Aircraft Structures

CHAPER 10.
Energy methods

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10.7 Development of a finite element formulation for trusses

◦ PMTPE(Principle of Minimum Total Potential Energy)

: powerful tool for the analysis of trusses

- Manageable for simple trusses consisting of only a few bars

  -> algebraic manipulations become increasingly tedious as the number of bars increases

- However, the method is very systematic and reduces the problem to the solution of a set of simultaneous linear equations

  -> large sets of simultaneous linear equations are easily solved with the help of computers

- Attention is directed to generating the equilibrium equations of the problem.

10.7 Development of a finite element formulation for trusses

◦ Individual truss member, i.e., an axially loaded bar

- Strain energy and potential of the externally applied loads for each individual bar are generated

- Total P.E of the entire truss is obtained by summing up the contribution from each bar

- Equilibrium equations are then generated by applying PMTPE

10.7 Development of a finite element formulation for trusses

- Additive property of strain energy

 - For trusses, each bar is an “element” of the system

 - Simple computation is repeated for each element of the truss

 - “Assembly process” through matrix operations

10.7 Development of a finite element formulation for trusses

◦ This approach is ideally suited for computer implementation

1) Strain energy in each bar

2) Contribution of each bar is added to the total strain energy

3) Large system of linear equations resulting from the application of PMTPE is solved

- Linear algebra and matrix notation greatly simplifies the computer implementation

◦ Such approach is basically an introduction to the “Finite Element Method”

-> The tool of choice for the solution of complex structured problems

10.7 Development of a finite element formulation for trusses

10.7.1 General description of the problem

◦ 11-bar,7- node planar trusses

- Node number circled, each bar number in a square box
- Truss is in a plane defined by unit vector $\vec{i}_1$ and $\vec{i}_2$
- Truss is pinned to the ground at nodes 1 and 7

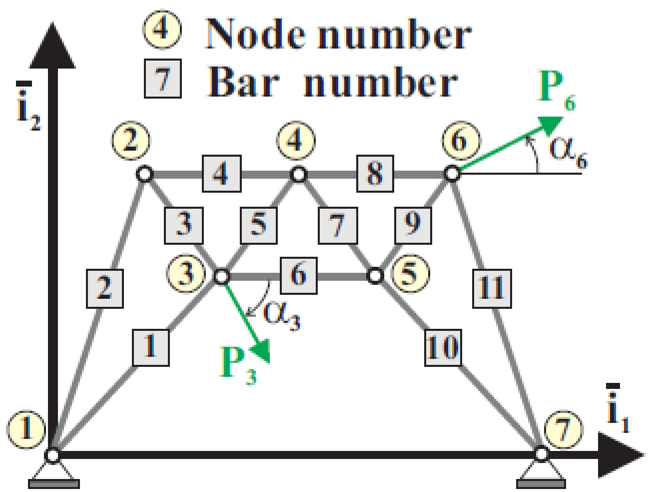


Fig. 10.22

- Two concentrated loads P_3 , P_6 at angles α_3 and α_6 w.r.t. the horizontal
- Stiffness properties of each bar : $\dfrac{E_{(i)}A_{(i)}}{L_{(i)}}$

  $E_{(i)}$: Young's modulus

  $A_{(i)}$: Cross-sectional area

  $L_{(i)}$: Length

  $(\cdot)_{(i)}$: Quantities pertaining to the i-th bar or element

10.7 Development of a finite element formulation for trusses

- Position vector of the i-th node : $p_1 = \{x_1 \quad y_1\}^T$

$$p_1 = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}, \; p_2 = \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix}, \ldots\ldots\ldots\ldots\ldots \;, p_n = \begin{Bmatrix} x_n \\ y_n \end{Bmatrix} \qquad (10.60)$$

$(\cdot)_i$: quantities pertaining to the i-th node

- Generalized coordinates of the problem will be selected as the horizontal and vertical displacement of each of the 7 nodes

$$q_1 = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}, \; q_2 = \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix}, \ldots\ldots\ldots\ldots\ldots \;, q_n = \begin{Bmatrix} x_n \\ y_n \end{Bmatrix} \qquad (10.61)$$

- “global displacement array” $\underline{q}$

$$\underline{q} = \left\{ q_1^T, \quad q_2^T, \quad q_3^T, \quad q_4^T, \quad q_5^T, \quad q_6^T, \quad q_7^T \right\} \qquad (10.62)$$

10.7 Development of a finite element formulation for trusses

- F.E.M first focuses on a "generic element" of the system to evaluate the strain energy stored in that specific element
- 2 nodes : a root node, "Node 1"
  : a tip node, "Node 2" } <- "local nodes"

- "global nodes" will be considered when the complete truss is considered

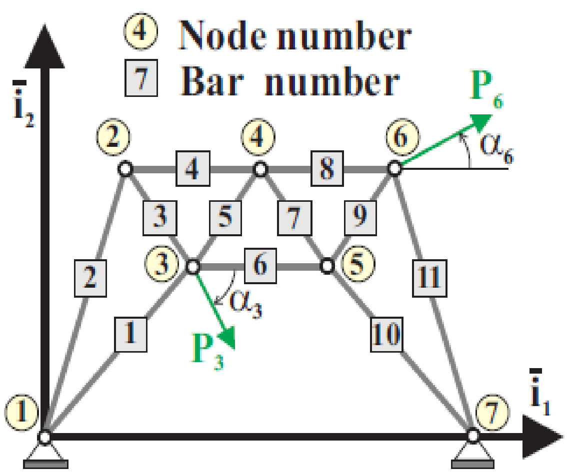


Fig. 10.22

	Local nodes	Global nodes
Bar No.4	Node 1	Node 2
	Node 2	Node 4
Bar No.9	Node 1	Node 5
	Node 2	Node 6

10.7.2 Kinematics of an element

◦ a single bar with local nodes "Node 1", "Node 2"

- "local coordinate system", unit vector $\vec{j}_1$ aligned with the axis of the bar

