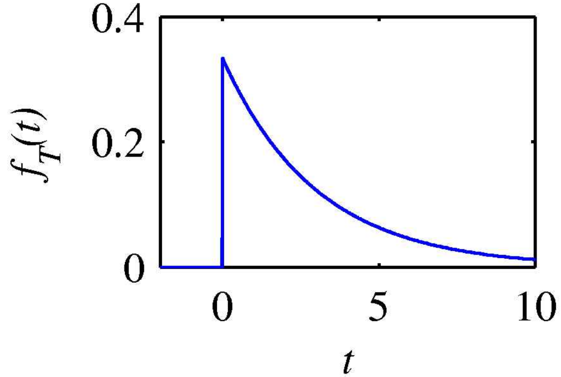
$$E[X|L] = \int_{-\infty}^{\infty} x f_{X|L}(x)\, dx = \int_{1/2}^{1} 2x\, dx = 3/4 \text{ meters.}$$

$$\text{Var}[X|L] = E\left[X^2|L\right] - (E[X|L])^2 = \frac{7}{12} - \left(\frac{3}{4}\right)^2 = 1/48 \text{ m}^2.$$

The conditional standard deviation is $\sigma_{X|L} = \sqrt{\text{Var}[X|L]} = 0.144$ meters. Example 3.9 derives $\sigma_X = 0.289$ meters. That is, $\sigma_X = 2\sigma_{X|L}$. It follows that learning that the pointer is on the left side of the circle leads to a set of typical values that are within 0.144 meters of 0.75 meters. Prior to learning which half of the circle the pointer is in, we had a set of typical values within 0.289 of 0.5 meters.

Example 3.34 Problem

Suppose the duration T (in minutes) of a telephone call is an exponential $(1/3)$ random variable:



$$f_T(t) = \begin{cases} (1/3)e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

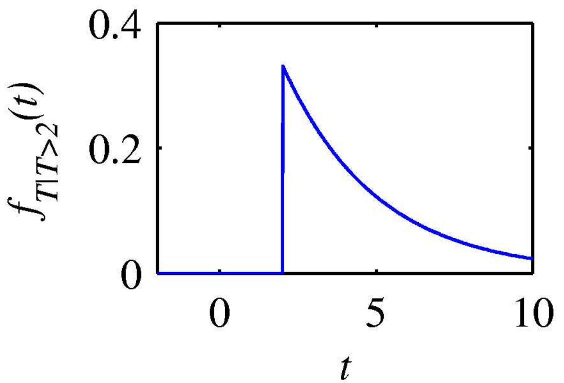
For calls that last at least 2 minutes, what is the conditional PDF of the call duration?

Example 3.34 Solution

In this case, the conditioning event is $T > 2$. The probability of the event is

$$P[T > 2] = \int_2^\infty f_T(t)\, dt = e^{-2/3}.$$

The conditional PDF of T given $T > 2$ is



$$f_{T|T>2}(t) = \begin{cases} \frac{f_T(t)}{P[T>2]} & t > 2, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} \frac{1}{3}e^{-(t-2)/3} & t > 2, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $f_{T|T>2}(t)$ is a time-shifted version of $f_T(t)$. In particular, $f_{T|T>2}(t) = f_T(t-2)$. An interpretation of this result is that if the call is in progress after 2 minutes, the duration of the call is 2 minutes plus an exponential time equal to the duration of a new call.