Seoul National University

457.309 Hydraulics and Laboratory
.05.02 Uncertainty Analysis (2)

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Today’s objectives

- Understanding the propagation or accumulation of errors
- Learning the way to determine the total uncertainty.

Uncertainty Analysis

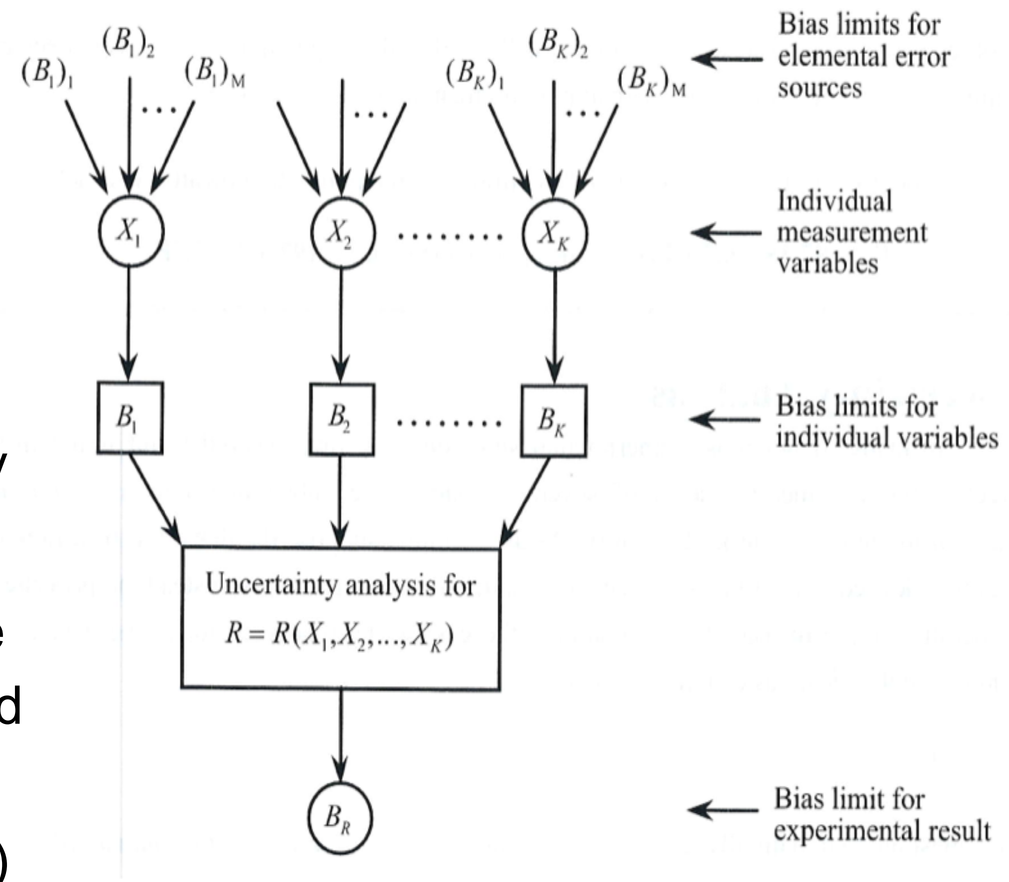
- In many expreimental situations, the final desited result is not measured directly. Instead, measurments of several varaibles are substituted into a *data reduction equation* to obtain the desired quantity.
- As an exmple, suppose the density of a flowing gas stream is dsired.Direct measurements of gas density are difficult, so instead the pressure P and temperature T are measured and, assuming the gas can be treated as ideal, the denisty can be found from the ideal gas euqation of state $P = \rho RT$
- The question that natually arises is: *How do the uncertainites in the individual measured variables P and T propagate through the data reduction equation into the final result for density?*
- The purpose of uncertainty analysis is to answer this important question.

Uncertainty Analysis

- As discussed in previous section, bias is a fixed error that can be reduced by calibration.
- On the other hand, precision error is a random error that can be reduced by obtaining multiple measurements.
- Because of the differing nature of these two components of uncertainty, it is desirable to investigate their propagation into the experimental result separately.
- This approach is consistent with that recommended in the ANSI/ASME Standard on Measurement Uncertainty.