  $\vec{j}_2$ normal to the bar

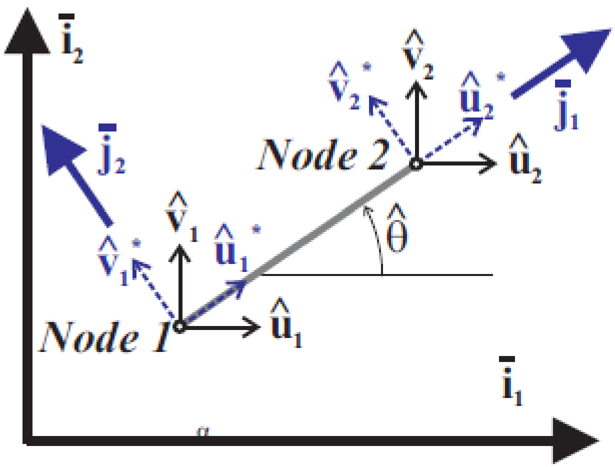


Fig. 10.23

$$\mathcal{J} = \left(\vec{j}_1,\ \vec{j}_2\right)\ldots$$

: Rotation of the global coordinate system,

$$\mathcal{I} = \left(\vec{i}_1,\ \ \vec{i}_2\right),$$

by an angle $\hat{\theta}$, which is the angle between the bar and the horizontal axis, $\vec{i}_1$

10.7 Development of a finite element formulation for trusses

- Position vector of the two local nodes of the element

$$\underline{\hat{p}}_1 = \begin{Bmatrix} \hat{x}_1 \\ \hat{y}_1 \end{Bmatrix} \qquad \underline{\hat{p}}_2 = \begin{Bmatrix} \hat{x}_2 \\ \hat{y}_2 \end{Bmatrix} \tag{10.63}$$

quantities pertaining to an element -> with a caret $\hat{(\cdot)}$

to distinguish them from their global counterparts

- Displacement of the two nodes of the element, resolved in $\mathcal{I}$ and $\mathcal{J}$

$$\underbrace{q_1 = \begin{Bmatrix} u_1 \\ \hat{v}_1 \end{Bmatrix}, \quad q_2 = \begin{Bmatrix} u_2 \\ \hat{v}_2 \end{Bmatrix}}_{\text{Resolved in the global coordinate system}} \qquad \underbrace{q_1^* = \begin{Bmatrix} u_1^* \\ \hat{v}_1^* \end{Bmatrix}, \quad q_2^* = \begin{Bmatrix} u_2^* \\ \hat{v}_2^* \end{Bmatrix}}_{\text{Resolved in the local coordinate system}} \tag{10.64}$$

10.7 Development of a finite element formulation for trusses

- Relationship between quantities resolved in two distinct orthonormal bases

$$\underline{\hat{q}}_1 = \underline{\underline{\hat{R}}}\,\underline{\hat{q}}_1^* \tag{10.65}$$

element rotation matrix $\underline{\underline{\hat{R}}}$

$$\underline{\underline{\hat{R}}} = \begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} \\ \sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix} \tag{10.66}$$

$$\underline{\hat{q}}_2 = \underline{\underline{\hat{R}}}\,\underline{\hat{q}}_2^*$$

- Length of the bar

$$\hat{L} = \|\underline{\hat{p}}_2 - \underline{\hat{p}}_1\| = \sqrt{(\hat{x}_2 - \hat{x}_1)^2 + (\hat{y}_2 - \hat{y}_1)^2} \tag{10.67}$$

10.7 Development of a finite element formulation for trusses

- Orientation angle, θ , can be found from nodal position vector using the definition of the scalar product

$$\bar{\imath}_1 \cdot (\hat{\underline{p}}_2 - \hat{\underline{p}}_1) = \|\bar{\imath}_1\| \, \|(\hat{\underline{p}}_2 - \hat{\underline{p}}_1)\| \cos\hat{\theta}$$

$$\bar{\imath}_2 \cdot (\hat{\underline{p}}_2 - \hat{\underline{p}}_1) = \|\bar{\imath}_2\| \, \|(\hat{\underline{p}}_2 - \hat{\underline{p}}_1)\| \sin\hat{\theta} \qquad (10.68)$$

$$\cos\hat{\theta} = \frac{\bar{\imath}_1 \cdot (\hat{\underline{p}}_2 - \hat{\underline{p}}_1)}{\hat{L}}, \quad \sin\hat{\theta} = \frac{\bar{\imath}_2 \cdot (\hat{\underline{p}}_2 - \hat{\underline{p}}_1)}{\hat{L}}$$

10.7 Development of a finite element formulation for trusses

- It will be convenient to combine the displacement of the element's two nodes into single array -> "element displacement array"

$$\underline{\hat{q}} = \begin{Bmatrix} \underline{\hat{q}}_1 \\ \underline{\hat{q}}_2 \end{Bmatrix} \qquad \underline{\hat{q}}^* = \begin{Bmatrix} \underline{\hat{q}}_1^* \\ \underline{\hat{q}}_2^* \end{Bmatrix} \tag{10.69}$$

- Relationship between these two arrays <- Eq.(10.65)

$$\underline{\hat{q}} = \begin{Bmatrix} \hat{q}_1 \\ \hat{q}_2 \end{Bmatrix} = \left[\begin{array}{cc} \underline{\underline{\hat{R}}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\hat{R}}} \end{array}\right] \begin{Bmatrix} \hat{q}_1^* \\ \hat{q}_2^* \end{Bmatrix} = \underline{\underline{\hat{T}}}\, \underline{\hat{q}}^* \tag{10.70}$$

