Field and Wave Electromagnetics

Chapter 3. Static Electric Fields

3-1 Introduction

- Coulomb's law → Experimental law

$$\vec{F}_{12} = \hat{a}_{R_{12}} k \frac{q_1 q_2}{R_{12}^2}$$

3.2 Fundamental Postulates of Electrostatics in Free Space

- Force : $\vec{F}$ (N)

- Electric field intensity (전계 강도)

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \quad [\mathrm{V/m}] \text{ or } [\mathrm{N/C}]$$

why? - Not to disturb the charge distribution of the source

3.2 Fundamental Postulates of Electrostatics in Free Space

❖ Two fundamental postulates of electrostatics

$$\begin{cases} \nabla \bullet \vec{E} = \dfrac{\rho}{\varepsilon_0} & \text{—①} \quad \text{(MKS)} \\ \nabla \times \vec{E} = 0 \rightarrow \text{irrotational} & \text{—②} \end{cases}$$

note $\nabla \bullet \vec{E} = 4\pi\rho$ (cgs)

$$① \rightarrow \int_v \nabla \bullet \vec{E} dv = \frac{1}{\varepsilon_0} \int_v \rho dv$$

$$= \boxed{\oint_S \vec{E} \bullet d\vec{s} = \frac{Q}{\varepsilon_0}}$$ Gauss's law

$$② \rightarrow \oint_c \vec{E} \bullet d\vec{l} = 0$$

$$\because \oint_S (\nabla \times \vec{E}) \bullet d\vec{s} = \underbrace{\oint \vec{E} \bullet d\vec{l} = 0}$$

→ Kirchhoff's voltage law

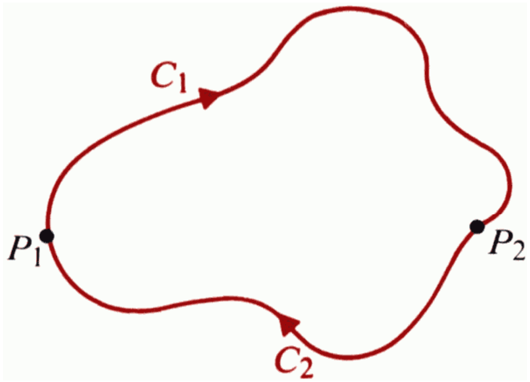
3.2 Fundamental Postulates of Electrostatics in Free Space

❖ Conservative field

$$\int_{C_1} \vec{E} \bullet d\vec{l} + \int_{C_2} \vec{E} \bullet d\vec{l} = 0$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2} \vec{E} \bullet d\vec{l}} = - \underset{\text{Along } C_2}{\int_{P_2}^{P_1} \vec{E} \bullet d\vec{l}}$$

$$= \underset{\text{Along } C_2}{\int_{P_1}^{P_2} \vec{E} \bullet d\vec{l}}$$



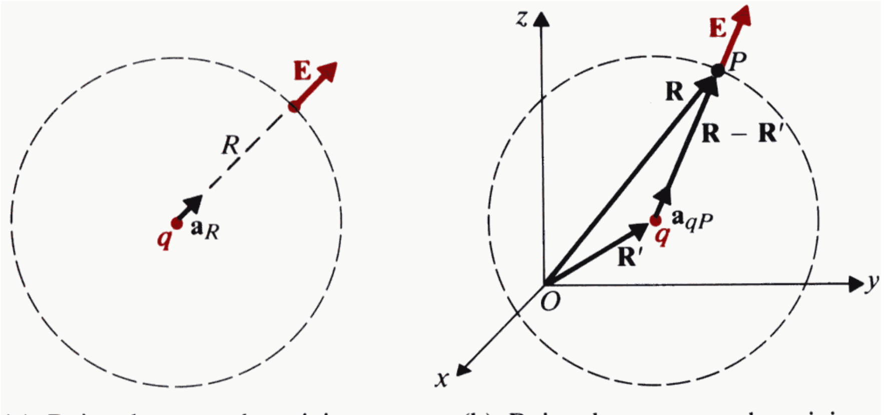
Why?

(Usually $\int \vec{E} \bullet d\vec{l}$ depends on integral path)

cf) $\because \vec{E}$ is irrotational!!

