# Module #2:
# **Basic Proof Methods**

Rosen 5th ed., §§1.5 & 3.1

29 slides, ~2 lectures

# Nature & Importance of Proofs

- In mathematics, a *proof* is:
  - a *correct* (well-reasoned, logically valid) and *complete* (clear, detailed) argument that rigorously & undeniably establishes the truth of a mathematical statement.
- Why must the argument be correct & complete?
  - *Correctness* prevents us from fooling ourselves.
  - *Completeness* allows anyone to verify the result.
- In this course (& throughout mathematics), a very high standard for correctness and completeness of proofs is demanded!!

# Overview of §§1.5 & 3.1

- Methods of mathematical argument (*i.e.*, proof methods) can be formalized in terms of *rules of logical inference*.
- Mathematical *proofs* can themselves be represented formally as discrete structures.
- We will review both correct & fallacious inference rules, & several proof methods.

# Applications of Proofs

- An exercise in clear communication of logical arguments in any area of study.
- The fundamental activity of mathematics is the discovery and elucidation, through proofs, of interesting new theorems.
- Theorem-proving has applications in program verification, computer security, automated reasoning systems, *etc.*
- Proving a theorem allows us to rely upon on its correctness even in the most critical scenarios.

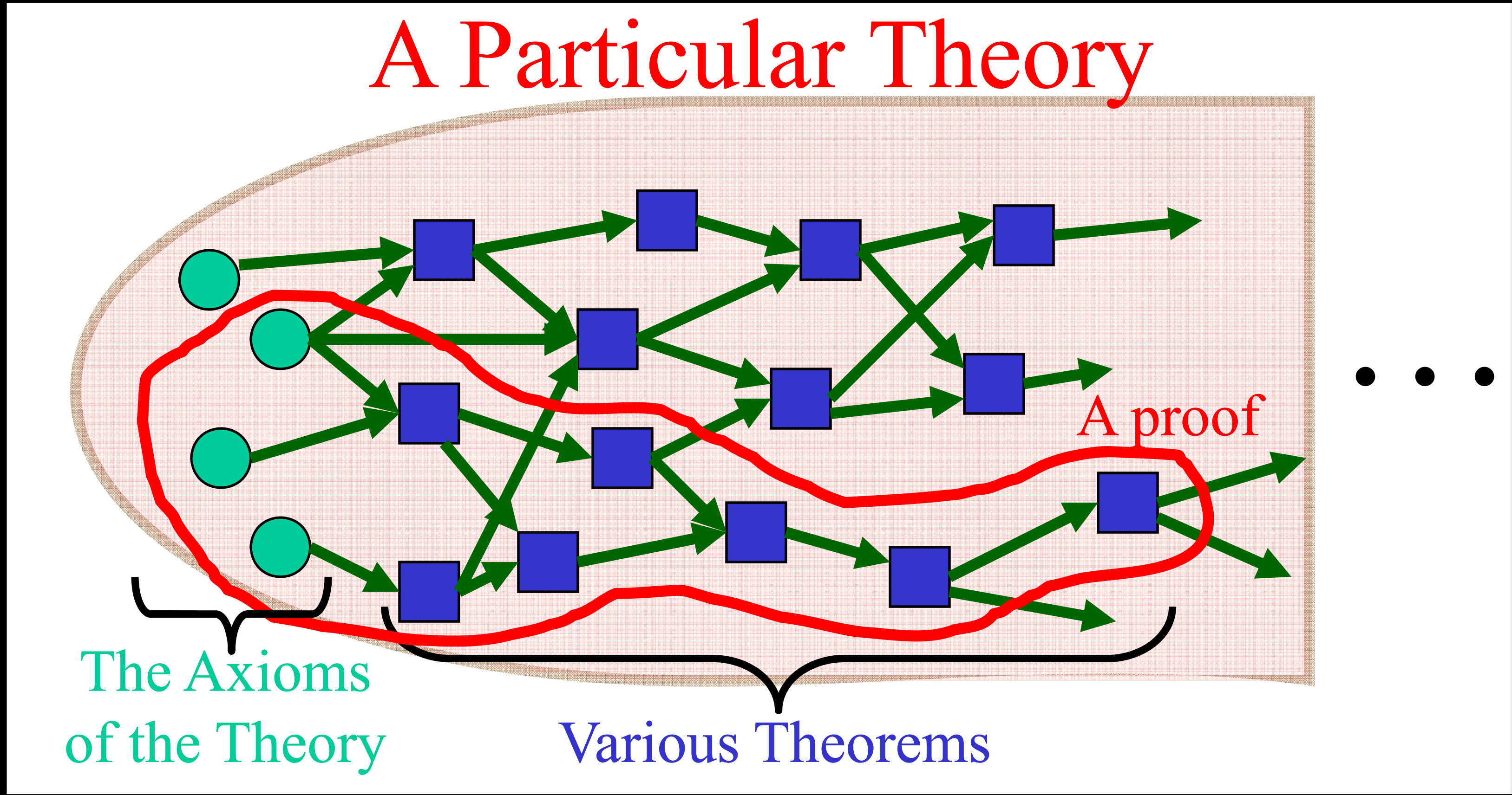
# Proof Terminology

- *Theorem*
  - A statement that has been proven to be true.
- *Axioms*, *postulates*, *hypotheses*, *premises*
  - Assumptions (often unproven) defining the structures about which we are reasoning.
- *Rules of inference*
  - Patterns of logically valid deductions from hypotheses to conclusions.

# More Proof Terminology

- *Lemma* - A minor theorem used as a stepping-stone to proving a major theorem.
- *Corollary* - A minor theorem proved as an easy consequence of a major theorem.
- *Conjecture* - A statement whose truth value has not been proven. (A conjecture may be widely believed to be true, regardless.)
- *Theory* – The set of all theorems that can be proven from a given set of axioms.