10.7 Development of a finite element formulation for trusses

$\underline{\underline{0}}$: 2 by 2 null matrix

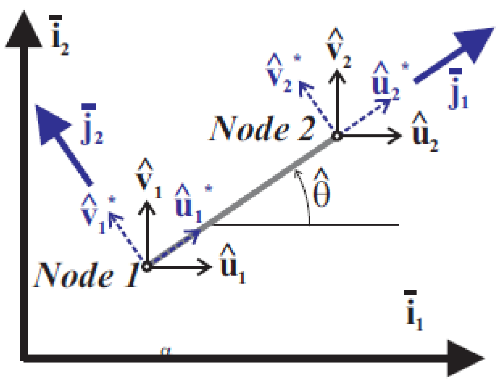
$$\underline{\underline{\hat{T}}} = \begin{bmatrix} \underline{\underline{\hat{R}}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\hat{R}}} \end{bmatrix} \tag{10.70}$$

$\underline{\underline{\hat{R}}}$: Orthogonal matrix, $\underline{\underline{\hat{T}}}\,\underline{\underline{\hat{T}}}^T = I$

$$\underline{\hat{q}}^* = \underline{\underline{\hat{T}}}^{-1} \underline{\hat{q}} = \underline{\underline{\hat{T}}}^T \underline{\hat{q}} \tag{10.72}$$

10.7 Development of a finite element formulation for trusses

10.7.3 Element elongation and force



Elongation, $\hat{e}$ of the bar

$$\hat{e} = u_2^* - u_1^* = \{-1,0,1,0\}\underline{q}^* = \underline{\hat{b}}^{*T}\underline{q}^*$$

Where $\hat{b}^* = \{-1,0,1,1\}^T$: array that relates the element elongation to the nodal displacement $\underline{q}^*$

- But it is also possible to express the elongations in terms of the displacement components resolved in the global coordinate system

$$\hat{e} = \underline{\hat{b}}^{*T}\underline{\hat{q}}^* = \underline{\hat{b}}^{*T}\underline{\underline{\hat{T}}}^T\underline{\hat{q}} = \underline{\hat{b}}^T\underline{\hat{q}} \qquad (10.73)$$

- where $\underline{\hat{b}} = \underline{\underline{\hat{T}}}\,\underline{\hat{b}}^*$

- Bar force, $\hat{F}$, is obtained by multiplying the elongation by the bar's axial stiffness to find

$$\hat{F} = \frac{\hat{E}\hat{A}}{\hat{L}}\hat{e} = \frac{\hat{E}\hat{A}}{\hat{L}}\underline{\hat{b}}^{*T}\underline{\hat{q}}^* = \frac{\hat{E}\hat{A}}{\hat{L}}\underline{\hat{b}}^T\underline{\hat{q}}. \qquad (10.74)$$

10.7.4 Element strain energy and stiffness matrix

10.7.4 Element strain energy and stiffness matrix

- Strain energy stored in a typical bar of the truss, Eq.(10.21)

$$\hat{A} = \frac{1}{2}\frac{\hat{E}\hat{A}}{\hat{L}}\hat{e}^2 = \frac{1}{2}\frac{\hat{E}\hat{A}}{\hat{L}}\hat{e}\cdot\hat{e} = \frac{1}{2}\frac{\hat{E}\hat{A}}{\hat{L}}(\underline{b}^{*T}\underline{\hat{q}}^{*})^T(\underline{b}^{*T}\underline{\hat{q}}^{*})$$

Regrouping

$$\hat{A} = \frac{1}{2}\underline{\hat{q}}^{*T}\left[\frac{\hat{E}\hat{A}}{\hat{L}}(\underline{b}^{*}\underline{b}^{*T})\right]\underline{\hat{q}}^{*} = \frac{1}{2}\underline{\hat{q}}^{*T}\underline{\underline{\hat{k}}}^{*}\underline{\hat{q}}^{*}$$

$\underline{\underline{\hat{k}}}^{*}$: element stiffness matrix expressed in the local coordinate system

- Since $\underline{b} = \{-1, 0, 1, 0\}^T$

$$\underline{\underline{\hat{k}}}^{*} = \frac{\hat{E}\hat{A}}{\hat{L}}(\underline{b}^{*}\underline{b}^{*T}) = \frac{\hat{E}\hat{A}}{\hat{L}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (10.75)$$

- It is also possible to evaluate the components of the same stiffness matrix expressed in the global coordinate system $\underline{\hat{q}}^{*} = \underline{\underline{\hat{T}}}^{T}\underline{\hat{q}}$

10.7.4 Element strain energy and stiffness matrix

$$\hat{A} = \frac{1}{2}\underline{\hat{q}}^{*T}\underline{\underline{\hat{k}}}^{*}\underline{\hat{q}}^{*} = \frac{1}{2}\left(\underline{\hat{q}}^{T}\underline{\underline{\hat{T}}}\right)\underline{\underline{\hat{k}}}^{*}\left(\underline{\underline{\hat{T}}}^{T}\underline{\hat{q}}\right) = \frac{1}{2}\underline{\hat{q}}^{T}\left(\underline{\underline{\hat{T}}}\,\underline{\underline{\hat{k}}}^{*}\underline{\underline{\hat{T}}}^{T}\right)\underline{\hat{q}} = \frac{1}{2}\underline{\hat{q}}^{T}\underline{\underline{\hat{k}}}\underline{\hat{q}} \tag{10.76}$$

where $\underline{\underline{\hat{k}}} = \underline{\underline{\hat{T}}}\,\underline{\underline{\hat{k}}}^{*}\underline{\underline{\hat{T}}}^{T}$

$$\underline{\underline{\hat{k}}} = \frac{\hat{E}\hat{A}}{\hat{L}}\begin{bmatrix} \cos^2\hat{\theta} & \sin\hat{\theta}\cos\hat{\theta} & -\cos^2\hat{\theta} & -\sin\hat{\theta}\cos\hat{\theta} \\ \sin\hat{\theta}\cos\hat{\theta} & \sin^2\hat{\theta} & -\sin\hat{\theta}\cos\hat{\theta} & -\sin^2\hat{\theta} \\ -\cos^2\hat{\theta} & -\sin\hat{\theta}\cos\hat{\theta} & \cos^2\hat{\theta} & \sin\hat{\theta}\cos\hat{\theta} \\ -\sin\hat{\theta}\cos\hat{\theta} & -\sin^2\hat{\theta} & \sin\hat{\theta}\cos\hat{\theta} & \sin^2\hat{\theta} \end{bmatrix} \tag{10.77}$$