Propagation of Bias Errors

- The steps involved in determining the *bias limit* B_R for the experimental results are sketched to the next.
- Each of the individual measurement variables (X_1, X_2, …, X_K) is subject to several, say M, elemental bias errors.
- The bias limits for each of these elemental sources are combined in some manner to obtain the overall bias limit (B_1, B_2, …, B_K) for each variable.



Propagation of Bias Errors

- Perhaps the most difficult step in this process is the identification and quantification of the element bias limits that affect each of the measurement variables.
- As discussed in an earlier section, element bias error sources can generally be placed in three categories: calibration, data acquisition, and data reduction. However, assigning magnitudes to these sources is not a straightforward task.
- Unlike estimating the precision limit from computation of the precision index S, there is no statistical calculation that can be done to estimate the bias limit. Since it is a fixed error, the bias is the same for each measurement. However, its magnitude, being the difference between the sample mean and the true value, is unknown because the true values is always unknown in any realistic experiment.

Propagation of Bias Errors

- Thus the element bias limits must always be estimated. In addition, the bias limit estimates are made at a 95% confidence level for con sistency with the precision limit determinations. This can be interpr eted to mean that the magnitude of the bias β is less than or equal to the bias limit *B* at a 95% level of confidence.
- Information on bias errors can be inferred from comparison of inde pendent measurements that depend on different physical principle s or that have been independently calibrated.
- Bias limit estimates can also be made based on previous experien ce of the experimenter or other individuals, instrument manufacture r's information and specifications, and comparison of measurement s with known values.

Propagation of Bias Errors

- Once the bias limits for the elemental error sources are estimated, they must be combined in some manner to obtain the bias limit for each measured variable.
- The preferred method for doing this is the root-sum-square (RSS) t echnique. For measurement variable X_K this is given by:

$$B_K = \left[\left(B_K\right)_1^2 + \left(B_K\right)_2^2 + \cdots + \left(B_K\right)_M^2 \right]^{1/2}$$

- Where $B_{K,}$ is the 95% confidence estimate of the bias limit for mea surement.
- The next step in the procedure is to apply uncertainty analysis to d etermine how the bias limits (B_1, B_2,…, B_K) for individual variables propagate through the data reduction equation to form the bias limi t B_R for the experimental result.

Propagation of Bias Errors

- The data reduction equation is taken to be of the form

$$R = R\left(X_1, X_2, \cdots, X_K\right)$$

- Where it is assumed that this relation is continuous and has continuous derivatives in the domain of interest and that the bias limit B_i for the measurement variables are independent of one another. Under these conditions, the bias limit for the result is given by the uncertainty analysis expression.

$$\left(\frac{B_R}{R}\right)^2 = \left(\frac{1}{R}\frac{\partial R}{\partial X_1}B_1\right)^2 + \left(\frac{1}{R}\frac{\partial R}{\partial X_2}B_2\right)^2 + \cdots + \left(\frac{1}{R}\frac{\partial R}{\partial X_K}B_K\right)^2$$

Propagation of Precision Errors

- The procedure for determining the *precision limit, P_R* for an experimental result is similar to that for determining the bias limit B_R. In fact, a sketch of the procedure would look identical to the one given previously for the bias limit with the substitution of "P" for "B" and "precision" for "bias."
- The measurement of each variable (X_1, X_2,…, X_K) is influenced by precision errors from a number of element error sources. These ra ndom errors combine to cause the precision error in the measurem ent of each variable; the latter is quantified by determining the prec ision limit (P_1, P_2,…, P_K) of each measured variable.
- These individual measurement precision limits are then propagate d through the uncertainty analysis to obtain the P_R for the results.

Propagation of Precision Errors

- The manner of determining precision limits of the individual measruement variales depends on the type of experiment and the phase underconsideratin.
- For example, on the *design phase* of a new experiment, before any equipment has been specified or data obtained, *esimtates* for the precision limits are made based on all available infromation: the experimenter’s experience, that of others, manufacturer’s specifications, etc.

Propagation of Precision Errors

- At this stage in the experiment, the precision limit associated with the measurement system may be the only precision error sources considered.
- As a general rule-of-thumb, the precision limit resulting from the readability of an analog instrument can be taken as one-half of the least digit in the output. Likewise, for a digital output, the precision limit associated with the readability is one-half of the least digit in the output.
- For cases in which the precision limit P_i is estimated, the estimate should be that band which will contain the mean value of the variable with 95% confidence.