# Graphical Visualization



# Inference Rules - General Form

- *Inference Rule* –
  - Pattern establishing that if we know that a set of *antecedent* statements of certain forms are all true, then a certain related *consequent* statement is true.
- *antecedent 1*
  *antecedent 2 …*
  ∴ *consequent*

"∴" means "therefore"

# Inference Rules & Implications

- Each logical inference rule corresponds to an implication that is a tautology.
- $\dfrac{\textit{antecedent 1} \\ \textit{antecedent 2} \ \ldots}{\therefore\ \textit{consequent}}$ Inference rule
- Corresponding tautology:

  $((\textit{ante. 1}) \wedge (\textit{ante. 2}) \wedge \ldots) \rightarrow \textit{consequent}$

# Some Inference Rules

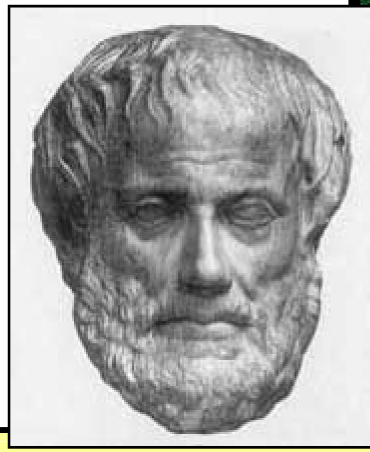
- $$\frac{p}{\therefore\ p \lor q}$$ Rule of Addition
- $$\frac{p \land q}{\therefore\ p}$$ Rule of Simplification
- $$\begin{array}{c} p \\ q \\ \hline \therefore\ p \land q \end{array}$$ Rule of Conjunction

# Modus Ponens & Tollens

- $$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$
  Rule of *modus ponens* (a.k.a. *law of detachment*)
  "the mode of affirming"
- $$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$$
  Rule of *modus tollens*
  "the mode of denying"

# Syllogism Inference Rules

- $$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$ Rule of hypothetical syllogism
- $$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array}$$ Rule of disjunctive syllogism



Aristotle
(ca. 384-322 B.C.)

# Formal Proofs

- A formal proof of a conclusion *C*, given premises $p_1, p_2, \ldots, p_n$ consists of a sequence of *steps*, each of which applies some inference rule to premises or to previously-proven statements (as antecedents) to yield a new true statement (the consequent).
- A proof demonstrates that *if* the premises are true, *then* the conclusion is true.

# Formal Proof Example

- Suppose we have the following premises:
  **"It is not sunny and it is cold."**
  **"We will swim only if it is sunny."**
  **"If we do not swim, then we will canoe."**
  **"If we canoe, then we will be home early."**
- Given these premises, prove the theorem **"We will be home early"** using inference rules.