- Eqs. (10.75) and (10.77) ⋯ 4X4 element stiffness matrix is partitioned into four, 2X2 sub-matrices

$\left\{ \begin{array}{l} \text{First 2 rows and columns} \cdots \text{ stiffness associated with the 2 D.O.F's (2 displacement Components) at "Node 1"} \\ \text{Last 2 rows and columns} \cdots \text{ stiffness associated with the 2 D.O.F's (2 displacement Components) at "Node 2"} \end{array} \right.$

10.7.4 Element strain energy and stiffness matrix

Eq. (10.75) ⋯ D.O.F's are displacement components resolved in the "local" coordinate system

Eq. (10.77) ⋯ D.O.F's are displacement components resolved in the "global" coordinate system

- Eq. (10.75) is far simpler than Eq. (10.77). But why is it desirable to derive element stiffness matrices in the global system?

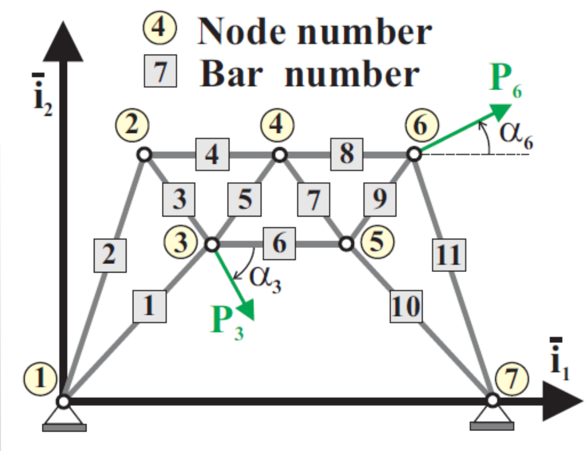


Fig. 10.22

Ans)

- in Fig. 10.22, bars 1, 3, 5, and 6 all connect to a single node, node 3.

- local coordinate systems of these four bars are all different

→ 4 corresponding element stiffness matrices expressed in their individual local

- systems are associated with local orthogonal displacement components resolved in four different systems

10.7.4 Element strain energy and stiffness matrix

In contrast, 4 element stiffness matrices are expressed in the global system are associated with orthogonal displacement components all resolved in the same global system

→ This latter form of the stiffness matrix will considerably simplify the assembly procedure

- $\hat{A}$: positive-definite quantity because the strain energy density for axially loaded bar is positive definite

10.7.5 Element external potential and load array

- It is assumed that the externally applied loads act only at the nodes.
- in Fig. 10.23, $\underline{\hat{F}_1}, \underline{\hat{F}_2}$: concentrated loads acting at local nodes, Node 1 and Node 2

  when resolved in the global coordinate system $\begin{cases} \underline{\hat{F}_1} = \hat{f}_1\vec{i}_1 + \hat{g}_1\vec{i}_2 \\ \underline{\hat{F}_2} = \hat{f}_2\vec{i}_1 + \hat{g}_2\vec{i}_2 \end{cases}$

- Potential of the externally applied loads

$$\hat{\Phi} = -\left[\hat{f}_1\hat{u}_1 + \hat{g}_1\hat{v}_1\right] - \left[\hat{f}_2\hat{u}_2 + \hat{g}_2\hat{v}_2\right] = -\left\{\hat{f}_1, \hat{g}_1, \hat{f}_2, \hat{g}_2\right\}\underline{\hat{q}} = -\underline{\hat{f}}^T\underline{\hat{q}} \qquad (10.78)$$

where "*element load array*" $\underline{\hat{f}} = \left\{\hat{f}_1, \hat{g}_1, \hat{f}_2, \hat{g}_2\right\}^T$ (10.79)

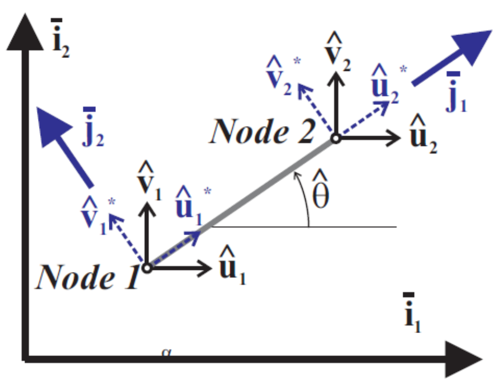


Fig. 10.23

- concentrated load of magnitude P and orientation α w.r.t the horizontal, acting at Node 1

  $\rightarrow \quad \hat{f}_1 = P\cos\alpha, \quad \hat{g}_1 = P\sin\alpha, \quad \hat{f}_2 = \hat{g}_2 = 0$

- weight of the bar as an externally applied load.

  ∵ gravity acts along the negative axis $\vec{i}_2$ direction

  $\rightarrow \quad \hat{f}_1 = 0, \quad \hat{g}_1 = -\dfrac{\hat{m}g}{2}, \quad \hat{f}_2 = 0, \quad \hat{g}_2 = -\dfrac{\hat{m}g}{2}$

10.7.6 Assembly procedure

- Eq. (10.77) : "element" stiffness matrix
- Eq. (10.79) : "element" load array

  ⋯ Since both strain energy and external potential are scalar quantities, their combined total will be evaluated simply by summing up the contributions from the individual elements.