Propagation of Precision Errors

- During *execution phase* of an experiment, previous measurements may be available with which to determine each of the P_i values. In other cases, multiple measurements of the variable may be made during the actual experiment, from which the precision index S_i and precision limit P_i can be *calculated* from a sample of N readings.
- Several comments may clarify this procedure. From the discussion in preceding sections, recall the appropriate precision limit to use with a variable X_i determined from a *single* reading is the precision index of the sample population times factor taken from the t-distribution table for N<31 or t=2.0 for N>30,

$$P_{X_i} = tS_{X_i}.$$

Propagation of Precision Errors

- Of course, for a single reading P_{xi} must be estimated or must be available from previous measurements. As discussed previously, the $\pm P_{xi}$ band around the measurement X_i contains the mean value of the measured variable with 95% confidence. Therefore, in the uncertainty analysis equations given below X_i and P_i should be interpreted as X_i and P_{X_i}
  when the value of X_i used in the data reduction equation is determined from a single reading.
- When the value of the measurement is determined as the mean $\bar{X}_i$ of N separate readings, then the precision limit of the sample mean

$$P_{\bar{X}_i} = tS_{\bar{X}_i} = tS_{X_i} \Big/ \sqrt{N}$$

should be used.

Propagation of Precision Errors

- In such case, the $\pm P_{\bar{X}_i}$ band around the sample mean $\bar{X}_i$ contains the mean value of the measured variable with 95% confidence. Therefore, if the value of the variable that is used in the data reduction euqation is determined as the *mean of N separate measurements*, the values

$$\bar{X}_i \text{ and } P_{\bar{X}_i}$$

should be used in the uncertainity anlaysis equations.

Propagation of Precision Errors

- When several separate factors can be identified as causing the precision error in a measured variable, it may sometimes be desirable to determine the precision limit by considering the contributions of the bias limt.
- For the X_k measurement variable, suppose that M element precision error sources are identified and their 95% confidence precision limmts are determined as $(P_K)_1, (P_K)_2, ..., (P_K)_M$. Then at 95% confidence, the overall precision limit s given

$$P_K = \left[\left(P_K\right)_1^2 + \left(P_K\right)_2^2 + \cdots + \left(P_K\right)_M^2 \right]^{1/2}.$$

Propagation of Precision Errors

- Another factor that must be considered in estimating precision limit s is the time period over which the sample population is obtained.
- The following rule should be observed:

  Data sets for estimating precision indices should be acquired over a period that is long relative to the time scale of any factor with a s ignificant influence on the data and the precision errors.

- If this rule is not followed, the precision limit estimations would not i nclude long time (low frequency) variations that affect the measure ment, and therefore these limit estimates would be inaccurate.

Propagation of Precision Errors

- Once the 95% precision limit, P_i for each measured variable, X_i in the data reduction equation, $R = R\left(X_1, X_2, \cdots, X_K\right)$ is determined, then 95% precision limit for the experimental result P_R is found from the uncertainty analysis expression (Coleman and Steele, 1989)

$$\left(\frac{P_R}{R}\right)^2 = \left(\frac{1}{R}\frac{\partial R}{\partial X_1}P_1\right)^2 + \left(\frac{1}{R}\frac{\partial R}{\partial X_2}P_2\right)^2 + \cdots + \left(\frac{1}{R}\frac{\partial R}{\partial X_K}P_K\right)^2$$

- Note that this expression is identical in form to the one used for determination of the bias limit B_R of the result. It is assumed that the data reduction equation is continuous and has continuous derivatives in the domain of interest and that the precision limit P_i for the measured variables are independent of one another.

$$U_R = \left(B_R^2 + P_R^2\right)^{1/2}$$

Propagation of Precision Errors

- In addition, it is assumed that the result R is determined from the re duction equation only once at a given experimental condition using either a single measurement X_i or the mean value $\bar{X}_i$ of N repeated measurements.

Uncertainty of the Experimental Result

- In order to determine the *overall uncertainty* U_R of the experimental result, the bias and precision limits B_R *and* P_R must be combined. This is accomplished using the root-sum-square (RSS) method

$$U_R = \left(B_R^2 + P_R^2\right)^{1/2}$$

thereby providing 95% coverage of the true value.