# Proof Example *cont.*

- Let us adopt the following abbreviations:
  - *sunny* = **"It is sunny"**; *cold* = **"It is cold"**; *swim* = **"We will swim"**; *canoe* = **"We will canoe"**; *early* = **"We will be home early"**.
- Then, the premises can be written as:

(1) $\neg sunny \wedge cold$ (2) $swim \rightarrow sunny$
(3) $\neg swim \rightarrow canoe$ (4) $canoe \rightarrow early$

# Proof Example *cont.*

| Step | Proved by |
|---|---|
| 1. $\neg sunny \wedge cold$ | Premise #1. |
| 2. $\neg sunny$ | Simplification of 1. |
| 3. $swim \rightarrow sunny$ | Premise #2. |
| 4. $\neg swim$ | Modus tollens on 2,3. |
| 5. $\neg swim \rightarrow canoe$ | Premise #3. |
| 6. $canoe$ | Modus ponens on 4,5. |
| 7. $canoe \rightarrow early$ | Premise #4. |
| 8. $early$ | Modus ponens on 6,7. |

# Inference Rules for Quantifiers

- $\dfrac{\forall x\, P(x)}{\therefore P(o)}$ Universal instantiation (substitute *any* object *o*)
- $\dfrac{P(g)}{\therefore \forall x\, P(x)}$ (for *g* a *general* element of u.d.) Universal generalization
- $\dfrac{\exists x\, P(x)}{\therefore P(c)}$ Existential instantiation (substitute a *new constant c*)
- $\dfrac{P(o)}{\therefore \exists x\, P(x)}$ (substitute any extant object *o*) Existential generalization

# Common Fallacies

- A *fallacy* is an inference rule or other proof method that is not logically valid.
  - May yield a false conclusion!
- Fallacy of *affirming the conclusion*:
  - "$p \rightarrow q$ is true, and $q$ is true, so $p$ must be true." (No, because **F**→**T** is true.)
- Fallacy of *denying the hypothesis*:
  - "$p \rightarrow q$ is true, and $p$ is false, so $q$ must be false." (No, again because **F**→**T** is true.)

# Circular Reasoning

- The fallacy of (explicitly or implicitly) assuming the very statement you are trying to prove in the course of its proof. Example:
- Prove that an integer $n$ is even, if $n^2$ is even.
- Attempted proof: "Assume $n^2$ is even. Then $n^2=2k$ for some integer $k$. Dividing both sides by $n$ gives $n = (2k)/n = 2(k/n)$. So there is an integer $j$ (namely $k/n$) such that $n=2j$. Therefore $n$ is even."

*Begs the question: How do you show that $j=k/n=n/2$ is an integer, without first assuming $n$ is even?*

# Removing the Circularity

Suppose $n^2$ is even $\therefore 2|n^2 \ \therefore \ n^2 \bmod 2 = 0$. Of course $n \bmod 2$ is either 0 or 1. If it's 1, then $n \equiv 1 \pmod 2$, so $n^2 \equiv 1 \pmod 2$, using the theorem that if $a \equiv b \pmod m$ and $c \equiv d \pmod m$ then $ac \equiv bd$ (mod m), with $a=c=n$ and $b=d=1$. Now $n^2 \equiv 1 \pmod 2$ implies that $n^2 \bmod 2 = 1$. So by the hypothetical syllogism rule, ($n \bmod 2 = 1$) implies ($n^2 \bmod 2 = 1$). Since we know $n^2 \bmod 2 = 0 \neq 1$, by *modus tollens* we know that $n \bmod 2 \neq 1$. So by disjunctive syllogism we have that $n \bmod 2 = 0 \ \therefore 2|n \ \therefore \ n$ is even.

# Proof Methods for Implications

For proving implications $p \rightarrow q$, we have:

- *Direct* proof: Assume $p$ is true, and prove $q$.
- *Indirect* proof: Assume $\neg q$, and prove $\neg p$.
- *Vacuous* proof: Prove $\neg p$ by itself.
- *Trivial* proof: Prove $q$ by itself.
- Proof by cases:
  Show $p \rightarrow (a \vee b)$, and $(a \rightarrow q)$ and $(b \rightarrow q)$.

