Capacitance and Capacitors

❖ Capacitance

① Conducting surface : Equipotential and only surface charge

② Suppose the potential V due to the total charge Q

③ Increasing the total charge by some factor k

$$\Rightarrow kQ \Rightarrow k\rho_s \quad (\because \text{ Only surface charge and area } \Delta S \text{ is fixed})$$

④ $V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{S'} \frac{\rho_s}{R} ds' \; [\mathrm{V}]$

$$\therefore \; V \Rightarrow kV \quad \text{and} \quad \vec{E} = -\nabla V \Rightarrow k\vec{E}$$

⑤ $\vec{E} = \hat{n}\dfrac{\rho_s}{\varepsilon_0} \; (\because E_t = 0) \Rightarrow \hat{n}\dfrac{k\rho_s}{\varepsilon_0} = k\vec{E}$

⑥ The ratio $\dfrac{Q}{V}$, therefore remain unchanged

$$\frac{Q}{V} = C \Rightarrow Q = CV$$

constant of proportionality C
$\Rightarrow$ Capacitance of the isolated conducting body

cf) unit of C : $[\mathrm{C/V}]$ or $[\mathrm{F{:}farad}]$

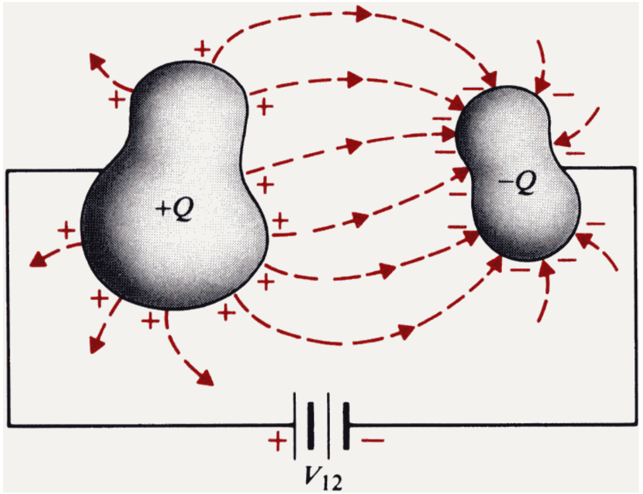
Capacitance and Capacitors

- Capacitor
  - Two conductors separated by free space or dielectric medium

    cf) field lines are perpendicular to the conductor surfaces

$$C = \frac{Q}{V_{12}}$$

depends on
- the geometry of the conductors
- the permittivity of the medium
- not the charge Q
- not the potential difference



cf) method to determine C

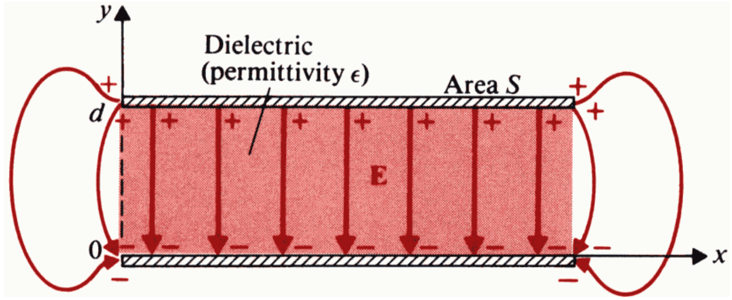
- Assuming V_{12} and determining Q in terms of V_{12}
- Assuming Q and determining V_{12} in terms of Q

Capacitance and Capacitors

- Procedure to find C
  - Choose a coordinate system for the given geometry
  - Assume charges $+Q$ and $-Q$ on the conductors
  - Find $\vec{E}$ from Q
  - Find V_{12} by $V_{12} = -\int_2^1 \vec{E} \bullet d\vec{l}$

    from the conductor having $-Q$ to the other having $+Q$
  - Find C by taking the ratio $\dfrac{Q}{V_{12}}$

Ex) 3-17. parallel-plate capacitor



- Area S separation distance d, constant permittivity ε; then the capacitance
  - Cartesian coordinate system
  - Assume $+Q$ and $-Q$ on the upper and lower conducting plates
  - So $\rho_s, -\rho_s$ on each plate $\rho_s = \dfrac{Q}{S}$