3-3 Coulomb's Law

- Coulomb's law



(a) Point charge at the origin. (b) Point charge not at the origin.

$$\oint_S \vec{E} \bullet d\vec{s} = \oint_S (\hat{R}\, E_R) \bullet \hat{R}\, ds = \frac{q}{\varepsilon_0}$$

$$\therefore E_R = \frac{q}{4\pi\varepsilon_0 R^2}$$

$$\vec{E} = \hat{R}\, E_R = \hat{R}\frac{q}{4\pi\varepsilon_0 R^2} \quad [\mathrm{V/m}] \text{ or } [\mathrm{N/C}]$$

General Expression

$$\vec{E}_p = \frac{q}{4\pi\varepsilon_0 \left|\vec{R}-\vec{R}'\right|^2}\,\hat{a}_{qp} = \frac{q(\vec{R}-\vec{R}')}{4\pi\varepsilon_0 \left|\vec{R}-\vec{R}'\right|^3}$$

cf) $\vec{F} = \hat{R}\dfrac{q_1 q_2}{4\pi\varepsilon_0 R^2}$ [N]

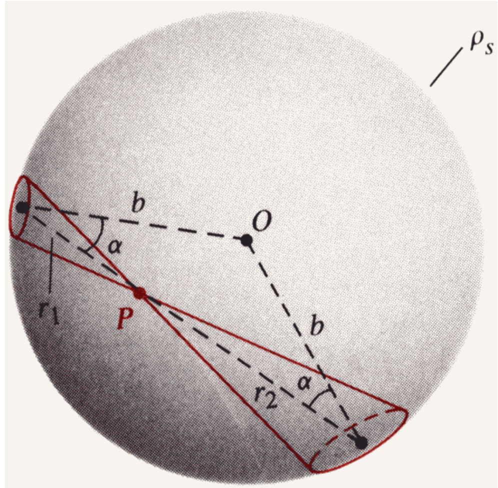
3-3 Coulomb's Law

❖ Charged Shell

$$dE = \frac{\rho_s}{4\pi\varepsilon_0}\left(\frac{ds_1}{r_1^2} - \frac{ds_2}{r_2^2}\right)$$

solid angle $\dfrac{d\Omega\cdot r_1^2}{ds_1} = \cos\alpha = \dfrac{d\Omega\cdot r_2^2}{ds_2}$

$$\therefore\ dE = 0$$

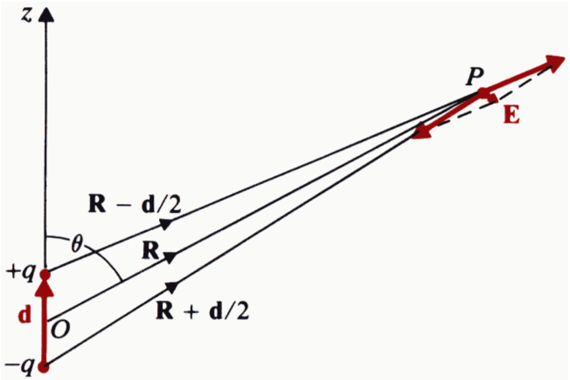


❖ Discrete Charges

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k(\vec{R}-\vec{R}_k')}{\left|\vec{R}-\vec{R}_k'\right|^3} \quad [\mathrm{V/m}]$$

3-3.1 Electric Field due to a System of Discrete Charges

❖ Electric dipole



$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left\{\frac{\vec{R}-\dfrac{\vec{d}}{2}}{\left|\vec{R}-\dfrac{\vec{d}}{2}\right|^3}-\frac{\vec{R}+\dfrac{\vec{d}}{2}}{\left|\vec{R}+\dfrac{\vec{d}}{2}\right|^3}\right\}$$

$$\left|\vec{R}-\frac{\vec{d}}{2}\right|^{-3}=\left[\left(\vec{R}-\frac{\vec{d}}{2}\right)\cdot\left(\vec{R}-\frac{\vec{d}}{2}\right)\right]^{-\frac{3}{2}}=\left[R^2-\vec{R}\cdot\vec{d}+\frac{d^2}{4}\right]^{-\frac{3}{2}} \quad , \; R\gg d$$

$$\cong R^{-3}\left[1-\frac{\vec{R}\cdot\vec{d}}{R^2}\right]^{-\frac{3}{2}}\cong R^{-3}\left[1+\frac{3}{2}\frac{\overbrace{\vec{R}\cdot\vec{d}}^{Rd\cos\theta}}{R^2}\right] \quad \text{(binomial expansion)}$$

note $\quad (1\pm x)^n=1\pm nx\pm\dfrac{n(n-1)}{2!}x^2\pm\dfrac{n(n-1)(n-2)}{3!}x^3\pm\cdots$

$$\left|\vec{R}+\frac{\vec{d}}{2}\right|^{-3}\cong R^{-3}\left[1-\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}\right] \qquad \therefore \boxed{\vec{E}\cong\frac{q}{4\pi\varepsilon_0 R^3}\left[3\cdot\frac{\vec{R}\cdot\vec{d}}{R^2}\vec{R}-\vec{d}\right]}$$