- total strain energy, A

$$A = \sum_{i=1}^{N_e} \hat{A}_{(i)} = \frac{1}{2} \sum_{i=1}^{N_e} \underline{\hat{q}}_{(i)}^T \underline{\underline{\hat{k}}}_{(i)} \underline{\hat{q}}_{(i)} \tag{10.80}$$

  N_e is the number of bars in the truss (N_e = 11 in Fig. 10.23)

  also necessary to add the element identification subscript to $\underline{\underline{\hat{k}}}_{(i)}$ and $\underline{\hat{q}}_{(i)}$

- but it is not easy to manipulate because each term in the sum is expressed in terms of a different set of D.O.F's

  ex) in Fig. 10.22 : element 8 is connected to global nodes 4 and 6

  $\underline{\underline{\hat{k}}}_{(i)}$, corresponding element displacement array

$$\underline{\hat{q}}_{(8)}^T = \left\{ \underline{\hat{q}}_1^T, \underline{\hat{q}}_2^T \right\} = \left\{ \underline{\hat{q}}_4^T, \underline{\hat{q}}_6^T \right\} = \left\{ u_4, v_4, u_6, v_6 \right\}^T$$

10.7.6 Assembly procedure

- To remedy this situation, the connectivity matrix, $\underline{\underline{C}}_{(i)}$, for the i^{th} element is introduced

  ⋯ to extract the element displacement array, $\hat{q}_{(i)}$, from the global displacement array, q, Eq. (10.62).

$$\underline{\hat{q}}_{(i)} = \underline{\underline{C}}_{(i)} \underline{q} \tag{10.81}$$

ex) in Fig. 10.22 : bar 6 local nodes → global node numbers 3 and 5

$$\underline{\hat{q}}_1 = \underline{q}_3, \quad \underline{\hat{q}}_2 = \underline{q}_5$$

$$\underline{\hat{q}}_{(6)} = \begin{Bmatrix} \underline{\hat{q}}_1 \\ \underline{\hat{q}}_2 \end{Bmatrix}_{(6)} = \begin{Bmatrix} \underline{q}_3 \\ \underline{q}_5 \end{Bmatrix} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{I}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{I}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \begin{Bmatrix} \underline{q}_1 \\ \underline{q}_2 \\ \underline{q}_3 \\ \underline{q}_4 \\ \underline{q}_5 \\ \underline{q}_6 \\ \underline{q}_7 \end{Bmatrix} = \underline{\underline{C}}_{(6)} \underline{q}$$

$\underline{\underline{C}}_{(6)}$: connectivity matrix, "Boolean matrix"

10.7.6 Assembly procedure

- total strain energy, Eq. (10.80), now becomes

$$A = \frac{1}{2}\sum_{i=1}^{N_e}\left(\underline{\hat{q}}^T \underline{\underline{C}}^T{}_{(i)}\right)\underline{\underline{\hat{k}}}_{(i)}\left(\underline{\underline{C}}_{(i)}\underline{q}\right) = \frac{1}{2}\underline{\hat{q}}^T\left[\sum_{i=1}^{N_e}\underline{\underline{C}}^T{}_{(i)}\underline{\underline{\hat{k}}}_{(i)}\underline{\underline{C}}_{(i)}\right]\underline{q}$$

$$A = \frac{1}{2}\underline{\hat{q}}^T\underline{\underline{K}}\,\underline{q} \qquad (10.82)$$

- $\underline{\underline{K}}$: global stiffness matrix

$$\underline{\underline{K}} = \sum_{i=1}^{N_e}\underline{\underline{C}}^T{}_{(i)}\underline{\underline{\hat{k}}}_{(i)}\underline{\underline{C}}_{(i)} \qquad (10.83)$$

- potential of the externally applied loads, Φ

$$\Phi = \sum_{i=1}^{N_e}\hat{\Phi}_{(i)} = -\sum_{i=1}^{N_e}\underline{\hat{q}}_{(i)}^T\underline{\hat{f}}_{(i)} \qquad (10.84)$$

- it is convenient to use the connectivity matrix in Eq. (10.81)

$$\Phi = -\sum_{i=1}^{N_e}\left(\underline{\underline{C}}_{(i)}\underline{q}\right)^T\underline{\hat{f}}_{(i)} = -\underline{q}^T\left\{\sum_{i=1}^{N_e}\underline{\underline{C}}_{(i)}{}^T\underline{\hat{f}}_{(i)}\right\}$$

$$\Phi = -\underline{q}^T\underline{Q} \qquad (10.85)$$

10.7.6 Assembly procedure

- global load array, $\underline{Q}$

$$\underline{Q} = \sum_{i=1}^{N_e} \underline{\underline{C}}^T{}_{(i)} \underline{\hat{f}}_{(i)}$$

$$A = \frac{1}{2} \underline{\hat{q}}^T \underline{\underline{K}} \underline{q} \qquad (10.86)$$

- global load array, Π

$$\Pi = A + \Phi = \frac{1}{2} \underline{q}^T \underline{\underline{K}} \underline{q} - \underline{q}^T \underline{Q} \qquad (10.87)$$

quadratic form of the generalized coordinates (positive-definite)

linear form of "

10.7.7 Alternative description of the assembly procedure

- assembly procedure described in terms of the connectivity matrix is formally correct, but it is $\begin{cases} \text{not easy to understand} \\ \text{not computationally efficient for realistic trusses} \end{cases}$

  for large trusses, $\underline{\underline{C}}_{(i)}$ becomes very large with zero entries

  Eq. (10.83) ··· triple product, increasingly expensive, very wasteful

- more graphical visualization of the assembly process → Fig. 10.24

  global stiffness matrix : 7 rows and columns, each node has two D.O.F's, each of the 49 entries is 2 X 2 matrix, the size of the global stiffness matrix is 14 X 14

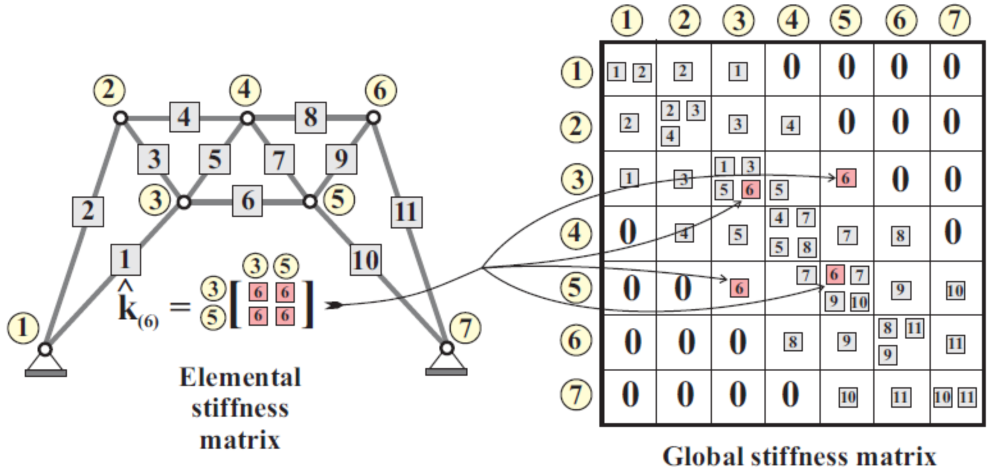


Fig. 10.24. Illustration of the assembly procedure.