# Direct Proof Example

- **Definition:** An integer $n$ is called *odd* iff $n=2k+1$ for some integer $k$; $n$ is *even* iff $n=2k$ for some $k$.
- **Axiom:** Every integer is either odd or even.
- **Theorem:** (For all numbers $n$) If $n$ is an odd integer, then $n^2$ is an odd integer.
- **Proof:** If $n$ is odd, then $n = 2k+1$ for some integer $k$. Thus, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Therefore $n^2$ is of the form $2j + 1$ (with $j$ the integer $2k^2 + 2k$), thus $n^2$ is odd. □

# Indirect Proof Example

- **Theorem:** (For all integers $n$)
  If $3n+2$ is odd, then $n$ is odd.
- **Proof:** Suppose that the conclusion is false, *i.e.*, that $n$ is even. Then $n=2k$ for some integer $k$. Then $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$. Thus $3n+2$ is even, because it equals $2j$ for integer $j = 3k+1$. So $3n+2$ is not odd. We have shown that $\neg(n \text{ is odd}) \rightarrow \neg(3n+2 \text{ is odd})$, thus its contra-positive $(3n+2 \text{ is odd}) \rightarrow (n \text{ is odd})$ is also true. □

# Vacuous Proof Example

- **Theorem:** (For all $n$) If $n$ is both odd and even, then $n^2 = n + n$.
- **Proof:** The statement "$n$ is both odd and even" is necessarily false, since no number can be both odd and even. So, the theorem is vacuously true. □

# Trivial Proof Example

- **Theorem:** (For integers *n*) If *n* is the sum of two prime numbers, then either *n* is odd or *n* is even.
- **Proof:** *Any* integer *n* is either odd or even. So the conclusion of the implication is true regardless of the truth of the antecedent. Thus the implication is true trivially. □

# Proof by Contradiction

- A method for proving $p$.
- Assume $\neg p$, and prove both $q$ and $\neg q$ for some proposition $q$.
- Thus $\neg p \rightarrow (q \wedge \neg q)$
- $(q \wedge \neg q)$ is a trivial contradition, equal to **F**
- Thus $\neg p \rightarrow \mathbf{F}$, which is only true if $\neg p = \mathbf{F}$
- Thus $p$ is true.

# Review: Proof Methods So Far

- *Direct*, *indirect*, *vacuous*, and *trivial* proofs of statements of the form $p \rightarrow q$.
- *Proof by contradiction* of any statements.
- Next: *Constructive* and *nonconstructive existence proofs*.

# Proving Existentials

- A proof of a statement of the form $\exists x\, P(x)$ is called an *existence proof*.
- If the proof demonstrates how to actually find or construct a specific element $a$ such that $P(a)$ is true, then it is a *constructive* proof.
- Otherwise, it is *nonconstructive*.

# Constructive Existence Proof

- **Theorem:** There exists a positive integer $n$ that is the sum of two perfect cubes in two different ways:
  - equal to $j^3 + k^3$ and $l^3 + m^3$ where $j, k, l, m$ are positive integers, and $\{j,k\} \neq \{l,m\}$
- **Proof:** Consider $n = 1729$, $j = 9$, $k = 10$, $l = 1$, $m = 12$. Now just check that the equalities hold.

# Another Constructive Existence Proof

- **Theorem:** For any integer $n>0$, there exists a sequence of $n$ consecutive composite integers.
- Same statement in predicate logic: $\forall n>0\ \exists x\ \forall i\ (1\leq i\leq n)\rightarrow(x+i$ is composite$)$
- Proof follows on next slide…

# The proof...

- Given $n>0$, let $x = (n + 1)! + 1$.
- Let $i \geq 1$ and $i \leq n$, and consider $x+i$.
- Note $x+i = (n + 1)! + (i + 1)$.
- Note $(i+1)|(n+1)!$, since $2 \leq i+1 \leq n+1$.
- Also $(i+1)|(i+1)$. So, $(i+1)|(x+i)$.
- $\therefore$ $x+i$ is composite.
- $\therefore$ $\forall n\ \exists x\ \forall 1\leq i\leq n : x+i$ is composite. Q.E.D.