✓ $\vec{E} = \hat{n}\dfrac{\rho_s}{\varepsilon}, \qquad \therefore\ \vec{E} = -\hat{y}\dfrac{\rho_s}{\varepsilon} = -\hat{y}\dfrac{Q}{\varepsilon S}$

$$V_{12} = -\int_{y=0}^{y=d} \vec{E}\bullet d\vec{l} = -\int_0^d \left(-\hat{y}\frac{Q}{\varepsilon S}\right)\bullet(\hat{y}\,dy) = \frac{Q}{\varepsilon S}d$$

$$\therefore\ C = \frac{Q}{V_{12}} = \varepsilon\frac{S}{d}$$

☞ note 1 C is independent of Q and V.

☞ note 2 C depends on d and S.

Ex) 3-17. parallel-plate capacitor

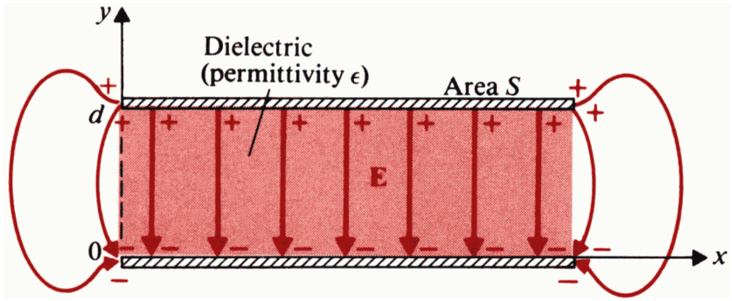
- cf) Assumption
  - Electric field intensity between the plates is uniform and equals.

$$\vec{E} = -\hat{y}\frac{V_{12}}{d}$$

$$\rho_s = \varepsilon E_y = \varepsilon \frac{V_{12}}{d}$$

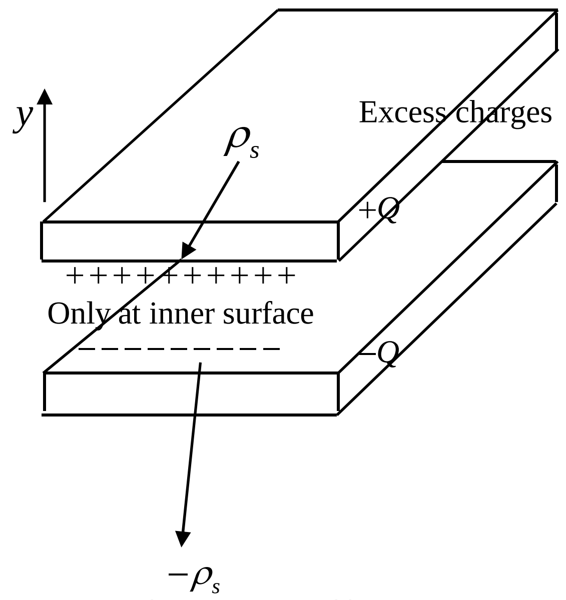
$$Q = \rho_s S = \left(\varepsilon \frac{S}{d}\right) V_{12} = C \cdot V_{12}$$

$$C = \varepsilon \frac{S}{d}$$



Ex) parallel-plate capacitor

- Capacitance of a two-conductor capacitor is defined as the ratio of the magnitude of charge on one of the conductors to the magnitude of the potential difference between the two conductors



$$\begin{cases} \vec{E}_{\text{upper}}\Big|_{\text{cond}} = \hat{y}\dfrac{\rho_s}{\varepsilon}, \qquad \rho_s = \dfrac{Q}{A} \\ \vec{E}_{\text{lower}}\Big|_{\text{cond}} = -\hat{y}\dfrac{\rho_s}{\varepsilon} \end{cases}$$

Assume $\vec{E}$ is uniform between two conductors and ignore fringing effect.