10.7.7 Alternative description of the assembly procedure

- bar 6 ⋯ local nodes are associated with the global node No. 3 and 5

  $\underline{\underline{K}}_{(6)}$ partitioned into four 2 X 2 matrices → can simply be added to entries

  $\underline{\underline{K}}(3, 3)$, $\underline{\underline{K}}(5, 5)$, $\underline{\underline{K}}(3, 5)$, $\underline{\underline{K}}(5, 3)$ in the global stiffness matrix

- careful interpretation ⋯ square boxes defines a 2 X 2 sub-matrix extracted from the corresponding element stiffness matrix
- Another way : diagonal entry $\underline{\underline{K}}$ (2, 2) collects contributions from bars 2, 3, 4 because these 3 bars are all physically connected to node 2.
- off-diagonal entries, entries $\underline{\underline{K}}$ (1, 3) and $\underline{\underline{K}}$ (3, 1) each collect the single contribution stemming from bar 1, because bar 1 connects nodes 1 and 3.
- symmetry of the local stiffness matrix, Eq. (10.77), symmetry of the assembly process,

  → global stiffness matrix also a symmetric matrix.
- non-zero entries in the global stiffness matrix concentrate near the diagonal

  → “banded matrix”
- alternative assembly process ⋯ easy to program and efficient to execute.

10.7.8 Derivation of the governing equations

- total potential E of the truss : Eq. (10.87)

  → application of PMTPE, Eq. (10.17)

$$\frac{\partial \Pi}{\partial \underline{q}} = \frac{\partial}{\partial \underline{q}}\left(\frac{1}{2}\underline{q}^T \underline{\underline{K}}\,\underline{q} - \underline{q}^T \underline{Q}\right) = \underline{\underline{K}}\,\underline{q} - \underline{Q} = 0 \qquad (10.88)$$

  → linear system of eqns.

$$\underline{\underline{K}}\,\underline{q} = \underline{Q} \qquad (10.89)$$

  ⋯ element-oriented version of the displacement or stiffness methods described in sec. 4. 3. 2

Each line of Eq. (10.89) ⋯ equilibrium eqn of the problem

First line ⋯ equilibrium eqn obtained by imposing the vanishing of the sum of the horizontal forces acting at node 1

10.7 Development of a finite element formulation for trusses

10.7.9 Solution procedure

- Linear system in Eq.(10.89) cannot be solved because the global stiffness matrix is singular.
- Element stiffness matrix are each singular

  ··· Eq.(10.75) : Element stiffness matrix contains 2 rows of zeros , and the 3rd row is simply (-1) times the 1st row.

  4X4 matrix is three times singular. $\rightarrow$ rank 1 , rank deficiency of 3 , determinant 0.

  Eq.(10.77) has the some rank deficiency

  $\rightarrow \underline{\underline{K}}$ also presents a rank deficiency of 3, not invertible

10.7 Development of a finite element formulation for trusses

- Eigenvalues and eigenvectors of the element stiffness matrix

  ⋯ unit eigenvectors

$$\underline{n_1} = \frac{1}{\sqrt{2}}\begin{Bmatrix}1\\0\\1\\0\end{Bmatrix}, \quad \underline{n_2} = \frac{1}{\sqrt{2}}\begin{Bmatrix}0\\1\\0\\1\end{Bmatrix}, \quad \underline{n_3} = \frac{1}{\sqrt{2}}\begin{Bmatrix}\sin\hat{\theta}\\-\cos\hat{\theta}\\-\sin\hat{\theta}\\\cos\hat{\theta}\end{Bmatrix}, \quad \underline{n_4} = \frac{1}{\sqrt{2}}\begin{Bmatrix}\cos\hat{\theta}\\\sin\hat{\theta}\\-\cos\hat{\theta}\\-\sin\hat{\theta}\end{Bmatrix}$$

corresponding eigenvector $\lambda_1 = \lambda_2 = \lambda_3 = 0$,

$$\lambda_4 = 2\frac{\hat{E}\hat{A}}{\hat{L}}$$

$$\left.\begin{array}{l}\underline{n_1} : \text{horizontal rigid body translation of the bar}\\ \underline{n_2} : \text{vertical rigid body translation of the bar}\end{array}\right\} \begin{array}{l}\text{associated}\\ \lambda_i = 0\end{array}$$

$\underline{n_3}$: rigid body rotation

↳ creates no deformation or straining $\rightarrow$ zero strain E

10.7 Development of a finite element formulation for trusses

- Eq.(10.76) $\rightarrow$ strain E associated with $\underline{n}_1$, $\hat{A} = \frac{1}{2} \underline{n}_1^T \underline{\underline{\hat{k}}}\, \underline{n}_1 = 0$

  since the definition of eigenvectors , $\underline{\underline{\hat{k}}}\, \underline{n}_1 = \lambda_1 \underline{n}_1 = 0$

- 3 rigid body modes $\rightarrow$ rank deficiency of 3 for the element $\underline{\underline{k}}$,

  entire truss also presents 3 rigid body modes $\rightarrow$ global $\underline{\underline{K}}$ also features

  a rank deficiency of 3.