Example

- The drag coefficient, $C_D = F_D / \left[\left(\rho V^2 / 2 \right) \left(\pi D^2 / 4 \right) \right]$ is to be reported for the flow of water over a strut-mounted sphere. The drag force F_D is measured directly with a force transducer, the freestream velocity V is measured with a pitot-static probe, and the sphere diameter D is measured with a micrometer. The table below fives nominal values of the measurement variables and the water density ρ, as well as estimates for the bias and precision limits of each variable at the 95% confidence level. The bias limits have been estimated based on manufacturer's specifications and previous experience with the instruments during independent calibrations. The precision limits, on the other hand, have been determined from multiple measurements of each variable, together with computation of the precision indices S_{X_i} and $P_{\bar{X}_i} = tS_{X_i} / \sqrt{N}$. Estimate the overall uncertainty in the reported drag coefficient U_{C_D} at a confidence level of 95%.

Example

Measured variables X_i	*Nominal Value*	*Bias Limit, B_i*	*Precision Limit, P_i*
Drag force, F_D	0.5 N	0.02 N	0.032 N
Water density, ρ	998 kg/m^3	0.2 %	---
Freestream velocity, V	5 m/s	0.1 m/s	0.18 m/s
Sphere diameter	10 mm	0.1 mm	0.05 mm

- Before beginning the details of the solution, discussion of the percentage uncertainty listed for the water density is in order. When using tabular or curve-fit reference values for quantities such as material properties, it tis important to remember that these are not true values. Rather , they are “best estimates: based on experimental data that have uncertainties associated with them. However, once a table or curve-fit equation has been chosen to determine a property, the same values will be obtained for a given experimental condition no matter how many times the table or equation is used.

Example

Measured variables X_i	*Nominal Value*	*Bias Limit, B_i*	*Precision Limit, P_i*
Drag force, F_D	0.5 N	0.02 N	0.032 N
Water density, ρ	998 kg/m^3	0.2 %	---
Freestream velocity, V	5 m/s	0.1 m/s	0.18 m/s
Sphere diameter	10 mm	0.1 mm	0.05 mm

- Thus, the precision limit associated with a property value determined from a table or equation is zero. All of the uncertainty in experiment al property data is combined into a bias limit that is the best estimate of the overall uncertainty in the data used to generate the table or eq uation. In the current cased, the 95% bias limit estimate for density (expressed here as a percentage) is quite low, 0.2%, since the densit y of water is well known and is also relatively insensitive to environm ental factors such as temperature variations.

Example

- Writing the expression for the data reduction equation and the unce rtainty expression for the bias limit B_{C_D}

$$C_D = \frac{8}{\pi} F_D \rho^{-1} V^{-2} D^{-2}$$

$$\left(\frac{B_{C_D}}{C_D}\right)^2 = \left(\frac{1}{C_D}\frac{\partial C_D}{\partial F_D} B_{F_D}\right)^2 + \left(\frac{1}{C_D}\frac{\partial C_D}{\partial \rho} B_\rho\right)^2 + \left(\frac{1}{C_D}\frac{\partial C_D}{\partial V} B_V\right)^2 + \left(\frac{1}{C_D}\frac{\partial C_D}{\partial D} B_D\right)^2$$

Computing the derivatives:

$$\frac{1}{C_D}\frac{\partial C_D}{\partial F_D} B_{F_D} = \left(\frac{1}{8/\pi\ F_D \rho^{-1} V^{-2} D^{-2}}\right)\left(\frac{8}{\pi}\rho^{-1} V^{-2} D^{-2}\right) B_{F_D} = \frac{B_{F_D}}{F_D}$$

$$\frac{1}{C_D}\frac{\partial C_D}{\partial \rho} B_\rho = \left(\frac{1}{8/\pi\ F_D \rho^{-1} V^{-2} D^{-2}}\right)\left(-\frac{8}{\pi} F_D \rho^{-2} V^{-2} D^{-2}\right) B_\rho = -\frac{B_\rho}{\rho}$$

$$\frac{1}{C_D}\frac{\partial C_D}{\partial V} B_V = \left(\frac{1}{8/\pi\ F_D \rho^{-1} V^{-2} D^{-2}}\right)\left(-\frac{16}{\pi} F_D \rho^{-1} V^{-3} D^{-2}\right) B_V = \frac{-2B_V}{V}$$

$$\frac{1}{C_D}\frac{\partial C_D}{\partial D} B_D = \left(\frac{1}{8/\pi\ F_D \rho^{-1} V^{-2} D^{-2}}\right)\left(-\frac{16}{\pi} F_D \rho^{-1} V^{-2} D^{-3}\right) B_D = \frac{-2B_D}{D}$$