3-3.1 Electric Field due to a System of Discrete Charges

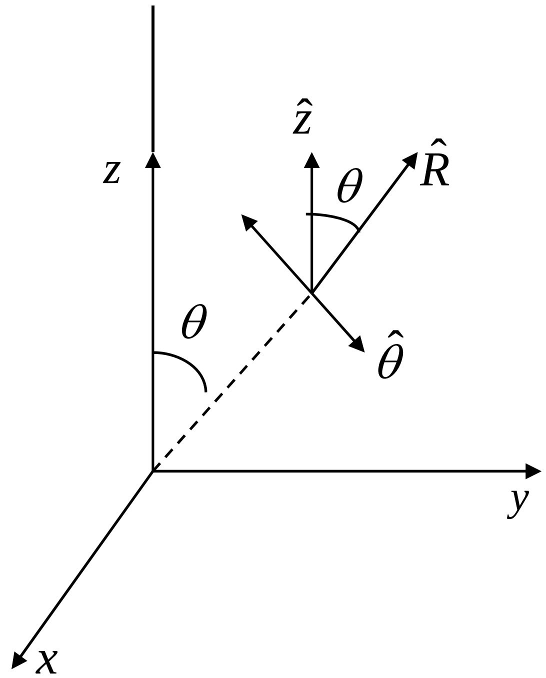
Define $\vec{p} = q\vec{d}$ $(\textit{electric dipole moment vector})$

then, $\vec{E} = \frac{1}{4\pi\varepsilon_0 R^3}\left[3 \cdot \frac{\vec{R}\bullet\vec{p}}{R^2}\vec{R} - \vec{p}\right]$, $\hat{z} = \cos\theta\,\hat{R} - \sin\theta\,\hat{\theta}$

$$\left(\vec{R}\bullet\vec{p} = Rp\cos\theta\right)$$

$$\vec{p} = p\left(\hat{R}\cos\theta - \hat{\theta}\sin\theta\right)$$

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{R}\,2\cos\theta + \hat{\theta}\sin\theta\right)\ [\mathrm{V/m}]$$



3-3.2 Electric field due to a continuous distribution of charge

❖ Continuous distribution

$$d\vec{E}=\hat{R}\frac{\rho\, dv'}{4\pi\varepsilon_0 R^2}\longrightarrow \vec{E}=\frac{1}{4\pi\varepsilon_0}\int_{v'}\hat{R}\frac{\rho}{R^2}dv',\ \text{where } \hat{R}=\frac{\vec{R}}{R}$$

$$\therefore\ \vec{E}=\frac{1}{4\pi\varepsilon_0}\int_{v'}\rho\frac{\vec{R}}{R^3}dv'\ \text{[V/m]}$$

cf) Surface & line charge density case

☞ $\vec{E}=\frac{1}{4\pi\varepsilon_0}\int_{S'}\hat{R}\frac{\rho_s}{R^2}ds'\ \text{[V/m]}$

☞ $\vec{E}=\frac{1}{4\pi\varepsilon_0}\int_{L'}\hat{R}\frac{\rho_l}{R^2}dl'\ \text{[V/m]}$

Ex 3-4 Infinitely long line charge

- Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_l in air

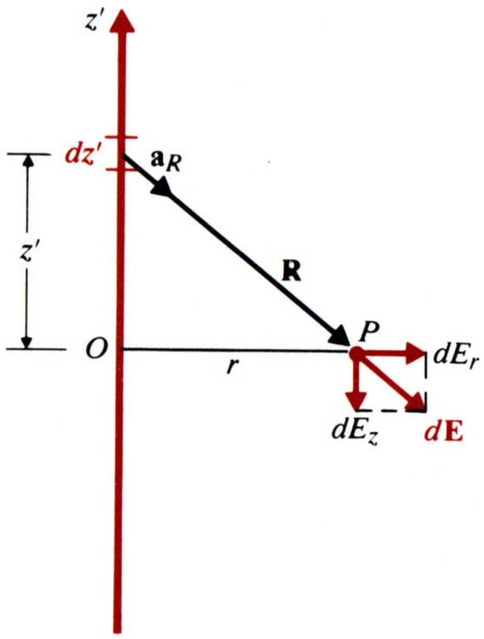
sol) $\vec{R} = \hat{r}\, r - \hat{z}\, z', \ \rho_l\, dl' = \rho_l\, dz'$