then, $\vec{E} = -\hat{y}\dfrac{\rho_s}{\varepsilon}$

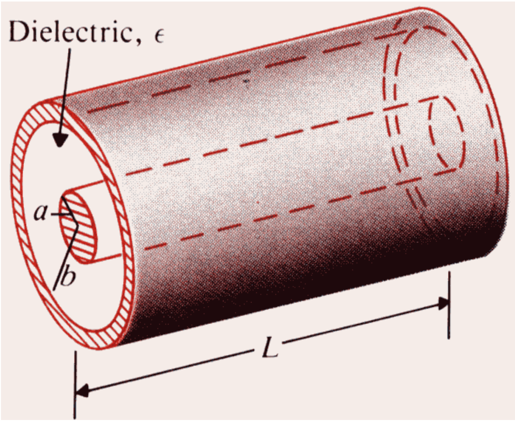
$$V_{12} = -\int_{y=0}^{d} \vec{E}\bullet d\vec{l} = -\int_{y=0}^{d}\left(-\hat{y}\frac{\rho_s}{\varepsilon}\right)\bullet(\hat{y}\,dy) = \frac{\rho_s}{\varepsilon}d$$

$$Q = \int_s \varepsilon\vec{E}\bullet d\vec{s} = \int_{\text{top}} \varepsilon\left(-\hat{y}\frac{\rho_s}{\varepsilon}\right)\bullet(-\hat{y})\,ds = \rho_s S$$

$$\boxed{\therefore C = \frac{Q}{V_{12}} = \varepsilon\frac{S}{d}}$$

Ex) 3-18. Cylindrical capacitor

- ✓ inner conductor radius : a
- ✓ inner radius of outer conductor : b



- Cylindrical coordinate system
  → z-axis along the cylinder
- Assume $+Q$ on the inner conductor,
  $-Q$ on the outer conductor
- Gauss's law
  - ✓ $\vec{E} = E_r \hat{r}$ at the conductor surface
    (only normal component on the conductor, tangential component is zero)
  - ✓ E_r is not constant anymore ($\because$ conductor surfaces are not planes)

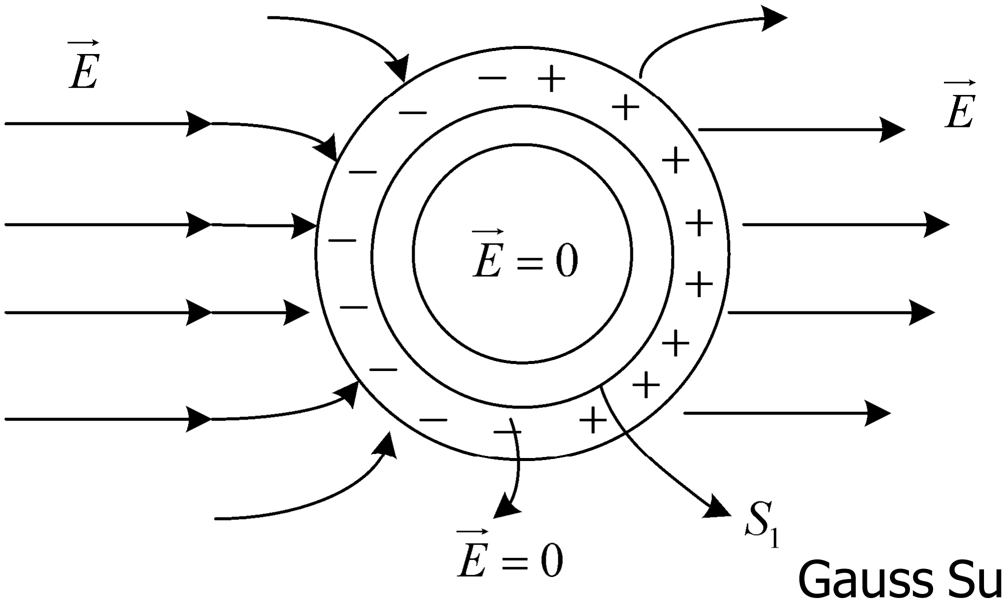
$$E_r = \frac{Q}{2\pi\varepsilon L r} \quad \left(\oint_S \vec{E} \bullet d\vec{s} = \int_0^L \int_0^{2\pi} E_r r \, d\phi dz = 2\pi r L E_r = \frac{Q}{\varepsilon} \right)$$

