Module #19:
Graph Theory

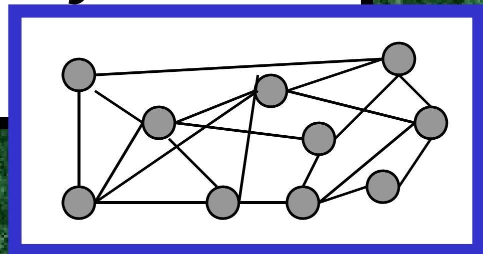
Rosen 5^{th} ed., chs. 8-9

~44 slides (more later), ~3 lectures

What are Graphs?

Not Our Meaning

- General meaning in everyday math: *A plot or chart of numerical data using a coordinate system.*
- Technical meaning in discrete mathematics: *A particular class of discrete structures (to be defined) that is useful for representing relations and has a convenient webby-looking graphical representation.*

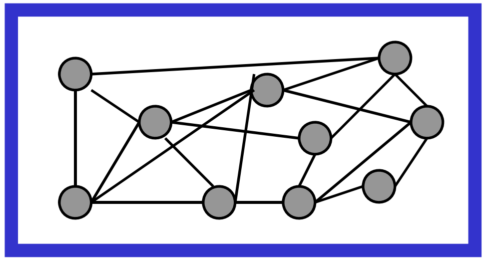


Applications of Graphs

- Potentially anything (graphs can represent relations, relations can describe the extension of any predicate).
- Apps in networking, scheduling, flow optimization, circuit design, path planning.
- Geneology analysis, computer game-playing, program compilation, object-oriented design, …

Simple Graphs

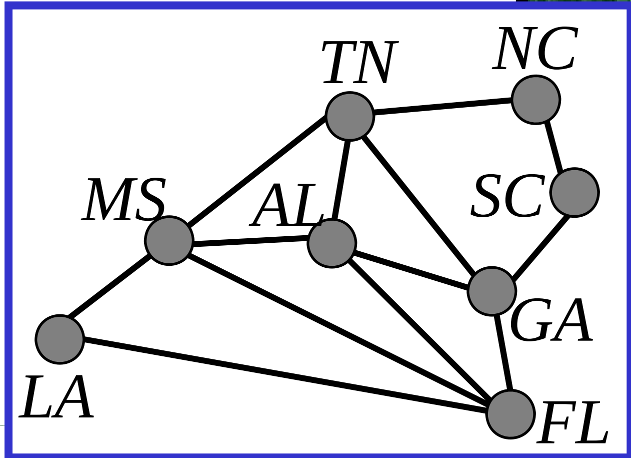
- Correspond to symmetric binary relations *R*.
- A *simple graph G*=(*V*,*E*) consists of:
  - a set *V* of *vertices* or *nodes* (*V* corresponds to the universe of the relation *R*),
  - a set *E* of *edges* / *arcs* / *links*: unordered pairs of [distinct?] elements *u*,*v* ∈ *V*, such that *uRv*.



Visual Representation of a Simple Graph

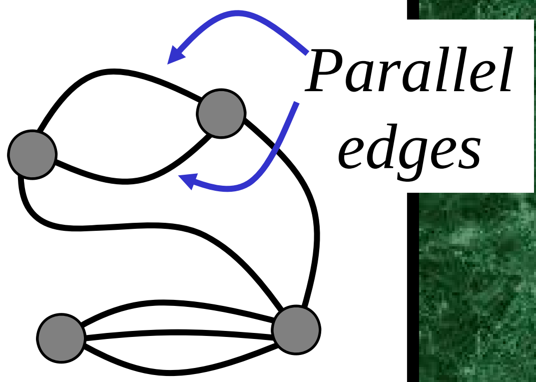
Example of a *Simple Graph*

- Let *V* be the set of states in the far-southeastern U.S.:
  - *V*={FL, GA, AL, MS, LA, SC, TN, NC}
- Let *E*={{*u*,*v*}|*u* adjoins *v*}
  ={{FL,GA},{FL,AL},{FL,MS},{FL,LA},{GA,AL},{AL,MS},{MS,LA},{GA,SC},{GA,TN},{SC,NC},{NC,TN},{MS,TN},{MS,AL}}



