Module #12: **Summations**

Rosen 5th ed., §3.2
~19 slides, ~1 lecture

Summation Notation

- Given a series $\{a_n\}$, an integer *lower bound* (or *limit*) $j \geq 0$, and an integer *upper bound* $k \geq j$, then the *summation of* $\{a_n\}$ *from j to k* is written and defined as follows:

$$\sum_{i=j}^{k} a_i :\equiv a_j + a_{j+1} + \ldots + a_k$$

- Here, *i* is called the *index of summation*.

Generalized Summations

- For an infinite series, we may write:

$$\sum_{i=j}^{\infty} a_i :\equiv a_j + a_{j+1} + \ldots$$

- To sum a function over all members of a set $X=\{x_1, x_2, \ldots\}$: $\sum_{x\in X} f(x) :\equiv f(x_1) + f(x_2) + \ldots$
- Or, if $X=\{x|P(x)\}$, we may just write:

$$\sum_{P(x)} f(x) :\equiv f(x_1) + f(x_2) + \ldots$$

Simple Summation Example

$$\sum_{i=2}^{4} i^2 + 1 = (2^2 + 1) + (3^2 + 1) + (4^2 + 1)$$

$$= (4+1) + (9+1) + (16+1)$$

$$= 5 + 10 + 17$$

$$= 32$$

More Summation Examples

- An infinite series with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \ldots = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots = 2$$

- Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \wedge \\ x<10}} x^2 = 2^2 + 3^2 + 5^2 + 7^2 = 4 + 9 + 25 + 49 = 87$$

Summation Manipulations

- Some handy identities for summations:

$$\sum_x cf(x) = c\sum_x f(x)$$ (Distributive law.)

$$\sum_x f(x) + g(x) = \left(\sum_x f(x)\right) + \sum_x g(x)$$ (Application of commut-ativity.)

$$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n)$$ (Index shifting.)

More Summation Manipulations

- Other identities that are sometimes useful:

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i)\right) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

(Series splitting.)

$$\sum_{i=j}^{k} f(i) = \sum_{i=0}^{k-j} f(k-i)$$

(Order reversal.)

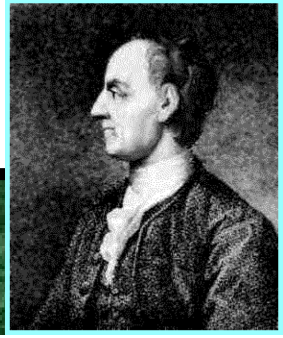
$$\sum_{i=0}^{2k} f(i) = \sum_{i=0}^{k} f(2i) + f(2i+1)$$

(Grouping.)

Example: Impress Your Friends

- Boast, "I'm so smart; give me any 2-digit number *n*, and I'll add all the numbers from 1 to *n* in my head in just a few seconds."
- *I.e.*, Evaluate the summation: $\sum_{i=1}^{n} i$
- There is a simple closed-form formula for the result, discovered by Euler at age 12!

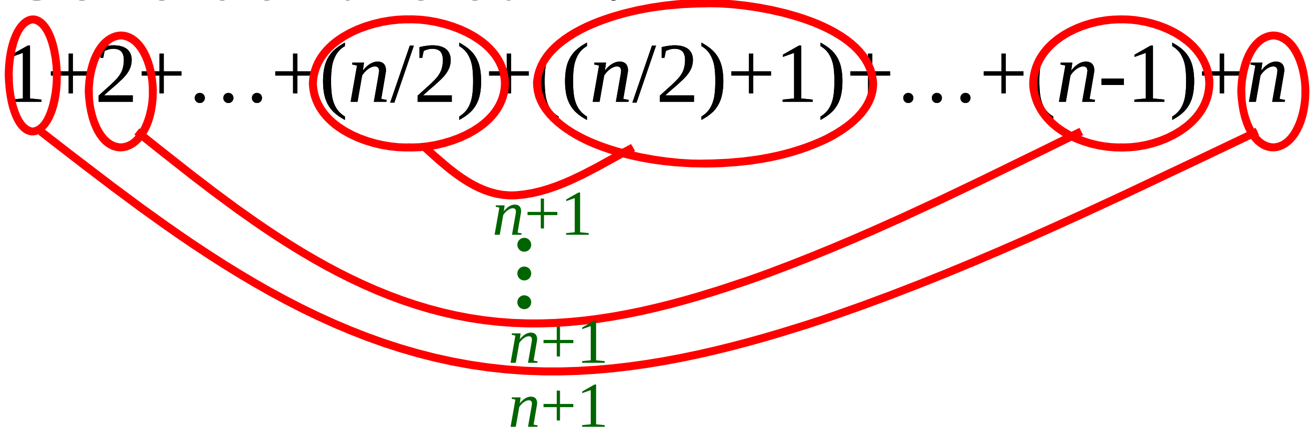
Leonhard Euler (1707-1783)