$$d\vec{E} = \frac{\rho_l\, dz'}{4\pi\varepsilon_0}\frac{\vec{R}}{R^3} = \frac{\rho_l\, dz'}{4\pi\varepsilon_0}\frac{\hat{r}\, r - \hat{z}\, z'}{(r^2 + z'^2)^{\frac{3}{2}}} = \hat{r}\, dE_r + \hat{z} dE_z$$

where $dE_r = \dfrac{\rho_l\, r\, dz'}{4\pi\varepsilon_0 (r^2 + z'^2)^{\frac{3}{2}}}$

$$dE_z = -\frac{\rho_l\, z' dz'}{4\pi\varepsilon_0 (r^2 + z'^2)^{\frac{3}{2}}}$$

$$\text{at } z = z', \ \begin{cases} dE_r = \dfrac{\rho_l\, r\, dz'}{4\pi\varepsilon_0 (r^2 + z'^2)^{\frac{3}{2}}} \\ dE_z\big|_{z=z'} = \dfrac{-\rho_l\, z' dz'}{4\pi\varepsilon_0 (r^2 + z'^2)^{\frac{3}{2}}} \end{cases}$$



Ex 3-4 Infinitely long line charge

$$\text{at } z=-z', \begin{cases} dE_r = \dfrac{\rho_l r dz'}{4\pi\varepsilon_0 (r^2+z'^2)^{\frac{3}{2}}} \\ dE_z\big|_{z=-z'} = \dfrac{\rho_l z' dz'}{4\pi\varepsilon_0 (r^2+z'^2)^{\frac{3}{2}}} \quad = -dE_z\big|_{z=z'} \end{cases}$$

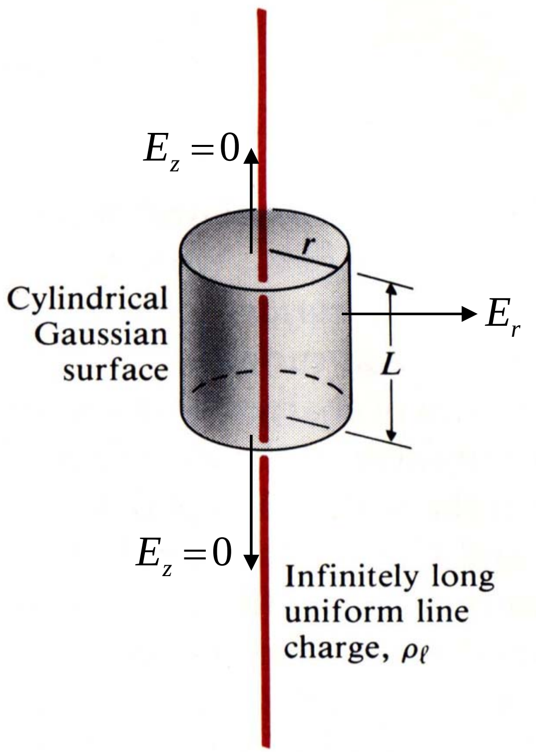
$$\therefore \vec{E} = \hat{r} E_r = \hat{r}\frac{\rho_l r}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dz'}{(r^2+z'^2)^{\frac{3}{2}}} = \hat{r}\frac{\rho_l}{2\pi\varepsilon_0 r} \quad \text{[V/m]}$$

HW. Integrate to have the above result

cf) Symmetry $\rightarrow$ Gauss law

3-4 Gauss's Law and Applications

❖ Cylindrical symmetry



- $\nabla \bullet \vec{E} = \dfrac{\rho}{\varepsilon_0} \Rightarrow \int_{v'} \left(\nabla \bullet \vec{E}\right) dv = \dfrac{Q}{\varepsilon_0}$

  $\xleftrightarrow{\text{By Divergence theorem}} \oint_s \vec{E} \bullet \vec{ds} = \dfrac{Q}{\varepsilon_0}$