- neglect the fringing effect

$$V_{ab} = -\int_{r=b}^{r=a} \vec{E} \bullet d\vec{l} = -\int_b^a \left(\hat{r} \frac{Q}{2\pi\varepsilon L r} \right) \bullet (\hat{r} dr) = \frac{Q}{2\pi\varepsilon L} \ln\left(\frac{b}{a}\right)$$

$$\therefore \; C = \frac{Q}{V_{ab}} = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

Example

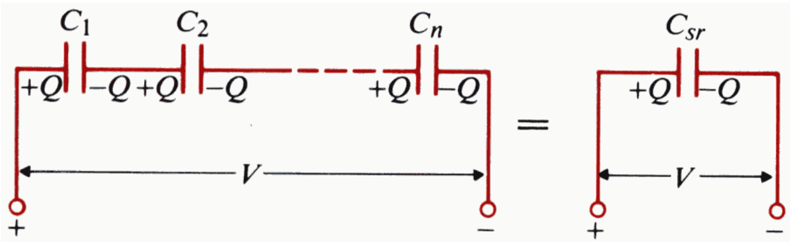


Gauss Surface

$\rightarrow$ No charge inside Gauss surface

$\Rightarrow \therefore$ No charge will be distributed on the inner surface

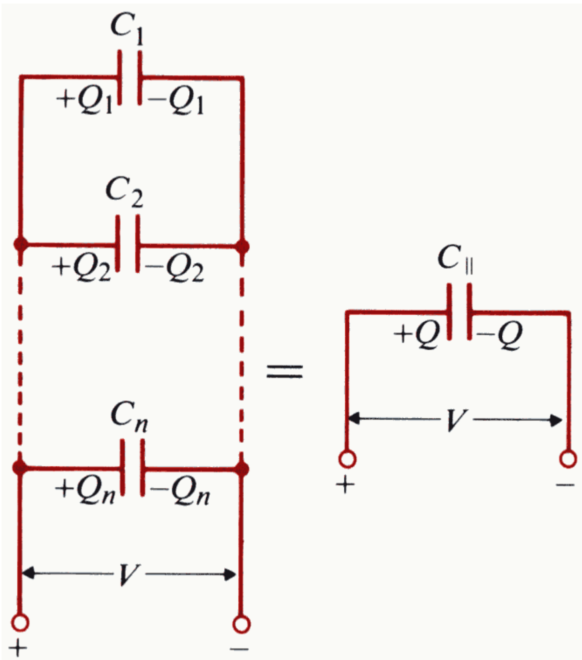
Series and Parallel Connections of Capacitors



$$V = V_1 + V_2 + V_3 + \cdots + V_n$$

$$V = \frac{Q}{C_{sr}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_n}$$

$$\therefore \boxed{\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}}$$



$$Q = Q_1 + Q_2 + \cdots + Q_n$$

$$= C_1 V + C_2 V + \cdots + C_n V$$

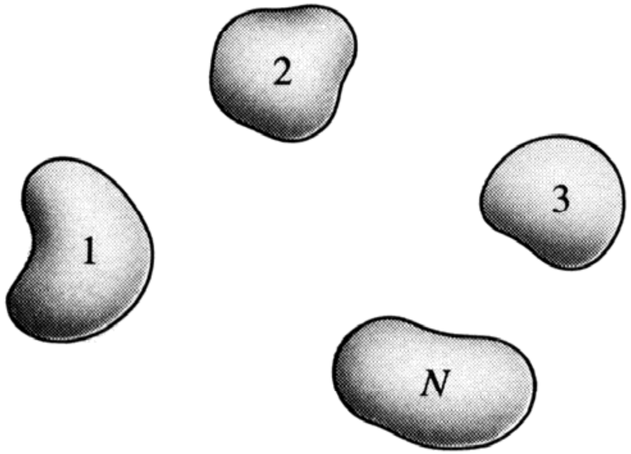
$$= C_{\|} V$$

$$\therefore \boxed{C_{\|} = C_1 + C_2 + \cdots + C_n}$$

Capacitances in Multiconductor Systems

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & \cdots & \cdots & p_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{bmatrix}$$