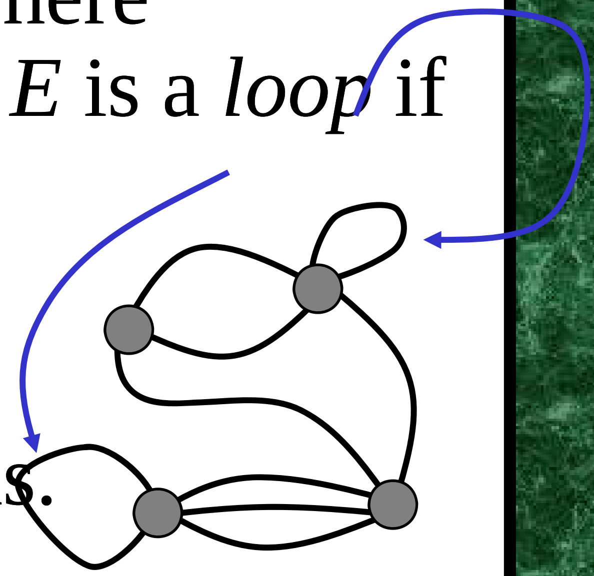
Multigraphs

- Like simple graphs, but there may be *more than one* edge connecting two given nodes.
- A *multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges (as primitive objects), and a function $f:E\rightarrow\{\{u,v\}|u,v\in V \wedge u\neq v\}$.
- E.g., nodes are cities, edges are segments of major highways.



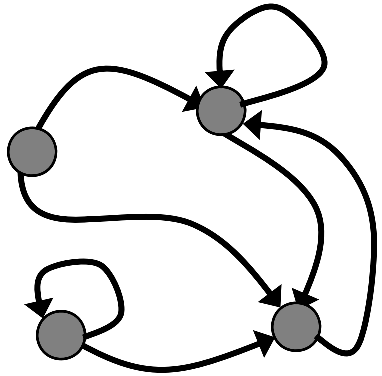
Pseudographs

- Like a multigraph, but edges connecting a node to itself are allowed.
- A *pseudograph* $G=(V, E, f)$ where $f:E\rightarrow\{\{u,v\}|u,v\in V\}$. Edge $e\in E$ is a *loop* if $f(e)=\{u,u\}=\{u\}$.
- *E.g.*, nodes are campsites in a state park, edges are hiking trails through the woods.



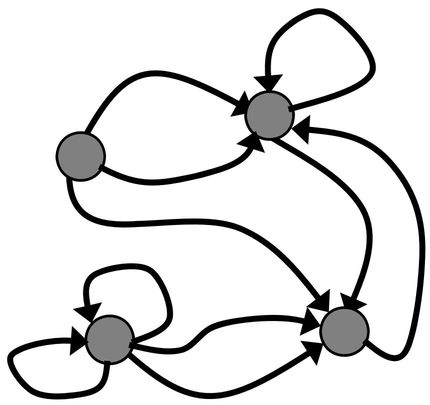
Directed Graphs

- Correspond to arbitrary binary relations *R*, which need not be symmetric.
- A *directed graph* (*V*,*E*) consists of a set of vertices *V* and a binary relation *E* on *V*.
- *E.g.*: *V* = people, *E*={(*x*,*y*) | *x* loves *y*}



Directed Multigraphs

- Like directed graphs, but there may be more than one arc from a node to another.
- A *directed multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges, and a function $f:E\rightarrow V\times V$.
- E.g., V=web pages, E=hyperlinks. *The WWW is a directed multigraph...*



Types of Graphs: Summary

- Summary of the book's definitions.
- Keep in mind this terminology is not fully standardized...