Euler's Trick, Illustrated

- Consider the sum:

$1+2+\ldots+(n/2)+((n/2)+1)+\ldots+(n-1)+n$



- $n/2$ pairs of elements, each pair summing to $n+1$, for a total of $(n/2)(n+1)$.

Symbolic Derivation of Trick

$$\sum_{i=1}^{n} i = \sum_{i=1}^{2k} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=k+1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} (i+(k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} ((n-(k+1)-i)+(k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=0}^{n-(k+1)} (n-i) = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{n-k} (n-(i-1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{n-k} (n+1-i) = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{k} (n+1-i) = \ldots$$

Concluding Euler's Derivation

$$\sum_{i=1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{k}(n+1-i) = \sum_{i=1}^{k}(i+n+1-i)$$

$$= \sum_{i=1}^{k}(n+1) = k(n+1) = \tfrac{n}{2}(n+1)$$

$$= n(n+1)/2$$

- So, you only have to do 1 easy multiplication in your head, then cut in half.
- Also works for odd *n* (prove this at home).

Example: Geometric Progression

- A *geometric progression* is a series of the form $a, ar, ar^2, ar^3, \ldots, ar^k$, where $a,r \in \mathbf{R}$.
- The sum of such a series is given by:

$$S = \sum_{i=0}^{k} ar^i$$

- We can reduce this to *closed form* via clever manipulation of summations...

Geometric Sum Derivation

- Here we go...

$$S = \sum_{i=0}^{n} ar^i$$

$$rS = r\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} rar^i = \sum_{i=0}^{n} arr^i = \sum_{i=0}^{n} ar^1 r^i$$

$$= \sum_{i=0}^{n} ar^{1+i} = \sum_{i=1}^{n+1} ar^{1+(i-1)} = \sum_{i=1}^{n+1} ar^i$$

$$= \left(\sum_{i=1}^{n} ar^i\right) + \sum_{i=n+1}^{n+1} ar^i = \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} = \ldots$$

Derivation example cont...

$$rS = \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} = (ar^0 - ar^0) + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1}$$

$$= ar^0 + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - ar^0$$

$$= \left(\sum_{i=0}^{0} ar^i\right) + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - a$$

$$= \left(\sum_{i=0}^{n} ar^i\right) + a(r^{n+1} - 1) = S + a(r^{n+1} - 1)$$

Concluding long derivation...

$$rS = S + a(r^{n+1} - 1)$$

$$rS - S = a(r^{n+1} - 1)$$

$$S(r-1) = a(r^{n+1} - 1)$$

$$S = a\left(\frac{r^{n+1} - 1}{r - 1}\right) \qquad \text{when } r \neq 1$$

$$\text{When } r = 1, S = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} a1^i = \sum_{i=0}^{n} a \cdot 1 = (n+1)a$$

Nested Summations

- These have the meaning you'd expect.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3)$$

$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4)$$

$$= 6 \cdot 10 = 60$$

- Note issues of free vs. bound variables, just like in quantified expressions, integrals, etc.

Some Shortcut Expressions

$$\sum_{k=0}^{n} ar^k = a(r^{n+1}-1)/(r-1), r \neq 1$$ Geometric series.

$$\sum_{k=1}^{n} k = n(n+1)/2$$ Euler's trick.

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$$ Quadratic series.

$$\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$$ Cubic series.

Using the Shortcuts

- Example: Evaluate $\sum_{k=50}^{100} k^2$.
  - Use series splitting.
  - Solve for desired summation.
  - Apply quadratic series rule.
  - Evaluate.

$$\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$$

$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338{,}350 - 40{,}425$$

$$= 297{,}925.$$

Summations: Conclusion

- You need to know:
  - How to read, write & evaluate summation expressions like:

$$\sum_{i=j}^{k} a_i \qquad \sum_{i=j}^{\infty} a_i \qquad \sum_{x \in X} f(x) \qquad \sum_{P(x)} f(x)$$

  - Summation manipulation laws we covered.
  - Shortcut closed-form formulas, & how to use them.