Dielectrics in Static Electric Field

- No free charge in dielectrics to make interior charge density and electric field vanish

- Dielectrics contain bound charge
  \( \Rightarrow \) effect on the electric field

- E-field \( \Rightarrow \) small displacement of positive and negative charges (bound charge) \( \Rightarrow \) polarize a dielectric material

  - a. Polar molecules: permanent dipole moments
    - \( \text{ex) } \text{H}_2\text{O (two or more dissimilar atoms) } \Rightarrow \, P \sim 10^{-30} \, \text{(C\cdot m)} \)
    - Individual dipoles are randomly oriented
      \( \Rightarrow \) macroscopically no net dipole
    - Some have a permanent dipole moment even in the absence of external field \( \Rightarrow \) electrets

  - b. nonpolar molecules: no permanent dipole moments
Equivalent Charge Distribution of Polarized Dielectrics

Define polarization vector, $\overrightarrow{P}$

$$\overrightarrow{P} = \lim_{\Delta v \to 0} \sum_{k=1}^{n_{\Delta v}} P_k \Delta v$$

: volume density of electric dipole moment

$d\overrightarrow{P}$ of an elemental volume $d\overrightarrow{p} = \overrightarrow{P}dv'$

$$dV = \frac{\overrightarrow{P} \cdot \hat{R}}{4\pi \varepsilon_0 R^2} dv'$$

(cf) potential due to a dipole

$$V = \frac{\overrightarrow{P} \cdot \hat{R}}{4\pi \varepsilon_0 R^2}$$

$$\therefore V = \frac{1}{4\pi \varepsilon_0} \int_{V'} \frac{\overrightarrow{P} \cdot \hat{R}}{R^2} dv'$$

where $R$ is the distance from $dv'$ to a fixed field point.

(cf) $R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ in Cartesian coordinate.

$$\nabla' \left( \frac{1}{R} \right) = \nabla' \left( \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right) = \frac{\overrightarrow{R}}{R^3} = \frac{\hat{R}}{R^2}$$

$$\overrightarrow{R} = (x-x') \hat{x} + (y-y') \hat{y} + (z-z') \hat{z}$$
Equivalent Charge Distribution of Polarized Dielectrics

\[ V = \frac{1}{4\pi\varepsilon_0} \int v' \bar{P} \cdot \nabla' \left( \frac{1}{R} \right) dv' \]

cf) \( \nabla' \cdot (f A) = f \nabla' \cdot A + A \nabla' f \) \( \Rightarrow \) \( \bar{P} \cdot \nabla' \left( \frac{1}{R} \right) = \nabla' \left( \frac{\bar{P}}{R} \right) - \frac{1}{R} (\nabla' \cdot \bar{P}) \)

\[ V = \frac{1}{4\pi\varepsilon_0} \left[ \int v' \nabla' \left( \frac{\bar{P}}{R} \right) dv' - \int v' \frac{\nabla' \cdot \bar{P}}{R} dv' \right] \]

\[ = \frac{1}{4\pi\varepsilon_0} \int_{s'} \frac{\bar{P} \cdot \hat{n}'}{R} ds' + \frac{1}{4\pi\varepsilon_0} \int_{v'} \left( -\nabla' \cdot \bar{P} \right) dv' \]

cf) potential due to surface and volume charge

\[ V = \frac{1}{4\pi\varepsilon_0} \int v' \frac{\rho_v}{R} dv' \quad (3-61), \quad V = \frac{1}{4\pi\varepsilon_0} \int_{s'} \frac{\rho_s}{R} ds' \quad (3-62) \]

\[ \rho_{ps} = \bar{P} \cdot \hat{n} \]

\[ \rho_p = -\nabla' \cdot \bar{P} \]