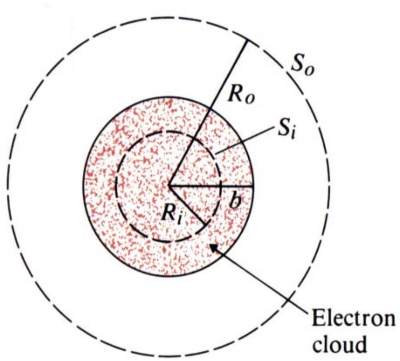
- $\oint_S \vec{E} \bullet \vec{ds} = \int_0^L \int_0^{2\pi} E_r \, r \, d\phi \, dz = 2\pi r L E_r = \dfrac{\rho_l L}{\varepsilon_0}$,

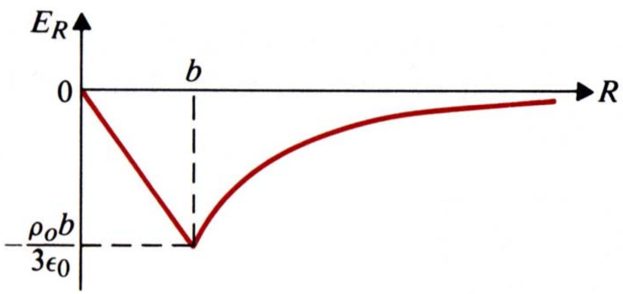
  where, $\vec{E} = \hat{r} E_r$, $E_Z = 0$

  $\therefore \boxed{E_r = \dfrac{\rho_l}{2\pi\varepsilon_0 r}}$

Ex 3-7 Spherical electron cloud

$$\begin{cases} \rho = -\rho_0 \ , & \text{for } 0 \le R \le b \\ \rho = 0 \ , & \text{for } R > b \end{cases}$$





❖ Spherical symmetry $\rightarrow$ i.e. $\vec{E} = \hat{r}\, E_r$

i) $0 \le R \le$ b

$$\oint_{S_i} \vec{E} \cdot \vec{ds} = E_R \int_{S_i} ds = E_R 4\pi R^2$$

$$Q = \int_v \rho\, dv = -\rho_0 \int_v dv = -\rho_0 \cdot \frac{4\pi}{3} R^3$$

$$\therefore \vec{E} = -\hat{R} \cdot \frac{\rho_0}{3\varepsilon_0} R, \quad 0 \le R \le b$$

ii) $R \ge$ b

$$Q = -\rho_0 \frac{4\pi}{3} b^3 \quad \text{(fixed)}$$

$$\therefore \vec{E} = -\hat{R} \frac{\rho_0 b^3}{3\varepsilon_0 R^2}, \quad R \ge b$$

3-5 Electric Potential

- $\nabla \times \vec{E} = 0 \rightarrow \vec{E} = -\nabla V$ (Vector Identity)

  $\frac{W}{q} = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}$ [J/C or V] (Work must be done against the field)

  $V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}$ [V] $\quad \because -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl} = -\int_{P_1}^{P_2} (-\nabla V) \bullet (\hat{l}\, dl) = \int_{P_1}^{P_2} dV$

  - Remember when we define gradient

    $$\frac{dV}{dl} = \frac{dV}{dn} \bullet \frac{dn}{dl} = \frac{dV}{dn} \cos\alpha = \frac{dV}{dn} \hat{n} \bullet \hat{l} = (\nabla V) \bullet \hat{l}$$

    $$\therefore \; dV = (\nabla V) \bullet \vec{dl}$$

cf) Refernece point of potential: the zero potential point is usually at infinity.

3-5.1 Electric potential due to a charge distribution

- Point charge at origin, potential at R w.r.t. that at infinity

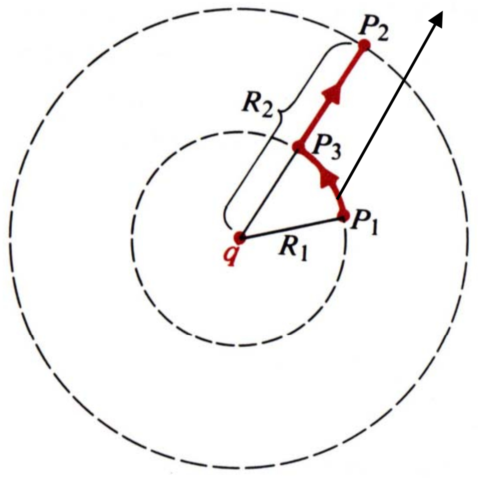
$$\vec{E} = \hat{R}\frac{q}{4\pi\varepsilon_0 R^2}$$

$$\therefore V = -\int_{\infty}^{R}\left(\hat{R}\frac{q}{4\pi\varepsilon_0 R^2}\right)\bullet(\hat{R}\,dR) = \frac{q}{4\pi\varepsilon_0 R} \text{ [V]}$$