- Physical interpretation ··· B.C. 's have not yet been applied

  Fig. 10.22 ··· nodes 1 and 7 are pinned to the ground , preventing any rigid

  body motion

10.7 Development of a finite element formulation for trusses

- B.C. : $\underline{q}_1 = \underline{q}_7 = 0$

  reaction forces at nodes 1 and 7 $(\underline{R}_1, \underline{R}_7)$ should be treated as externally applied forces.

  The equilibrium eqns associated with these 2 nodes can be removed from

  Eq. (10.89) in Fig. 10.24

$$\underline{\underline{K}}(1,1)\underline{q}_1 + \underline{\underline{K}}(1,2)\underline{q}_2 + \underline{\underline{K}}(1,3)\underline{q}_3 = \underline{R}_1 \qquad (10.90a)$$

$$\underline{\underline{K}}(7,5)\underline{q}_5 + \underline{\underline{K}}(7,6)\underline{q}_6 + \underline{\underline{K}}(7,7)\underline{q}_7 = \underline{R}_7 \qquad (10.90b)$$

  $\rightarrow$ will leave $14-4=10$ rows remaining in the set of eqns.

- Since the displacements at nodes 1 and 7 vanish, the corresponding terms

  vanish in Eq. (10.90) $\rightarrow$ can be solved for unknown reaction forces

$$\begin{aligned} \underline{R}_1 &= \underline{\underline{K}}(1,2)\underline{q}_2 + \underline{\underline{K}}(1,3)\underline{q}_3 \\ \underline{R}_7 &= \underline{\underline{K}}(7,5)\underline{q}_5 + \underline{\underline{K}}(7,6)\underline{q}_6 \end{aligned} \qquad (10.91)$$

  ⋯ the first two and last two columns of $\underline{\underline{K}}$, as well as the first two and last two entries

  in array $\underline{q}$ and $\underline{\underline{Q}}$ can be also eliminated

  $\rightarrow$ a reduced set of 10 eqns that can now be solved for the

  remaining 10 unknown nodal displacements

10.7 Development of a finite element formulation for trusses

- B.C. imposing process

(1) eliminate the rows and columns of $\underline{\underline{K}}$ corresponding to constrained D.O.F. 's

$\rightarrow$ reduced counterpart $\underline{\underline{\overline{K}}}$

(2) eliminate the rows of $\underline{q}$ corresponding to the constrained D.O.F. 's $\rightarrow$ $\underline{\overline{q}}$

(3) eliminate the rows of $\underline{Q}$ corresponding to the constrained D.O.F. 's $\rightarrow$ $\underline{\overline{Q}}$

$$\rightarrow \qquad \underline{\underline{\overline{K}}}\,\underline{\underline{\overline{q}}} = \underline{\overline{Q}} \qquad (10.92)$$

$\because$ $\underline{\underline{\overline{K}}}$ will now be non-singular , and the solution of the problem is found by

$$\underline{\overline{q}} = \underline{\underline{\overline{K}}}^{-1}\,\underline{\overline{Q}}$$

10.7 Development of a finite element formulation for trusses

10.7.10 Solution procedure using partitioning

- More mathematic description of the previous process ⋯ based on "partitioning" of the same quantities that allows separate treatment of the constrained and unconstrained nodes
- Fig. 10.22 ⋯ node numbering sequence is arbitrary

  Fig. 10.25 ⋯ same truss, but a different node numbering , bounding conditions are to be applied at nodes 6 and 7 (last in the series)

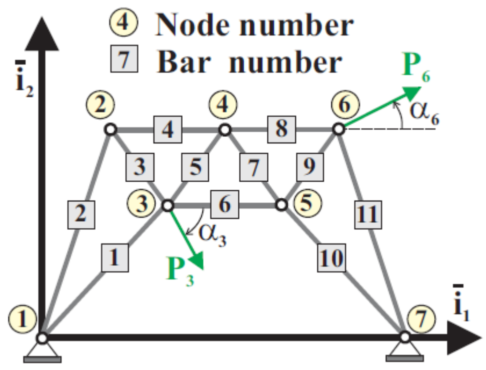


Fig. 10.22. Eleven-bar truss with node and element numbering.

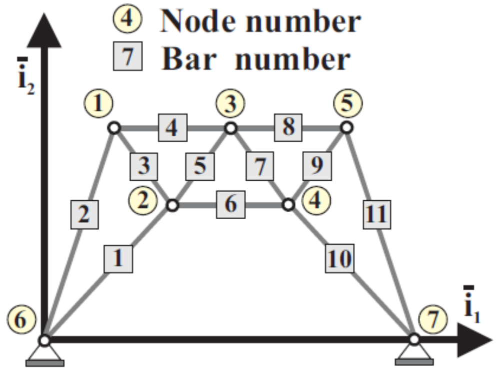


Fig. 10.25. Eleven-bar truss.

10.7 Development of a finite element formulation for trusses

- Global displacement array ⋯ now partitioned into 2 sub-arrays

$$\underline{q} = \{\underline{q}_1^T, \underline{q}_2^T, \underline{q}_3^T, \underline{q}_4^T, \underline{q}_5^T, |\underline{q}_6^T, \underline{q}_7^T\}^T = \{\underline{q}_u^T, \underline{q}_p^T\}^T \qquad (10.93)$$

$\underline{q}_u$: size N_u , unknown displacements at nodes 1 to 5

$\underline{q}_p$: size N_p , prescribed displacements at support nodes 6 to 7

in Fig. 10.25 , $N_u = 10$, $N_p = 4$

- Prescribed displacement at a node might be non-zero

ex) node 7 are prescribed to move by Δ along axis $\vec{i}_1$, $\underline{q}_7 = \{\Delta, 0\}^T$

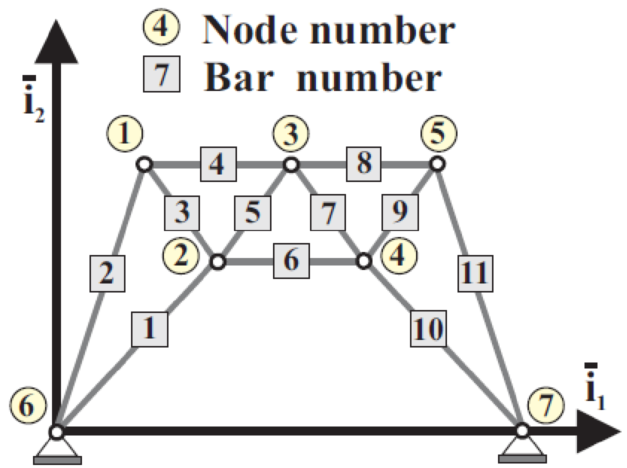


Fig. 10.25. Eleven-bar truss.