where p_{ij} : coefficient of potential



i.e.,

$$\begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & \cdots & \cdots & c_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N1} & \cdots & \cdots & c_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$

c_{ij} = coefficients of induction

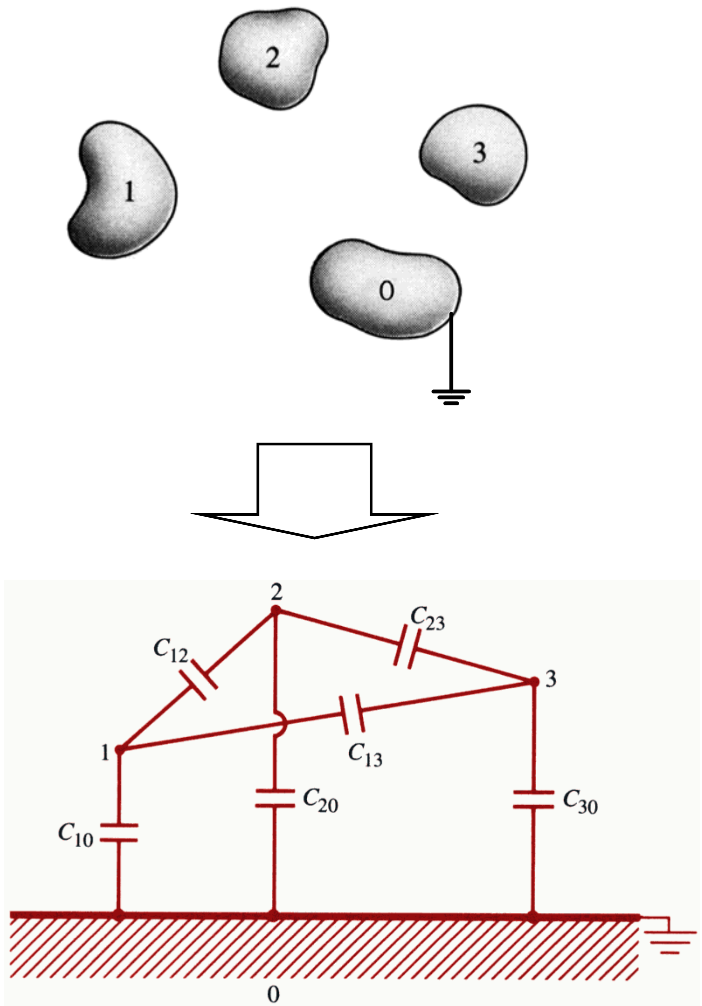
c_{ii} = coefficients of capacitance

$= \dfrac{Q_i}{V_i}$ with all other conductors grounded

$Q_1 + Q_2 + \cdots + Q_N = 0$ for isolated system

$p_{ij} = p_{ji},\ c_{ij} = c_{ji} \Rightarrow$ reciprocity condition

cf) Example of 4 conductors



$$\therefore \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad \text{or}$$

$$\begin{aligned} Q_1 &= C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3) \\ &= (C_{10} + C_{12} + C_{13})V_1 - C_{12}V_2 - C_{13}V_3 \\ Q_2 &= C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3) \\ Q_3 &= C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2) \end{aligned}$$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} C_{10} + C_{12} + C_{13} & -C_{12} & -C_{13} \\ -C_{12} & C_{20} + C_{12} + C_{23} & -C_{23} \\ -C_{13} & -C_{23} & C_{30} + C_{13} + C_{23} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\therefore \begin{cases} c_{11} = C_{10} + C_{12} + C_{13} \\ c_{22} = C_{20} + C_{12} + C_{23} \\ c_{33} = C_{30} + C_{13} + C_{23} \end{cases} \Rightarrow$$