- V_{21} : Potential difference between point P_1 and P_2

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

3-5.1 Electric potential due to a charge distribution



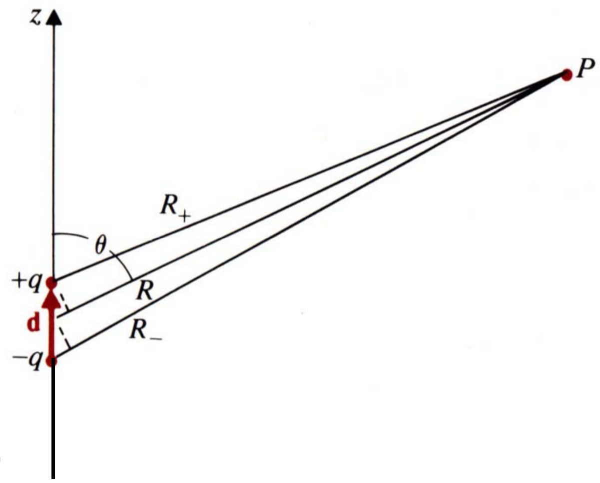
- $\vec{dl} = R\ d\phi \bullet \hat{\phi}$

  $\vec{F} = F_r\ \hat{r}$

  $\therefore\ \vec{F} \bullet \vec{dl} = 0$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k}{\left|\bar{R} - \bar{R}_k^{'}\right|} \quad [\mathrm{V}]}$$

Ex 3-8 An Electric dipole



- $$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

$$d \ll R, \quad \begin{cases} \dfrac{1}{R_+} \simeq \left(R - \dfrac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \dfrac{d}{2R}\cos\theta\right) \\ \dfrac{1}{R_-} \simeq \left(R + \dfrac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \dfrac{d}{2R}\cos\theta\right) \end{cases}$$

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{1}{4\pi\varepsilon_0 R^2}\vec{p}\cdot\hat{R}\ , \text{ where } \vec{p} = q\vec{d} \text{ : dipole moment vector}$$

$$\vec{E} = -\nabla V = -\hat{R}\frac{\partial V}{\partial R} - \hat{\theta}\frac{\partial V}{R\partial\theta} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\hat{R}2\cos\theta + \hat{\theta}\sin\theta\right)$$

$$cf)\ \nabla V = \hat{u}_1\frac{\partial V}{h_1\partial u_1} + \hat{u}_2\frac{\partial V}{h_2\partial u_2} + \hat{u}_3\frac{\partial V}{h_3\partial u_3}$$

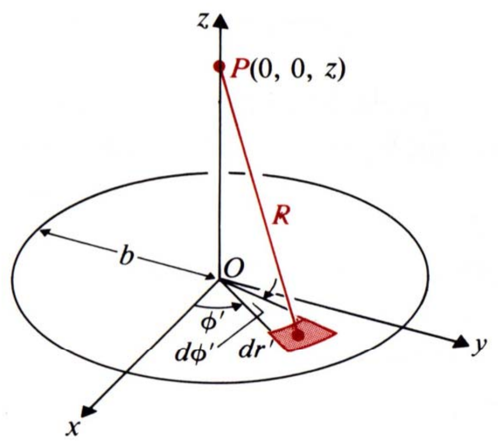
Note

1. Equipotential line : $R = c_V\sqrt{\cos\theta}$
2. E – field line : $R = c_E\sin^2\theta$

❖Potential due to continuous charge

- $V = \dfrac{1}{4\pi\varepsilon_0}\int_{v'} \dfrac{\rho}{R} dv'$ [V] : volume charge distribution
- $V = \dfrac{1}{4\pi\varepsilon_0}\int_{s'} \dfrac{\rho}{R} ds'$ [V] : surface charge distribution
- $V = \dfrac{1}{4\pi\varepsilon_0}\int_{L'} \dfrac{\rho_l}{R} dl'$ [V] : line charge distribution

