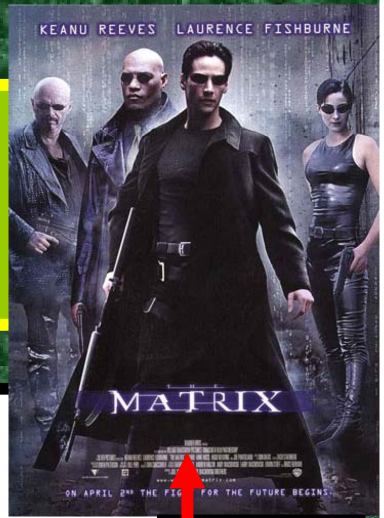
Module #9: **Matrices**

Rosen 5th ed., §2.7
~18 slides, ~1 lecture

§2.7 Matrices



Not our meaning!

- A *matrix* (say MAY-trix) is a rectangular array of objects (usually numbers).
- An $m \times n$ ("m by n") matrix has exactly m horizontal rows, and n vertical columns.
- Plural of matrix = *matrices* (say MAY-trih-sees)
- An $n \times n$ matrix is called a *square* matrix, whose *order* is n.

$$\begin{bmatrix} 2 & 3 \\ 5 & -1 \\ 7 & 0 \end{bmatrix}$$

a 3×2 matrix

Note: The singular form of "matrices" is "*matrix*," **not** "MAY-trih-see"!

Applications of Matrices

Tons of applications, including:

- Solving systems of linear equations
- Computer Graphics, Image Processing
- Models within Computational Science & Engineering
- Quantum Mechanics, Quantum Computing
- Many, many more…

Matrix Equality

- Two matrices **A** and **B** are equal iff they have the same number of rows, the same number of columns, and all corresponding elements are equal.

$$\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 & 0 \\ -1 & 6 & 0 \end{bmatrix}$$

Row and Column Order

- The rows in a matrix are usually indexed 1 to *m* from top to bottom. The columns are usually indexed 1 to *n* from left to right. Elements are indexed by row, then column.

$$\mathbf{A} = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Matrices as Functions

- An $m \times n$ matrix $\mathbf{A} = [a_{i,j}]$ of members of a set S can be encoded as a partial function
  $$f_{\mathbf{A}}: \mathbb{N}\times\mathbb{N}\rightarrow S,$$
  such that for $i<m$, $j<n$, $f_{\mathbf{A}}(i, j) = a_{i,j}$.
- By extending the domain over which $f_{\mathbf{A}}$ is defined, various types of infinite and/or multidimensional matrices can be obtained.

Matrix Sums

- The *sum* **A**+**B** of two matrices **A**, **B** (which **must** have the same number of rows, and the same number of columns) is the matrix (also with the same shape) given by adding corresponding elements.
- **A**+**B** = $[a_{i,j}+b_{i,j}]$

$$\begin{bmatrix} 2 & 6 \\ 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ -11 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ -11 & -5 \end{bmatrix}$$

Matrix Products

- For an $m \times k$ matrix **A** and a $k \times n$ matrix **B**, the *product* **AB** is the $m \times n$ matrix:

$$\mathbf{AB} = \mathbf{C} = [c_{i,j}] \equiv \left[\sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j} \right]$$

- *I.e.*, element (i,j) of **AB** is given by the vector *dot product* of the <u>*i*th row of **A**</u> and the <u>*j*th column of **B**</u> (considered as vectors).
- Note: Matrix multiplication is **not** commutative!

Matrix Product Example

- An example matrix multiplication to practice in class:

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -5 & -1 \\ 3 & -2 & 11 & 3 \end{bmatrix}$$

Identity Matrices

- The *identity matrix of order n*, $\mathbf{I}_n$, is the order-*n* matrix with 1's along the upper-left to lower-right diagonal and 0's everywhere else.

$$\boldsymbol{I}_n = \left[\begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Review: §2.6 Matrices, so far

Matrix sums and products:

$$\mathbf{A}+\mathbf{B} = [a_{i,j}+b_{i,j}]$$

$$\mathbf{AB} = \mathbf{C} = [c_{i,j}] \equiv \left[\sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j} \right]$$

Identity matrix of order n:

$\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i \neq j$.

Matrix Inverses

- For some (but not all) square matrices **A**, there exists a unique multiplicative *inverse* $\mathbf{A}^{-1}$ of **A**, a matrix such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.
- If the inverse exists, it is unique, and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1}$.
- We won't go into the algorithms for matrix inversion...

Matrix Multiplication Algorithm

procedure *matmul*(matrices **A**: $m \times k$, **B**: $k \times n$)

for i := 1 **to** m } $\Theta(m)\cdot$

  for j := 1 **to** n **begin** } $\Theta(n)\cdot($

    $c_{ij} := 0$ } $\Theta(1)+$

    for q := 1 **to** k } $\Theta(k)\cdot$

      $c_{ij} := c_{ij} + a_{iq}b_{qj}$ } $\Theta(1))$

  end {**C**=[c_{ij}] is the product of **A** and **B**}

What's the Θ of its time complexity?

Answer: $\Theta(mnk)$

Powers of Matrices

If **A** is an $n \times n$ square matrix and $p \geq 0$, then:

- $\mathbf{A}^p \equiv \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{p\text{ times}} \quad (\mathbf{A}^0 \equiv \mathbf{I}_n)$
- Example:

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$$

Matrix Transposition

- If $\mathbf{A}=[a_{ij}]$ is an $m \times n$ matrix, the *transpose* of $\mathbf{A}$ (often written $\mathbf{A}^{t}$ or $\mathbf{A}^{T}$) is the $n \times m$ matrix given by $\mathbf{A}^{t} = \mathbf{B} = [b_{ij}] = [a_{ji}]$ $(1 \le i \le n, 1 \le j \le m)$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}^{t} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}$$

Flip across diagonal

Symmetric Matrices

- A square matrix $\mathbf{A}$ is *symmetric* iff $\mathbf{A}=\mathbf{A}^t$. *I.e.*, $\forall i,j \leq n$: $a_{ij} = a_{ji}$.
- Which is symmetric?

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

Zero-One Matrices

- Useful for representing other structures.
  - *E.g.*, relations, directed graphs (later in course)
- All elements of a *zero-one* matrix are 0 or 1
  - Representing **False** & **True** respectively.
- The *meet* of **A**, **B** (both $m \times n$ zero-one matrices):
  - $\mathbf{A} \wedge \mathbf{B} :\equiv [a_{ij} \wedge b_{ij}] = [a_{ij}\, b_{ij}]$
- The *join* of **A**, **B**:
  - $\mathbf{A} \vee \mathbf{B} :\equiv [a_{ij} \vee b_{ij}]$

Boolean Products

- Let $\mathbf{A}=[a_{ij}]$ be an $m\times k$ zero-one matrix, & let $\mathbf{B}=[b_{ij}]$ be a $k\times n$ zero-one matrix,
- The *boolean product* of **A** and **B** is like normal matrix ×, but using ∨ instead + in the row-column "vector dot product."

$$\mathbf{A}\odot\mathbf{B}=\mathbf{C}=[c_{ij}]=\left[\bigvee_{\ell=1}^{k} a_{i\ell}\wedge b_{\ell j}\right]$$

Boolean Powers

- For a square zero-one matrix **A**, and any $k \geq 0$, the *kth Boolean power of **A*** is simply the Boolean product of k copies of **A**.
- $$\mathbf{A}^{[k]} \equiv \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{k \text{ times}}$$