Term	Edge type	Multiple edges ok?	Self-loops ok?
Simple graph	Undir.	No	No
Multigraph	Undir.	Yes	No
Pseudograph	Undir.	Yes	Yes
Directed graph	Directed	No	Yes
Directed multigraph	Directed	Yes	Yes

§8.2: Graph Terminology

- *Adjacent, connects, endpoints, degree, initial, terminal, in-degree, out-degree, complete, cycles, wheels, n-cubes, bipartite, subgraph, union.*

Adjacency

Let G be an undirected graph with edge set E. Let $e \in E$ be (or map to) the pair $\{u,v\}$. Then we say:

- u, v are *adjacent* / *neighbors* / *connected*.
- Edge e is *incident with* vertices u and v.
- Edge e *connects* u and v.
- Vertices u and v are *endpoints* of edge e.

Degree of a Vertex

- Let G be an undirected graph, $v \in V$ a vertex.
- The *degree* of v, $\deg(v)$, is its number of incident edges. (Except that any self-loops are counted twice.)
- A vertex with degree 0 is *isolated*.
- A vertex of degree 1 is *pendant*.

Handshaking Theorem

- Let G be an undirected (simple, multi-, or pseudo-) graph with vertex set V and edge set E. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

- Corollary: Any undirected graph has an even number of vertices of odd degree.

Directed Adjacency

- Let *G* be a directed (possibly multi-) graph, and let *e* be an edge of *G* that is (or maps to) (*u*,*v*). Then we say:
  - *u* is *adjacent to v*, *v* is *adjacent from u*
  - *e comes from* u, e *goes to* v.
  - *e connects u to v*, *e goes from u to v*
  - the *initial vertex* of *e* is *u*
  - the *terminal vertex* of *e* is *v*

Directed Degree

- Let G be a directed graph, v a vertex of G.
  - The *in-degree* of v, $\deg^{-}(v)$, is the number of edges going to v.
  - The *out-degree* of v, $\deg^{+}(v)$, is the number of edges coming from v.
  - The *degree* of v, $\deg(v) \equiv \deg^{-}(v) + \deg^{+}(v)$, is the sum of v's in-degree and out-degree.

Directed Handshaking Theorem

- Let G be a directed (possibly multi-) graph with vertex set V and edge set E. Then:

$$\sum_{v\in V}\deg^{-}(v)=\sum_{v\in V}\deg^{+}(v)=\frac{1}{2}\sum_{v\in V}\deg(v)=|E|$$

- Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.

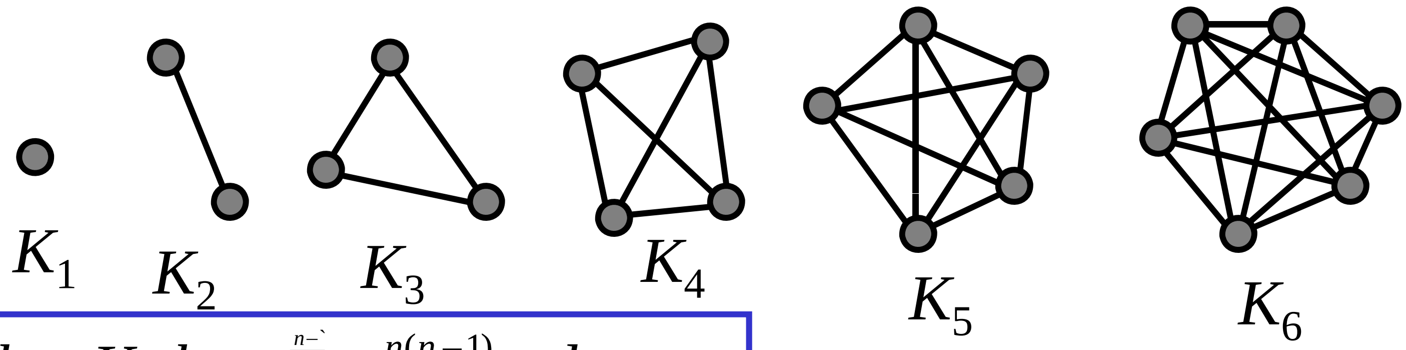
Special Graph Structures

Special cases of undirected graph structures:

- Complete graphs K_n
- Cycles C_n
- Wheels W_n
- n-Cubes Q_n
- Bipartite graphs
- Complete bipartite graphs $K_{m,n}$

Complete Graphs

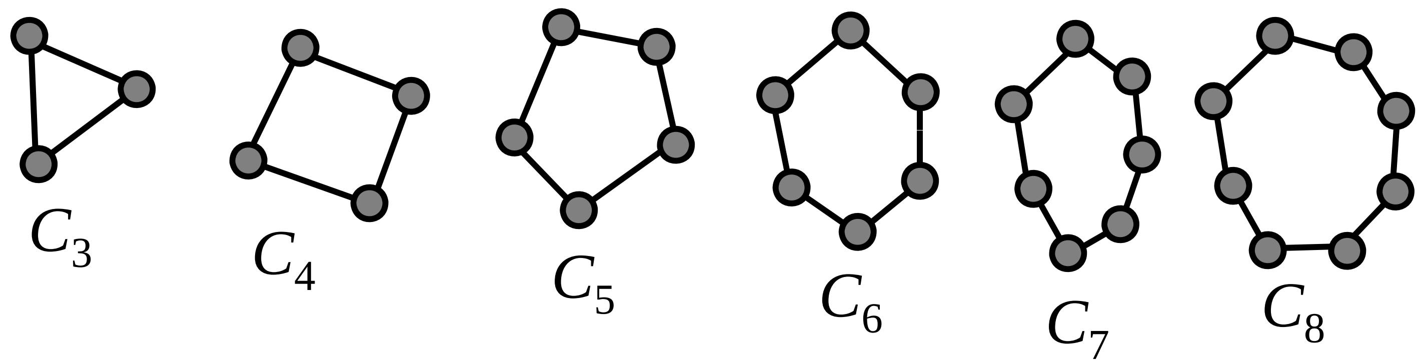
- For any $n \in \mathbf{N}$, a *complete graph* on n vertices, K_n, is a simple graph with n nodes in which every node is adjacent to every other node: $\forall u,v \in V$: $u \neq v \leftrightarrow \{u,v\} \in E$.



Note that K_n has $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ edges.

Cycles

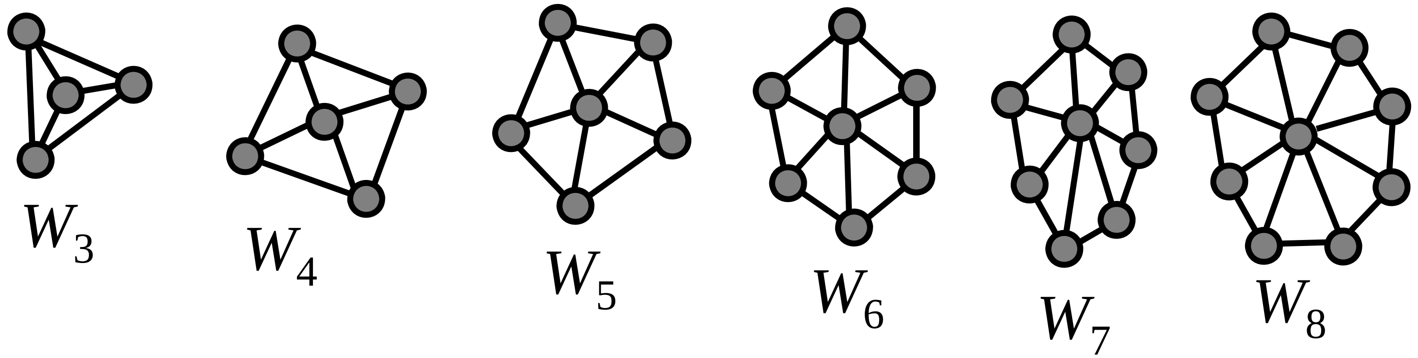
- For any $n \geq 3$, a *cycle* on n vertices, C_n, is a simple graph where $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}\}$.



How many edges are there in C_n?