Ex 3-9 A uniformly charged disk



❖ Obtain a formula for the electric field intensity

sol) $ds' = r'dr'd\phi'$ and $R = \sqrt{z^2 + r'^2}$

$$V = \frac{\rho_s}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{b}\frac{r'}{\left(z^2+r'^2\right)^{\frac{1}{2}}}dr'd\phi'$$

$$= \frac{\rho_s}{2\varepsilon_0}\left[\left(z^2+r'^2\right)^{\frac{1}{2}} - |z|\right]\Bigg|_0^b$$

cf) $\int \frac{r'}{\left(z^2+r'^2\right)^{\frac{1}{2}}}dr'$, substitute $z^2 + r'^2 = k^2$

$$\left.\begin{array}{l} ① \; r'=0,\; k^2 = z^2 \rightarrow k = |z| \\ ② \; r'=b,\; k^2 = z^2+b^2 \rightarrow k = \left(z^2+b^2\right)^{\frac{1}{2}} \\ ③ \; 2r'dr' = 2k\,dk \end{array}\right\} \Rightarrow \therefore \int \frac{k\,dk}{\left(k^2\right)^{\frac{1}{2}}} = \int dk = k\Big|_{|z|}^{\left(z^2+b^2\right)^{\frac{1}{2}}}$$

Ex 3-9 A uniformly charged disk

$$\vec{E} = -\nabla V = -\hat{z}\frac{\partial V}{\partial z} = \begin{cases} \hat{z}\dfrac{\rho_s}{2\varepsilon_0}\left[1 - z\left(z^2+b^2\right)^{-\frac{1}{2}}\right], & z>0 \\ -\hat{z}\dfrac{\rho_s}{2\varepsilon_0}\left[1 + z\left(z^2+b^2\right)^{-\frac{1}{2}}\right], & z<0 \end{cases}$$

❖ *Note*

For very large z, $\vec{E} = \hat{z}\dfrac{\pi b^2 \rho_s}{4\pi\varepsilon_0 z^2} = \begin{cases} \hat{z}\dfrac{Q}{4\pi\varepsilon_0 z^2}, & z>0 \\ -\hat{z}\dfrac{Q}{4\pi\varepsilon_0 z^2}, & z<0 \end{cases}$

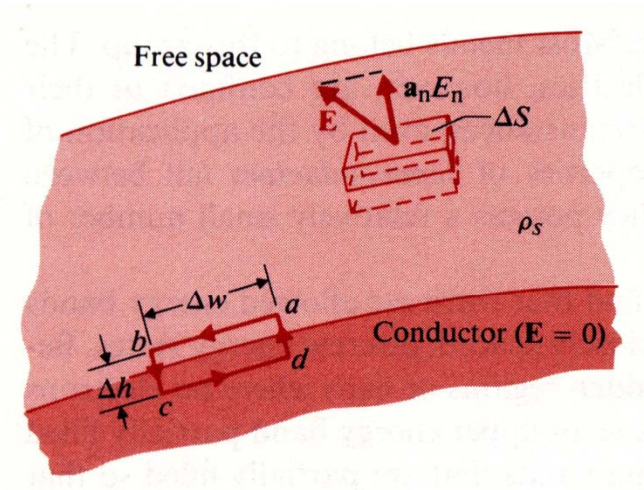
3-6 Conductors in Static Electric Field

- Conductor : Free Electrons
  - The electrons in the outermost shells of the atoms of conductors are free to move
- Insulator or dielectrics
  - The electrons in the atoms of insulators are confined to their orbits: no free electrons
- Semiconductors
  - A relatively small number of freely movable charge carriers
    → Allowed energy bands for electrons.
    → Forbidden band → Band gap

3-6 Conductors in Static Electric Field

- Charge introduced → Field will be set up → Charges will move away from each other to reach conductor's surface until the charge and the field inside vanish
  : Inside a conductor $\rho = 0, \vec{E} = 0$

- Under static conditions, the E field on a conductor surface is everywhere normal to the surface → Equipotential surface (No tangential component)



$$\oint_{abcda} \vec{E} \bullet \vec{dl} = E_t \, \Delta w = 0 \ (\because \text{E inside conductor is zero})$$

$$\therefore \boxed{E_t = 0}$$

$$\oint_s \vec{E} \bullet \vec{ds} = E_n \, \Delta s = \frac{\rho_s \, \Delta s}{\varepsilon_0}$$

$$\therefore \boxed{E_n = \frac{\rho_s}{\varepsilon_0}}$$

3-6 Conductors in Static Electric Field

- Boundary conditions at a conductor / Free space interface

$$\begin{cases} E_t = 0 \\ E_n = \dfrac{\rho_s}{\varepsilon_0} \end{cases} \qquad (\because E\ field = 0\ inside\ conductor)$$