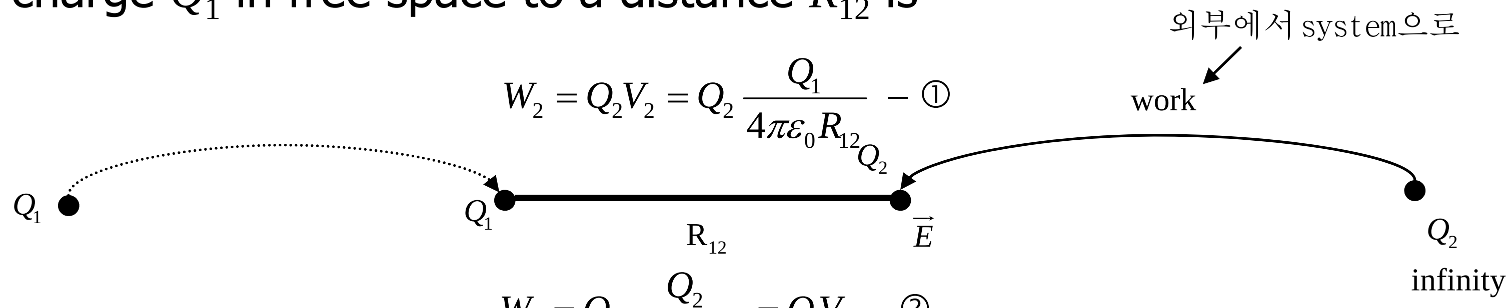
total capacitance between conductor 1 and all the other conductors connected together to ground

$$c_{12} = -C_{12},\ c_{23} = -C_{23},\ c_{13} = -C_{13}$$

Electrostatic Energy and Forces

❖ Work done by bring a charge Q_2 from infinity against the field of a charge Q_1 in free space to a distance R_{12} is

$$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\varepsilon_0 R_{12}} \quad -① $$



$$W_2 = Q_1 \frac{Q_2}{4\pi\varepsilon_0 R_{12}} = Q_1 V_1 \quad -②$$

combining ① and ②, $\qquad W_2 = \frac{1}{2}\left(Q_1 V_1 + Q_2 V_2\right)$

Bring another charge Q_3 from infinity to a point P_3

$$\Delta W = Q_3 V_3 = Q_3 \left(\frac{Q_1}{4\pi\varepsilon_0 R_{13}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}} \right)$$

$$W_3 = W_2 + \Delta W = \frac{1}{4\pi\varepsilon_0} \left(\frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_3}{R_{23}} \right)$$

$$= \frac{1}{2}\left[Q_1 \left(\frac{Q_2}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{13}} \right) + Q_2 \left(\frac{Q_1}{4\pi\varepsilon_0 R_{12}} + \frac{Q_3}{4\pi\varepsilon_0 R_{23}} \right) + Q_3 \left(\frac{Q_2}{4\pi\varepsilon_0 R_{23}} + \frac{Q_3}{4\pi\varepsilon_0 R_{13}} \right) \right]$$

$$= \frac{1}{2}\left[Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \right]$$

Electrostatic Energy and Forces

- General cases

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \text{ and } V_k = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1 \\ (j\neq k)}}^{N} \frac{Q_j}{R_{jk}}$$

mutual energy
Not self energy

cf) $1\ (\text{eV}) = (1.60\times 10^{-19})\times 1 = 1.60\times 10^{-19}\ (\text{J})$

$$\begin{aligned} \text{work (Joule)} &= \text{energy} \\ \text{voltage} &= \text{Joule/Coulomb} \\ e &= 1.6\times 10^{-19}\ \text{Coulomb} \\ \therefore\ \underline{\text{eV}} &= 1.6\times 10^{-19}\ \underline{\underline{\text{Joule}}} \end{aligned}$$