Wheels

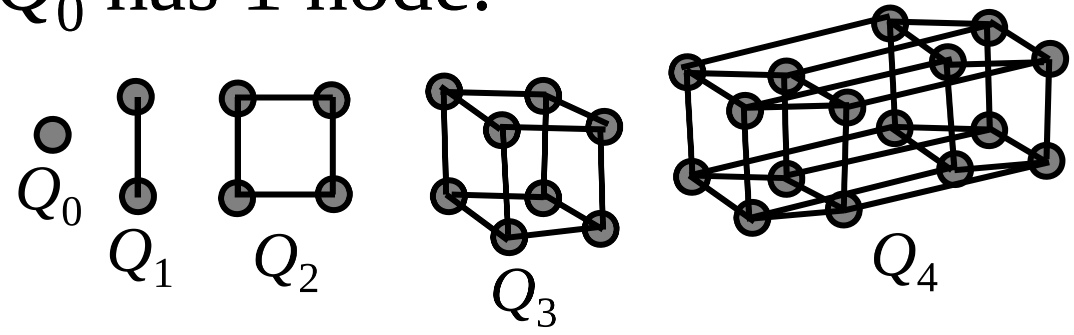
- For any $n \geq 3$, a *wheel* W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{\text{hub}},v_1\}, \{v_{\text{hub}},v_2\},\ldots,\{v_{\text{hub}},v_n\}\}$.



How many edges are there in W_n?

n-cubes (hypercubes)

- For any $n \in \mathbf{N}$, the hypercube Q_n is a simple graph consisting of two copies of Q_{n-1} connected together at corresponding nodes. Q_0 has 1 node.



Number of vertices: 2^n. *Number of edges:Exercise to try!*

n-cubes (hypercubes)

- For any $n \in \mathbf{N}$, the hypercube Q_n can be defined recursively as follows:
  - $Q_0=\{\{v_0\},\varnothing\}$ (one node and no edges)
  - For any $n \in \mathbf{N}$, if $Q_n=(V,E)$, where $V=\{v_1,\ldots,v_a\}$ and $E=\{e_1,\ldots,e_b\}$, then $Q_{n+1}=(V\cup\{v_1',\ldots,v_a'\}, E\cup\{e_1',\ldots,e_b'\}\cup\{\{v_1,v_1'\},\{v_2,v_2'\},\ldots, \{v_a,v_a'\}\})$ where $v_1',\ldots,v_a'$ are new vertices, and where if $e_i=\{v_j,v_k\}$ then $e_i'=\{v_j',v_k'\}$.

Bipartite Graphs

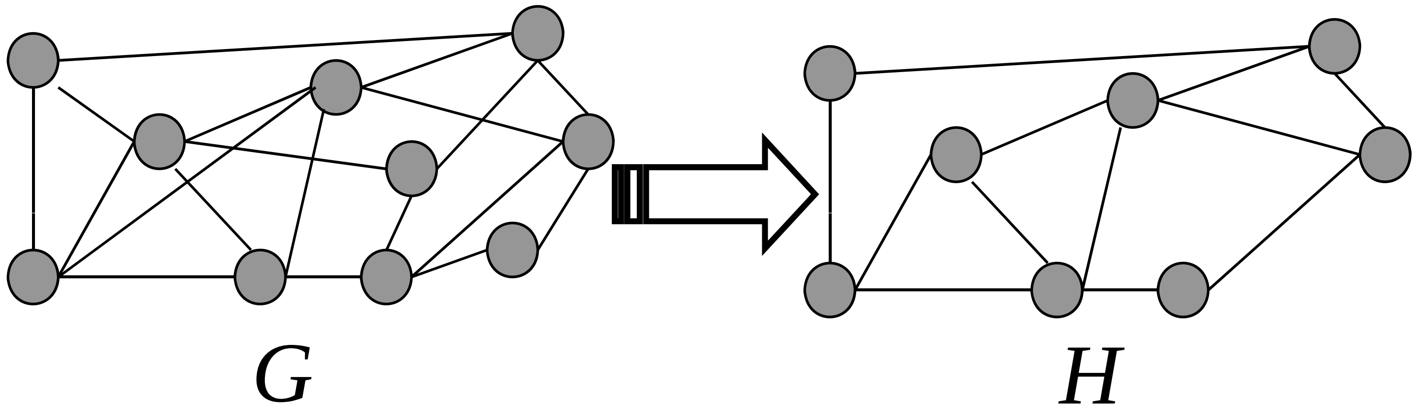
- Skipping this topic for this semester…

Complete Bipartite Graphs

- Skip...