- Uncharged conductor in a static field
  - Electrons move in a direction opposite to that of the field
  - Positive charges move in the direction of the field
  - So that induced free charge will distribute on the conductor surface and create an induced field in such a way to cancel the external field inside the conductor and tangent to its surface

Ex 3-11 A point charge +Q at the center of a conducting shell

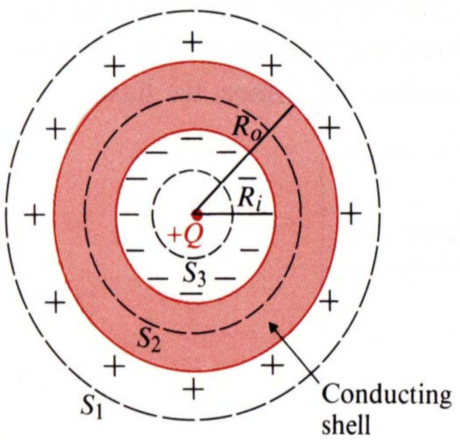
❖ Determine $\vec{E}$ and V as functions of the radial distance R

"Ungrounded"

(a) $R > R_0$ $\because$ Spherical Symmetry $\vec{E} = E_R \hat{R}$

$$\oint_S \vec{E} \cdot \vec{ds} = E_{R1} 4\pi R^2 = \frac{Q}{\varepsilon_0}$$

$$\therefore E_{R1} = \frac{Q}{4\pi\varepsilon_0 R^2},\ V_1 = -\int_\infty^R (E_{R1})\, dR = \frac{Q}{4\pi\varepsilon_0 R}$$



(b) $R_i < R < R_0$, Inside a conductor

$$\vec{E} = 0 \qquad \therefore E_{R2} = 0$$

$$V_2 = V_1\big|_{R=R_0} = \frac{Q}{4\pi\varepsilon_0 R_0},$$ whole conducting shell is an equipotential body

Note

An amount of negative charge equal to $-Q$ must be induced on the inner shell surface at $R = R_i$ in order to cancel out the field induced by positive charge $+Q$ at center.
This implies again that $+Q$ must be induced on the outer shell surface at $R = R_0$.

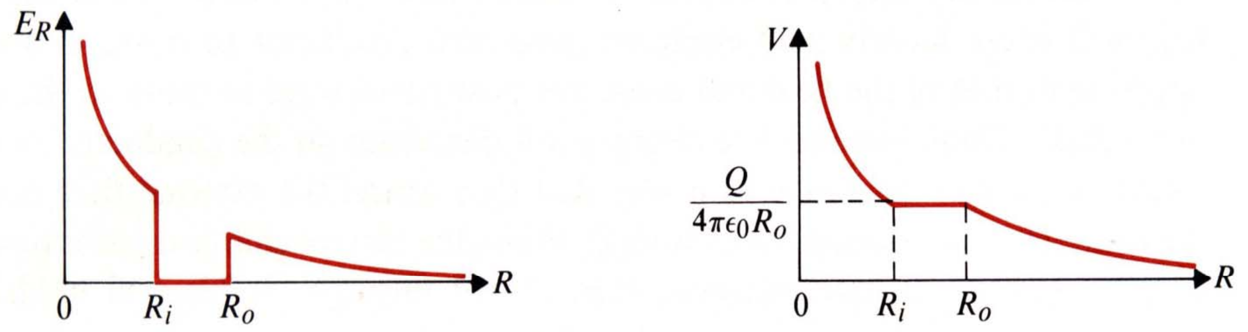
Ex 3-11 A point charge +Q at the center of a conducting shell

(c) $R < R_i$

$$E_{R3} = \frac{Q}{4\pi\varepsilon_0 R^2},\ V_3 = -\int E_{R3}\, dR + C = \frac{Q}{4\pi\varepsilon_0 R} + C$$

where, V_3 at $R = R_i$ is equal to V_2 at $R = R_0$

$$C = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_0} - \frac{1}{R_i}\right) \qquad \therefore V_3 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R} + \frac{1}{R_0} - \frac{1}{R_i}\right)$$



Electric field intensity and potential variations of
A point charge +Q at the center of a conducting shell