Equivalent polarization surface charge density

Equivalent polarization volume charge density

Polarization charge density or bound charge density
Equivalent Charge Distribution of Polarized Dielectrics

cf) Imaginary elemental surface $\Delta s$ of a nonpolar dielectric, net charge crossing the surface $\Delta s$ is,

$$\Delta Q = nq (\vec{d} \cdot \hat{n}) \Delta s,$$

where $n$ is the number of molecules per unit volume $nq\vec{d}$: dipole moment per unit volume $\Rightarrow$ polarization vector $\vec{P}$

$$\Delta Q = \vec{P} \cdot \hat{n} (\Delta s) \Rightarrow \rho_{ps} = \frac{\Delta Q}{\Delta s} = \vec{P} \cdot [\hat{n}] \text{ outward normal}$$

cf) For a surface $S$ bounding a volume $V$, the net total charge flowing out of $V$ = negative of the net charge remaining within the volume $V$

$$Q = - \oint_S \vec{P} \cdot \hat{n} \, ds = \int_V - (\nabla \cdot \vec{P}) \, dv = \int_V \rho_p \, dv$$
Electric Flux density and Dielectric Constant

\[ \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} (\rho + \rho_p) \]

\[ \rho_p = -\nabla \cdot \vec{P} \]

\[ \therefore \nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho \]

Define new fundamental quantity

\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \text{ (C/m}^2\text{)} \]: electric flux density or electric displacement

\[ \nabla \cdot \vec{D} = \rho \text{ (C/m}^3\text{)} \]

valid everywhere

\[ \text{cf} \) \nabla \times \vec{E} = 0 \]

Free charge density note no \( \varepsilon_0 \) appear

Two of static Maxwell equations

\[ \therefore \int_s \vec{D} \cdot d\vec{s} = Q \Rightarrow \text{Gauss's law} \left( \int_s \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0} \right) \]
Electric Flux density and Dielectric Constant

- Permittivity of dielectric material
  - linear and isotropic dielectric media:
    - Polarization is directly proportional to the electric field intensity
      \[ \vec{P} = \varepsilon_0 \chi_e \vec{E} \]
      where \( \chi_e \): electric susceptibility (dimensionless quantity)

  - cf) medium is linear if \( \chi_e \) is independent of \( \vec{E} \)
  - medium is homogeneous if \( \chi_e \) is independent of space coordinate.

\[
\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} \\
= \varepsilon_0 (1 + \chi_e) \vec{E} \\
= \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}
\]

\( \varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0} \): relative permittivity or dielectric constant of the medium
\( \varepsilon = \varepsilon_0 \varepsilon_r \): absolute permittivity (permittivity)
Electric Flux density and Dielectric Constant

- anisotropic medium
  - Dielectric constant is different for different directions of the electric field
    \[ \vec{D} \text{ and } \vec{E} \text{ vectors generally have different directions.} \]
    \[ \Rightarrow \text{ permittivity is a tensor.} \]
    \[
    \begin{bmatrix}
    D_x \\
    D_y \\
    D_z 
    \end{bmatrix} = \begin{bmatrix}
    \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
    \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
    \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
    \end{bmatrix}
    \begin{bmatrix}
    E_x \\
    E_y \\
    E_z 
    \end{bmatrix}
    \]

- Dielectric strength
  - Maximum electric field intensity that dielectric material can withstand without breakdown.
Ex) 3-13. Two connected conducting spheres

✓ Two spherical conductors with radius \( b_1 \) and \( b_2 \) (\( b_2 > b_1 \))
✓ Connected by a conducting wire
✓ Distance is large enough to ignore influence of each sphere on the other
✓ Total charge \( Q \) is deposited on the sphere

sol) Two conductors are at the same potential

\[ \frac{Q_1}{4\pi\varepsilon_0 b_1} = \frac{Q_2}{4\pi\varepsilon_0 b_2} \]

\[ \frac{Q_1}{Q_2} = \frac{b_1}{b_2}, \quad Q_1 + Q_2 = Q \]

\[ Q_1 = \frac{b_1}{b_1 + b_2} Q, \quad Q_2 = \frac{b_2}{b_1 + b_2} Q \]

\[ \therefore \frac{E_{1n}}{E_{2n}} = \left( \frac{b_2}{b_1} \right)^2 \frac{Q_1}{Q_2} = \frac{b_2}{b_1} \]

larger curvature \( \Rightarrow \) smaller sphere: higher electric field intensity
Boundary Conditions for Electrostatic Fields