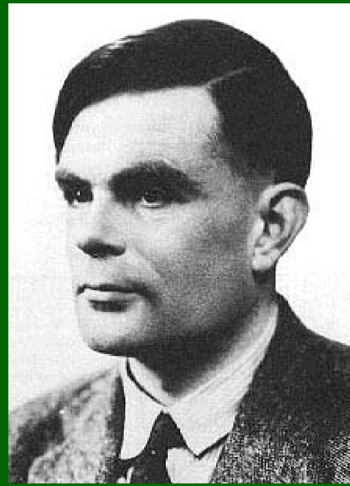
# Nonconstructive Existence Proof

- **Theorem:**
  "There are infinitely many prime numbers."
- Any finite set of numbers must contain a maximal element, so we can prove the theorem if we can just show that there is *no* largest prime number.
- *I.e.*, show that for any prime number, there is a larger number that is *also* prime.
- More generally: For *any* number, ∃ a larger prime.
- Formally: Show $\forall n\ \exists p>n : p$ is prime.

# The proof, using *proof by cases*...

- Given $n>0$, prove there is a prime $p>n$.
- Consider $x = n!+1$. Since $x>1$, we know ($x$ is prime)$\vee$($x$ is composite).
- **Case 1:** $x$ is prime. Obviously $x>n$, so let $p=x$ and we're done.
- **Case 2:** $x$ has a prime factor $p$. But if $p\leq n$, then $p$ mod $x = 1$. So $p>n$, and we're done.

# The Halting Problem (Turing'36)



Alan Turing
1912-1954

- The *halting problem* was the first mathematical function proven to have *no* algorithm that computes it!
  - We say, it is *uncomputable.*
- The desired function is *Halts*(*P*,*I*) :≡ the truth value of this statement:
  - *"Program P, given input I, eventually terminates."*
- **Theorem:** *Halts* is uncomputable!
  - I.e., There does *not* exist *any* algorithm *A* that computes *Halts* correctly for *all* possible inputs.
- Its proof is thus a *non*-existence proof.
- Corollary: General impossibility of predictive analysis of arbitrary computer programs.

# The Proof

- Given any *arbitrary* program *H*(*P*,*I*),
- Consider algorithm *Breaker*, defined as:

  **procedure** *Breaker*(*P*: a program)
  
  *halts* **:=** *H*(*P*,*P*)
  
  **if** *halts* **then while T begin end**

- Note that *Breaker*(*Breaker*) halts iff *H*(*Breaker*,*Breaker*) = **F**.
- So *H* does **not** compute the function *Halts*!

Breaker makes a liar out of *H*, by doing the opposite of whatever *H* predicts.

# Limits on Proofs

- Some very simple statements of number theory haven't been proved or disproved!
  - *E.g. Goldbach's conjecture*: Every integer $n \geq 2$ is exactly the average of some two primes.
  - $\forall n \geq 2$ $\exists$ primes $p,q$: $n=(p+q)/2$.
- There are true statements of number theory (or any sufficiently powerful system) that can *never* be proved (or disproved) (Gödel).

# More Proof Examples

- Quiz question 1a: Is this argument correct or incorrect?
  - "All TAs compose easy quizzes. Ramesh is a TA. Therefore, Ramesh composes easy quizzes."
- First, separate the premises from conclusions:
  - Premise #1: All TAs compose easy quizzes.
  - Premise #2: Ramesh is a TA.
  - Conclusion: Ramesh composes easy quizzes.

# Answer

Next, re-render the example in logic notation.

- Premise #1: All TAs compose easy quizzes.
  - Let U.D. = all people
  - Let $T(x)$ :≡ "$x$ is a TA"
  - Let $E(x)$ :≡ "$x$ composes easy quizzes"
  - Then Premise #1 says: $\forall x,\ T(x) \rightarrow E(x)$

# Answer cont…

- Premise #2: Ramesh is a TA.
  - Let R :≡ Ramesh
  - Then Premise #2 says: $T$(R)
  - And the Conclusion says: $E$(R)
- The argument is correct, because it can be reduced to a sequence of applications of valid inference rules, as follows:

# The Proof in Gory Detail

| Statement | How obtained |
|---|---|
| 1. $\forall x, T(x) \rightarrow E(x)$ | **(Premise #1)** |
| *2.* $T(\text{Ramesh}) \rightarrow E(\text{Ramesh})$ | **(Universal instantiation)** |
| *3.* $T(\text{Ramesh})$ | **(Premise #2)** |
| *4.* $E(\text{Ramesh})$ | **(*Modus Ponens* from statements #2 and #3)** |