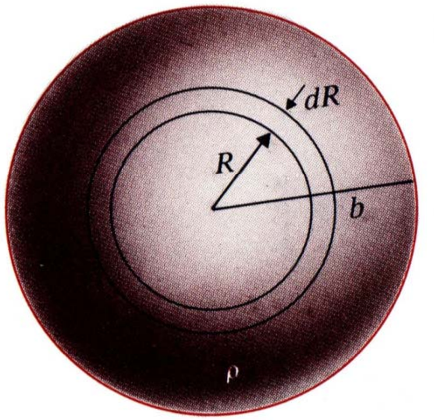
work or energy unit

Ex) 3-22. Assembling a uniform sphere of charge

- Energy required to assemble a uniform sphere of charge of radius b and volume charge density ρ

$$V_R = \frac{Q_R}{4\pi\varepsilon_0 R}, \quad \text{where} \quad Q_R = \rho \cdot \frac{4\pi R^3}{3}$$

$$\therefore \frac{dQ_R}{dR} = \rho \cdot 4\pi R^2 \quad \Rightarrow \quad dQ_R = \rho \cdot 4\pi R^2\, dR$$



the work (energy) done in bringing up dQ_R

$$dW = V_R \cdot dQ_R = \frac{1}{4\pi\varepsilon_0 R} \cdot \rho \cdot \frac{4}{3}\pi R^3 \cdot \rho\, 4\pi R^2 dR$$

$$= \frac{4\pi}{3\varepsilon_0}\rho^2 R^4 dR$$

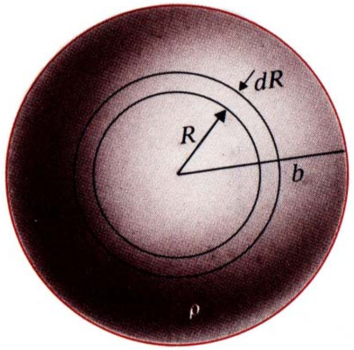
$$\therefore W = \int dW = \frac{4\pi}{3\varepsilon_0}\rho^2 \int_0^b R^4 dR = \frac{4\pi\rho^2 b^5}{15\varepsilon_0} \quad \text{and } Q = \rho\frac{4\pi}{3}b^3$$

$$\therefore W = \frac{3Q^2}{20\pi\varepsilon_0 b} \ [\text{J}]$$

Ex) 3-22. (continued)

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k, \qquad Q_k = \rho\, dv$$

$$\rightarrow W_e = \frac{1}{2}\int_{v'} \rho V dv$$



where, V is the potential at the point where the volume charge density

Electrostatic energy in terms of field quantities

- $$W_e = \frac{1}{2}\int_{v'} \rho V\, dv = \frac{1}{2}\int_{v'} \left(\nabla \cdot \vec{D}\right) V\, dv$$

$$\text{cf)}\ \nabla \cdot \left(V\vec{D}\right) = V\nabla \cdot \vec{D} + \vec{D} \cdot \nabla V$$

$$= \frac{1}{2}\int_{v'} \nabla \cdot \left(V\vec{D}\right) dv - \frac{1}{2}\int_{v'} \vec{D} \cdot \nabla V\, dv$$

$$= \frac{1}{2}\oint_{S'} V\vec{D} \cdot \hat{n}\, ds + \frac{1}{2}\int_{v'} \vec{D} \cdot \vec{E}\, dv$$

✓ Choose v' as any volume that includes all the charges

$\Rightarrow$ very large sphere with radius R

✓ let $R \to \infty$

$$\begin{cases} V \propto \dfrac{1}{R} \\ \vec{D} = \varepsilon \vec{E} \propto \dfrac{1}{R^2} \end{cases}$$

Electrostatic energy in terms of field quantities

Area $S' \propto R^2$

$$\therefore \oint_{S'} V \vec{D} \cdot \hat{n}\, ds \propto \frac{1}{R} \to 0 \quad \text{as } R \to \infty$$

$$\therefore \boxed{W_e = \frac{1}{2}\int_{v'} \vec{D} \cdot \vec{E}\, dv = \frac{1}{2}\int_{v'} \varepsilon \left|\vec{E}\right|^2 dv = \frac{1}{2}\int_{v'} \frac{\left|\vec{D}\right|^2}{\varepsilon} dv}$$