1. $\Delta h \to 0$
   \[
   \oint_{abcd} \mathbf{E} \cdot d\mathbf{l} = \mathbf{E}_1 \cdot \Delta \mathbf{w} + \mathbf{E}_2 \cdot (-\Delta \mathbf{w})
   = E_{1t} \Delta w - E_{2t} \Delta w = 0
   \]
   \[
   \therefore \mathbf{E}_{1t} = \mathbf{E}_{2t} \quad \text{or} \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}
   \]

   the tangential component of an $\mathbf{E}$ field is continuous across an interface.

2. Cylinder $\Delta h \to 0$
   \[
   \oint_s \mathbf{D} \cdot d\mathbf{s} = (\mathbf{D}_1 \cdot \mathbf{n}_2 + \mathbf{D}_2 \cdot \mathbf{n}_2) \Delta S = \mathbf{n}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) \Delta S = Q = \rho_s \Delta S
   \]
   \[
   \therefore \mathbf{n}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad \text{or} \quad \mathbf{n}_1 \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s
   
   \text{i.e.,} \quad D_{1n} - D_{2n} = \rho_s \, (C/m^2)
   \]

   reference normal is $\mathbf{n}_2$. 

(outward normal to medium 1)
Ex) 3-15 Boundary conditions

- Tangential $\vec{E}$ should be continuous at boundary.
  
  $$E_2 \sin \alpha_2 = E_1 \sin \alpha_1$$

- Normal $\vec{D}$ should be continuous at boundary.
  
  $$\varepsilon_2 E_2 \cos \alpha_2 = \varepsilon_1 E_1 \cos \alpha_1$$
  
  $$\therefore E_2 = \sqrt{E_{2t}^2 + E_{2n}^2} = \sqrt{(E_2 \sin \alpha_2)^2 + (E_2 \cos \alpha_2)^2}$$
  
  $$= \sqrt{(E_1 \sin \alpha_1)^2 + \left(\frac{\varepsilon_1}{\varepsilon_2} E_1 \cos \alpha_1\right)^2} = E_1 \left[\sin^2 \alpha_1 + \left(\frac{\varepsilon_1}{\varepsilon_2} \cos \alpha_1\right)^2\right]^{1/2}$$

**Note 1**

The normal component of $\vec{D}$ field is discontinuous across an interface where a surface charge exists.

**Note 2**

If $\rho_s = 0$, then $D_{1n} = D_{2n}$

Summary:

$$\begin{bmatrix} E_{1t} = E_{2t} \\ \hat{n}_1 \cdot (\vec{D}_2 - \vec{D}_1) = \rho_s \end{bmatrix}$$
Ex) 3-16. coaxial cable

- The radius of the inner conductor : 0.4 cm
- Concentric layers of rubber $\varepsilon_{rr} = 3.2$
- Polystyrene $\varepsilon_{rp} = 2.6$

Design a cable that is to work at a voltage rating of 20kV. $\rightarrow$ E field are not to exceed 25% of their dielectric strength.