# Another example

- Quiz question 2b: Correct or incorrect: At least one of the 280 students in the class is intelligent. Y is a student of this class. Therefore, Y is intelligent.
- First: Separate premises/conclusion, & translate to logic:
  - Premises: (1) $\exists x$ InClass($x$) $\wedge$ Intelligent($x$)
    (2) InClass(Y)
  - Conclusion: Intelligent(Y)

# Answer

- No, the argument is invalid; we can disprove it with a counter-example, as follows:
- Consider a case where there is only one intelligent student X in the class, and X≠Y.
  - Then the premise ∃***x*** **InClass(*x*)** ∧ **Intelligent(*x*)** is true, by existential generalization of **InClass(X)** ∧ **Intelligent(X)**
  - But the conclusion **Intelligent(Y)** is false, since X is the only intelligent student in the class, and Y≠X.
- Therefore, the premises *do not* imply the conclusion.

# Another Example

- Quiz question #2: Prove that the sum of a rational number and an irrational number is always irrational.
- First, you have to understand exactly what the question is asking you to prove:
  - "For all real numbers *x*,*y*, if *x* is rational and *y* is irrational, then *x*+*y* is irrational."
  - $\forall x,y$: Rational($x$) $\wedge$ Irrational($y$) $\rightarrow$ Irrational($x+y$)

# Answer

- Next, think back to the definitions of the terms used in the statement of the theorem:
  - $\forall$ reals $r$: Rational($r$) $\leftrightarrow$ $\exists$ Integer($i$) $\wedge$ Integer($j$): $r = i/j$.
  - $\forall$ reals $r$: Irrational($r$) $\leftrightarrow$ $\neg$Rational($r$)
- You almost always need the definitions of the terms in order to prove the theorem!
- Next, let's go through one valid proof:

# What you might write

- **Theorem:**
  $\forall x,y$: Rational($x$) $\land$ Irrational($y$) $\rightarrow$ Irrational($x+y$)
- **Proof:** Let $x$, $y$ be any rational and irrational numbers, respectively. … (universal generalization)
- Now, just from this, what do we know about $x$ and $y$? You should think back to the definition of rational:
- … Since $x$ is rational, we know (from the very definition of rational) that there must be some integers $i$ and $j$ such that $x = i/j$. So, let $i_x, j_x$ be such integers …
- We give them unique names so we can refer to them later.

# What next?

- What do we know about $y$? Only that $y$ is irrational: $\neg\exists$ integers $i,j$: $y = i/j$.
- But, it's difficult to see how to use a direct proof in this case. We could try indirect proof also, but in this case, it is a little simpler to just use proof by contradiction (very similar to indirect).
- So, what are we trying to show? Just that $x+y$ is irrational. That is, $\neg\exists i,j$: $(x + y) = i/j$.
- What happens if we hypothesize the negation of this statement?

# More writing…

- Suppose that $x+y$ were not irrational. Then $x+y$ would be rational, so ∃ integers $i,j$: $x+y = i/j$. So, let $i_s$ and $j_s$ be any such integers where $x+y = i_s/ j_s$.
- Now, with all these things named, we can start seeing what happens when we put them together.
- So, we have that $(i_x/j_x) + y = (i_s/j_s)$.
- Observe! We have enough information now that we can conclude something useful about *y*, by solving this equation for it.

# Finishing the proof.

- Solving that equation for $y$, we have:

  $$y = (i_s/j_s) - (i_x/j_x)$$
  $$= (i_s j_x - i_x j_s)/(j_s j_x)$$

  Now, since the numerator and denominator of this expression are both integers, $y$ is (by definition) rational. This contradicts the assumption that $y$ was irrational. Therefore, our hypothesis that $x+y$ is rational must be false, and so the theorem is proved.

# Example wrong answer

- 1 is rational. √2 is irrational. 1+√2 is irrational. Therefore, the sum of a rational number and an irrational number is irrational. (Direct proof.)
- Why does this answer merit no credit?
  - The student attempted to use an example to prove a universal statement. **This is always wrong!**
  - Even as an example, it's incomplete, because the student never even proved that $1+\sqrt{2}$ is irrational!