10.7 Development of a finite element formulation for trusses

- Fig. 10.25. governing eqns

$$\begin{bmatrix} \underline{\underline{K}}_{uu} & \underline{\underline{K}}_{up} \\ \underline{\underline{K}}_{up}{}^{T} & \underline{\underline{K}}_{pp} \end{bmatrix} \begin{Bmatrix} \underline{q}_u \\ \underline{q}_p \end{Bmatrix} = \begin{Bmatrix} \underline{Q}_u \\ \underline{Q}_p \end{Bmatrix} \tag{10.94}$$

subscripts u : "unconstrained" , p: "prescribed"

⋯ partitioned version of the general governing eqn. Eq. (10.89)

- Global load array

$$\underline{Q} = \{\underline{Q}_1^T, \underline{Q}_2^T, \underline{Q}_3^T, \underline{Q}_4^T, \underline{Q}_5^T, \Big| \underline{R}_6^T, \underline{R}_7^T\}^T = \{\underline{Q}_u^T, \underline{Q}_p^T\}^T \tag{10.95}$$

$\underline{Q}_u$: size N_u , $\underline{Q}_p$: size N_p

$\underline{\underline{K}}_{uu}$: size $(N_u \times N_u)$, $\underline{\underline{K}}_{uu}$: size $(N_p \times N_p)$, $\underline{\underline{K}}_{up}$: size $(N_u \times N_p)$

10.7 Development of a finite element formulation for trusses

- first N_u eqns of system (10.94)

$$\underline{\underline{K}}_{uu}\underline{q}_u = \underline{Q}_u - \underline{\underline{K}}_{up}\underline{q}_p \qquad (10.96)$$

$\underline{q}_p$ are known , then unknown nodal displacements are evaluated as

$$\underline{q}_u = \underline{\underline{K}}_{uu}^{-1}(\underline{Q}_u - \underline{\underline{K}}_{up}\underline{q}_p)$$

if B.C. 's consist solely of nodes rigidly connected to the ground , $\underline{q}_p = 0$

$$\underline{q}_u = \underline{\underline{K}}_{uu}^{-1}\underline{Q}_u \quad \rightarrow \quad \text{equivalent to Eq. (10.92)}$$

- last N_u eqns of system (10.94) $\rightarrow$ to equivalent the reactions

$$\underline{Q}_p = \underline{\underline{K}}_{up}^{T}\underline{q}_u + \underline{\underline{K}}_{pp}\underline{q}_p \qquad (10.97)$$

if B.C. 's solely of nodes rigidly connected to the ground , $\underline{q}_p = 0$

$$\underline{Q}_p = \underline{\underline{K}}_{up}^{T}\underline{q}_u$$

10.7 Development of a finite element formulation for trusses

- Reactions acting at nodes where the displacements are prescribed

  → "driving forces" : the force that must be applied at a node to achieve the prescribed displacements

  to represent mis-alignments in the supports due to non-ideal truss geometry

- In Eq. (10.94) , $\underline{\underline{K}}_{uu}$ is equivalent to the reduced stiffness matrix , $\overline{\underline{\underline{K}}}$,

  in Eq. (10.92) → rigorous justification of the previous procedure

- Renumbering ··· tedious to do by hand , but easily handled using computer programs

10.7.11 Post-processing

- Elongation of a bar ⋯ $\hat{e} = \underline{\hat{b}}^T \underline{\hat{q}}$, Eq. (10.73)

$$\hat{e}_{(i)} = \underline{\hat{b}}^T_{(i)} \underline{\hat{q}}_{(i)} = \underline{\hat{b}}^T_{(i)} \underline{\underline{C}}_{(i)} \underline{q} \qquad (10.98)$$

- bar force

$$\hat{F}_{(i)} = \frac{\hat{E}_{(i)} \hat{A}_{(i)}}{\hat{L}_{(i)}} \hat{e}_{(i)} \qquad (10.99)$$

ex) bar 6 in Fig. 10.22 , Node 1 $\rightarrow$ Node 3 , Node 2 $\rightarrow$ Node 5

orientation is parallel to axis $\vec{i}_1$, $\hat{\theta}_{(6)} = 0$, then $\underline{\underline{\hat{T}}}_{(6)} = \underline{\underline{I}}$

$$\underline{\hat{b}} = \underline{\underline{\hat{T}}} \underline{\hat{b}}^* = \underline{\hat{b}}^*$$

$$\hat{e}_{(6)} = \underline{\hat{b}}^T_{(6)} \underline{\underline{C}}_{(6)} \underline{q} = \underline{\hat{b}}^T_{(6)} \begin{Bmatrix} \underline{\hat{q}}_3 \\ \underline{\hat{q}}_5 \end{Bmatrix} = \{-1, 0, 1, 0\} \begin{Bmatrix} \underline{\hat{q}}_3 \\ \underline{\hat{q}}_5 \end{Bmatrix} = -u_3 + u_5$$

$$\hat{F}_{(6)} = \frac{\hat{E}_{(6)} \hat{A}_{(6)}}{\hat{L}_{(6)}} \hat{e}_{(6)} = \frac{\hat{E}_{(6)} \hat{A}_{(6)}}{\hat{L}_{(6)}} (-u_3 + u_5)$$