sol) Dielectric strength of rubber : $25 \times 10^6$ V/m  
Dielectric strength of polystyrene : $20 \times 10^6$ V/m  
Cylindrical symmetry $\rightarrow$ Consider only $E_r$ component

$$\text{max } E_r = 0.25 \times 25 \times 10^6 = \frac{\rho_i}{2\pi \varepsilon_0} \left( \frac{1}{3.2 \ r_i} \right)$$

$$\text{max } E_p = 0.25 \times 20 \times 10^6 = \frac{\rho_i}{2\pi \varepsilon_0} \left( \frac{1}{2.6 \ r_p} \right)$$

$$r_p = 1.54 r_i = 0.616 \ [\text{cm}]$$
Ex) 3-16. coaxial cable

Potential difference

\[-\int_{r_0}^{r_p} E_p \, dr - \int_{r_p}^{r_i} E_r \, dr = 20,000\]

\[
\frac{\rho_l}{2\pi\varepsilon_0} \left[ \frac{1}{\varepsilon_r} \left( -\int_{r_0}^{r_p} \frac{1}{r} \, dr \right) + \frac{1}{\varepsilon_r} \left( -\int_{r_p}^{r_i} \frac{1}{r} \, dr \right) \right] = \frac{\rho_l}{2\pi\varepsilon_0} \left( \frac{1}{2.6} \ln \frac{r_o}{r_p} + \frac{1}{3.2} \ln \frac{r_p}{r_i} \right) = 20,000
\]

\[r_i = 0.4, \quad r_p = 0.616,\]

\[\therefore \frac{\rho_l}{2\pi\varepsilon_0} = 0.25 \times 20 \times 10^6 \times 2.6r_p = 8 \times 10^4, \quad \therefore r_o = 2.08r_i = 0.832 \text{ [cm]}
\]

cf) \[
\frac{5}{4} = \frac{2.6r_p}{3.2r_i} \Rightarrow r_p = \frac{5}{4} \times \frac{3.2}{2.6} r_i
\]

If order is reversed,

\[0.25 \times 20 \times 10^6 = \frac{\rho_l}{2\pi\varepsilon_0} \cdot \frac{1}{2.6} \cdot \frac{1}{r_i}, \quad 0.25 \times 25 \times 10^6 = \frac{\rho_l}{2\pi\varepsilon_0} \cdot \frac{1}{3.2} \cdot \frac{1}{r_r}
\]

\[\frac{4}{5} = \frac{3.2r_r}{2.6r_i}, \quad \therefore r_r = \frac{4}{5} \times \frac{2.6}{3.2} r_i = \frac{10.4}{16} r_i, \quad r_r < r_i \Rightarrow \text{Non sense}\]
(Cheng 3 - 6) Two very small conducting spheres, each of a mass $1.0 \times 10^{-4} \text{ (kg)}$, are suspended at a common point by very thin nonconducting threads of a length 0.2 (m). A charge $Q$ is placed on each sphere. The electric force of repulsion separates the spheres, and an equilibrium is reached when the suspending threads make an angle of 10°. Assuming a gravitational force of 9.90 (N/kg) and a negligible mass of the threads, find $Q$.

(Cheng 3 - 12) Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b$ ($b > a$), carry surface charge densities $\rho_{sa}$ and $\rho_{sb}$, respectively.

a) Determine $\vec{E}$ everywhere.

b) What must be the relation between $a$ and $b$ in order that $E$ vanishes for $r > b$?
(Cheng 3 - 27) What are the boundary conditions that must be satisfied by the electrics potential at an interface between two perfect dielectrics with dielectric constants $\varepsilon_{r1}$ and $\varepsilon_{r2}$?

A point charge q is enclosed in a linear, isotropic, and homogeneous dielectric medium of infinite extent. Calculate the $\vec{E}$ field, the $\vec{D}$ field, the polarization vector $\vec{P}$, the bound surface charge density $\rho_{sb}$, and the bound volume charge density $\rho_{vb}$.

A very thin, finite and uniformly charged line of length 10 m carries a charge of 10 $\mu$C/m. Calculate the electric field intensity in a plane bisecting the line at $\rho = 5$m.

Show that the magnitude of the electric field intensity of an electric dipole is

$$E = \frac{p}{4\pi\varepsilon_0 r^3} \left[ 1 + 3 \cos^2 \theta \right]^{1/2}$$