$$W_e = \int_{v'} w_e\, dv$$

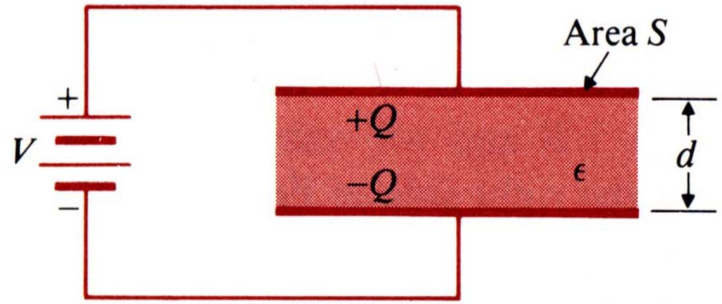
where, w_e : electrostatic energy density, $w_e = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{1}{2}\varepsilon E^2 = \frac{D^2}{2\varepsilon}$

cf) Two Conductor Capacitor

- $E = \frac{V}{d}$

$$\therefore W_e = \frac{1}{2}\int_{v'} \varepsilon \cdot \left(\frac{V}{d}\right)^2 dv = \frac{1}{2}\varepsilon\left(\frac{V}{d}\right)^2 \cdot Sd = \frac{1}{2}\left(\varepsilon\frac{S}{d}\right)V^2 = \frac{1}{2}CV^2$$

$$\therefore W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV$$



Electrostatic Forces

- Principle of virtual displacement
  - case 1 : isolated system of bodies with fixed charges
  - case 2 : system of conducting bodies with fixed potentials

case 1 : Imagine that the electric forces have displaced one of the bodies by $\vec{dl}$.

Work done by the system,

$$dW = \vec{F_Q} \cdot \vec{dl}$$

where, $\vec{F_Q}$ is the total electric force acting on the body under condition of constant charge.

$$dW = -dW_e = \vec{F_Q} \cdot \vec{dl}$$

Electrostatic Forces

cf) work done by the system implies that the stored electrostatic energy decreases by the same amount of work.

$$dW_e = (\nabla W_e) \cdot \vec{dl}$$

$$\therefore \boxed{\vec{F_Q} = -\nabla W_e}$$

cf) If the body under consideration is constrained to rotate about an axis(ex. z-axis), then the mechanical work done by the system for a virtual angular displacement $d\phi$ would be

$$dW = \left(T_Q\right)_z d\phi$$

$$\therefore \left(T_Q\right)_z = -\frac{\partial W_e}{\partial \phi} :$$

the z component of the torque acting on the body under condition of constant charges.

Electrostatic Forces

case 2 : System of conducting bodies with fixed potential
(External source supplies energy to maintain the potential)

The work done or energy supplied by the source to maintain the conductor at constant potential V_k by adding charge dQ_k to the conductor for the displacement $\vec{dl}$ of a conducting body

$$V_k dQ_k$$

The total energy supplied by the sources to the system

$$dW_s = \sum_k V_k dQ_k$$

Electrostatic Forces

The mechanical work done by the system for the virtual displacement

$$dW = \vec{F}_v \cdot \vec{dl}$$

The charge transfer change the electrostatic energy of the system

$$dW_e = \frac{1}{2}\sum_k V_k dQ_k = \frac{1}{2} dW_s \text{ and } dW + dW_e = dW_s$$: energy conservation

$$\therefore\ dW = dW_e$$

$$dW = \vec{F}_v \cdot \vec{dl} = dW_e$$

$$\therefore\ \vec{F}_v \cdot \vec{dl} = dW_e = (\nabla W_e) \cdot \vec{dl}$$

$$\therefore \vec{F}_v = \nabla W_e$$

(Constant potential, the force is in the direction of the increase of electrostatic energy)

$$(T_v)_z = \frac{\partial W_e}{\partial \phi}$$

Electrostatic Forces

$$W_e = \frac{1}{2}\sum_k Q_k V_k$$

$$\left(F_Q\right)_l = -\frac{\partial W_e}{\partial l} = -\frac{\partial}{\partial l}\left(\frac{Q^2}{2C}\right) = \frac{Q^2}{2C^2}\frac{\partial C}{\partial l}$$

$$\left(F_v\right)_l = \frac{\partial W_e}{\partial l} = \frac{\partial}{\partial l}\left(\frac{1}{2}CV^2\right) = \frac{V^2}{2}\frac{\partial C}{\partial l} = \frac{Q^2}{2C^2}\frac{\partial C}{\partial l}$$