Example

Substituting into the uncertainty expression

$$\left(\frac{B_{C_D}}{C_D}\right)^2 = \left(\frac{B_{F_D}}{F_D}\right)^2 + \left(\frac{B_\rho}{\rho}\right)^2 + 4\left(\frac{B_V}{V}\right)^2 + 4\left(\frac{B_D}{D}\right)^2$$

Thus the bias limit B_{C_D} depends most strongly on B_V and B_D, i.e., the factors in the data reduction equation with the largest exponents, which is always the case for reduction expressions of "power law" form. Substituting numerical values

$$\left(\frac{B_{C_D}}{C_D}\right)^2 = \left(\frac{0.02}{0.5}\right)^2 + (0.002)^2 + 4\left(\frac{0.1}{5}\right)^2 + 4\left(\frac{0.1}{10}\right)^2$$

$$\left(\frac{B_{C_D}}{C_D}\right)^2 = \left(1.6\times10^{-3}\right) + \left(4\times10^{-6}\right) + \left(1.6\times10^{-3}\right) + \left(4\times10^{-4}\right) = 3.604\times10^{-3}$$

$$\therefore \left(\frac{B_{C_D}}{C_D}\right) = 6.003\times10^{-2} \approx 6.0\%$$

Note that the bias limit contributions from the drag force and velocity measurements are equal and dominate the bias uncertainty for C_D and that the bias limit on density contributes negligibly.

Example

Now considering the precision limits, the uncertainty expression for P_{C_D} is:

$$\left(\frac{P_{C_D}}{C_D}\right)^2 = \left(\frac{1}{C_D}\frac{\partial C_D}{\partial F_D}P_{F_D}\right)^2 + \left(\frac{1}{C_D}\frac{\partial C_D}{\partial V}P_V\right)^2 + \left(\frac{1}{C_D}\frac{\partial C_D}{\partial V}B_V\right)^2 + \left(\frac{1}{C_D}\frac{\partial C_D}{\partial D}P_D\right)^2$$

The derivatives have already been carried out in considering the bias limit, so the following result can be written immediately:

$$\left(\frac{P_{C_D}}{C_D}\right)^2 = \left(\frac{P_{F_D}}{F_D}\right)^2 + 4\left(\frac{P_V}{V}\right)^2 + 4\left(\frac{P_D}{D}\right)^2$$

Substituting numerical values :

$$\left(\frac{P_{C_D}}{C_D}\right)^2 = \left(\frac{0.032}{0.5}\right)^2 + 4\left(\frac{0.18}{5}\right)^2 + 4\left(\frac{0.05}{10}\right)^2$$

$$\left(\frac{P_{C_D}}{C_D}\right)^2 = \left(4.096\times10^{-3}\right) + \left(5.184\times10^{-3}\right) + \left(1\times10^{-4}\right) = 9.38\times10^{-3}$$

$$\therefore \left(\frac{P_{C_D}}{C_D}\right) = 9.685\times10^{-2} \approx 9.7\%$$

Example

Combining the bias and precision limits by the RSS method:

$$\frac{U_{C_D}}{C_D} = \left[\left(\frac{B_{C_D}}{C_D}\right)^2 + \left(\frac{P_{C_D}}{C_D}\right)^2\right]^{1/2}$$

$$\frac{U_{C_D}}{C_D} = \left[\left(3.604\times10^{-3}\right)+\left(9.38\times10^{-3}\right)\right]^{1/2} = 0.114 = 11.4\%$$

Thus, under these conditions the total uncertainty in the drag coefficient is 11.4% at a 95% confidence level. Since the nominal values of C_D is

$$C_D = \frac{8}{\pi}\frac{F_D}{\rho V^2 D^2} = \left(\frac{8}{\pi}\right)\left[\frac{0.5}{998\cdot 5^2\cdot 0.01^2}\right] = 0.510$$

this results can also be written as

$$C_D = 0.510 \pm 0.058 \text{ at 95\% confidence.}$$