Subgraphs

- A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W\subseteq V$ and $F\subseteq E$.



Graph Unions

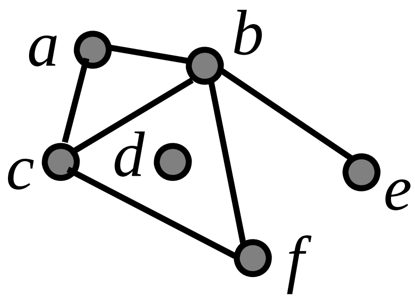
- The *union* $G_1 \cup G_2$ of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2,E_2)$ is the simple graph $(V_1 \cup V_2, E_1 \cup E_2)$.

§8.3: Graph Representations & Isomorphism

- Graph representations:
  - Adjacency lists.
  - Adjacency matrices.
  - Incidence matrices.
- Graph isomorphism:
  - Two graphs are isomorphic iff they are identical except for their node names.

Adjacency Lists

- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	*Adjacent Vertices*
a	*b*, *c*
b	*a*, *c*, *e*, *f*
c	*a*, *b*, *f*
d	
e	*b*
f	*c*, *b*

Directed Adjacency Lists

- 1 row per node, listing the terminal nodes of each edge incident from that node.

Adjacency Matrices

- Matrix **A**=[a_{ij}], where a_{ij} is 1 if {v_i, v_j} is an edge of G, 0 otherwise.

Graph Isomorphism

- Formal definition:
  - Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f{:}V_1\rightarrow V_2$ such that $\forall\ a,b\in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.
  - f is the "renaming" function that makes the two graphs identical.
  - Definition can easily be extended to other types of graphs.

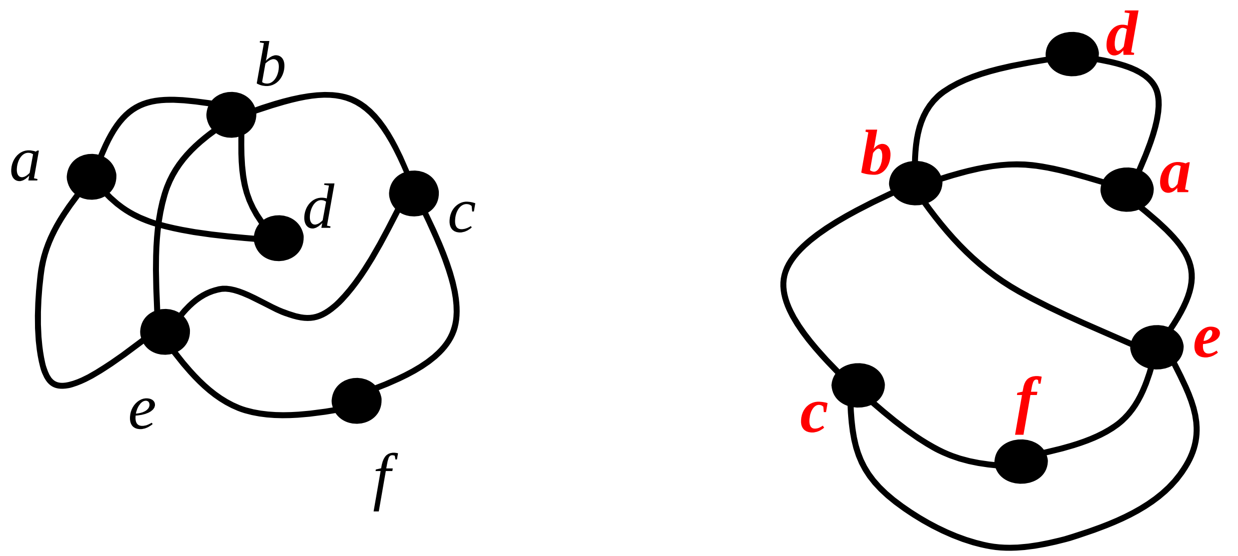
Graph Invariants under Isomorphism

Necessary but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:

- $|V1|=|V2|$, $|E1|=|E2|$.
- The number of vertices with degree n is the same in both graphs.
- For every proper subgraph g of one graph, there is a proper subgraph of the other graph that is isomorphic to g.

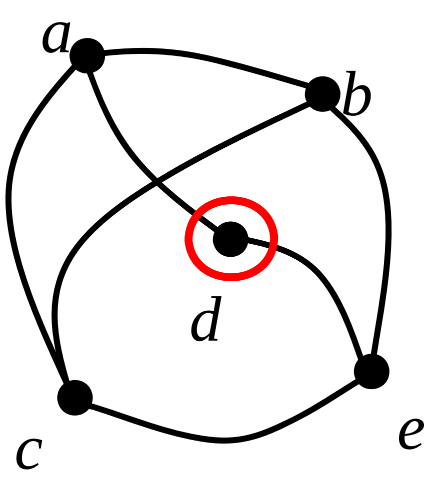
Isomorphism Example

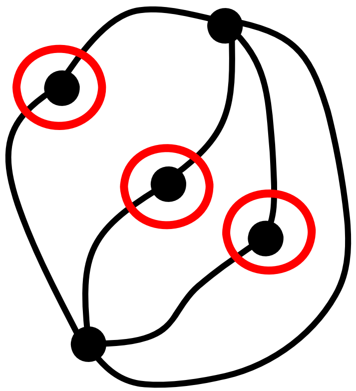
- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



Are These Isomorphic?

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.





** Same # of vertices*
** Same # of edges*
** Different # of verts of degree 2! (1 vs 3)*

§8.4: Connectivity

- In an undirected graph, a *path of length n from u to v* is a sequence of adjacent edges going from vertex u to vertex v.
- A path is a *circuit* if *u*=*v*.
- A path *traverses* the vertices along it.
- A path is *simple* if it contains no edge more than once.

Paths in Directed Graphs

- Same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness

- An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.
- Theorem: There is a *simple* path between any pair of vertices in a connected undirected graph.
- *Connected component*: connected subgraph
- A *cut vertex* or *cut edge* separates 1 connected component into 2 if removed.

Directed Connectedness

- A directed graph is *strongly connected* iff there is a directed path from *a* to *b* for any two verts *a* and *b*.
- It is *weakly connected* iff the underlying *undirected* graph (*i.e.*, with edge directions removed) is connected.
- Note *strongly* implies *weakly* but not vice-versa.

Paths & Isomorphism

- Note that connectedness, and the existence of a circuit or simple circuit of length *k* are graph invariants with respect to isomorphism.

Counting Paths w Adjacency Matrices

- Let **A** be the adjacency matrix of graph G.
- The number of paths of length k from v_i to v_j is equal to $(\mathbf{A}^k)_{i,j}$. (The notation $(\mathbf{M})_{i,j}$ denotes $m_{i,j}$ where $[m_{i,j}] = \mathbf{M}$.)

§8.5: Euler & Hamilton Paths

- An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.
- An *Euler path* in *G* is a simple path containing every edge of *G*.
- A *Hamilton circuit* is a circuit that traverses each vertex in *G* exactly once.
- A *Hamilton path* is a path that traverses each vertex in G exactly once.

Some Useful Theorems

- A connected multigraph has an Euler circuit iff each vertex has even degree.
- A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree.
- If (but <u>not</u> only if) *G* is connected, simple, has $n \geq 3$ vertices, and $\forall v$ $\deg(v) \geq n/2$, then *G* has a Hamilton circuit.