Magnetic field intensity and relative permeability

cf) Electric field ,

$$\nabla \bullet \vec{E} = \frac{1}{\varepsilon_0}(\rho + \rho_P) \Rightarrow \nabla \bullet (\varepsilon_0 \vec{E} + \vec{P}) = \rho$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

✓ $$\frac{1}{\mu_0}\nabla \times \vec{B} = \vec{J} + \vec{J}_m = \vec{J} + \nabla \times \vec{M} \Rightarrow \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad [A/m]$$: magnetic field intensity

$$\Rightarrow \boxed{\nabla \times \vec{H} = \vec{J}}$$ ($\vec{J}$: the volume density of free current)

$$\boxed{\int_S (\nabla \times \vec{H}) \bullet \vec{ds} = \int_S \vec{J} \bullet \vec{ds} \quad \text{or} \quad \int_C \vec{H} \bullet \vec{dl} = I}$$

When the magnetic properties of the medium are linear and isotropic

$\Rightarrow$ Magnetization is directly proportional to the magnetic field intensity $\vec{M} = \chi_m \vec{H}$

Magnetic field intensity and relative permeability

$\therefore \vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu_0\mu_r\vec{H}$ (when, $\mu_r = 1+\chi_m$) $= \mu\vec{H}$

(μ: absolute permeability; μ_r: Relative permeability)

μ of most of materials : μ_0

cf) ➢ Ferromagnetic materials(iron, nickel and cobalt): μ_r =50~5000 ≫ 1

☞ hysterisis : permeability depends not only on $\vec{H}$ but also on the previous history of the material

- ➢ Diamagnetic if $\mu_r \le 1$
- ➢ Paramagnetic if $\mu_r \ge 1$
- ➢ Ferromagnetic if $\mu_r >> 1$
- ➢ Anti-ferromagnet (No net magnetic moment due to alternate spins' direction)
- ➢ Ferrimagnet (between ferro and anti-ferro)
- ➢ Ferrite $\rightarrow$ low conductivity $\rightarrow$ small eddy-current (A kind of ferrite)

- $\nabla \times (\frac{\vec{B}}{\mu_0} - \vec{M}) = \vec{J}$, $\nabla \bullet (\varepsilon_0 \vec{E} + \vec{P}) = \rho$
- $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$, $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

✓ Duality

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

Magnetic circuits

✓ $\begin{cases} \text{Magnetic fluxes} \\ \text{Magnetic field intensity} \end{cases}$ $\Rightarrow$ Magnetic circuit from two basic equation

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

➢ $\oint_C \vec{H} \bullet \vec{dl} = NI = V_m$ $[A$ or $A \bullet turns]$: magnetomotive force(mmf)

$cf)\, V_e = \int_1^2 \vec{E_i} \bullet \vec{dl}$ $[V]$

Impressed electric field intensity

closed path is enclosing N turns of a winding carring a current I

Ex 6-10) Coil on ferromagnetic toroid with air gap

① N turns around toroidal

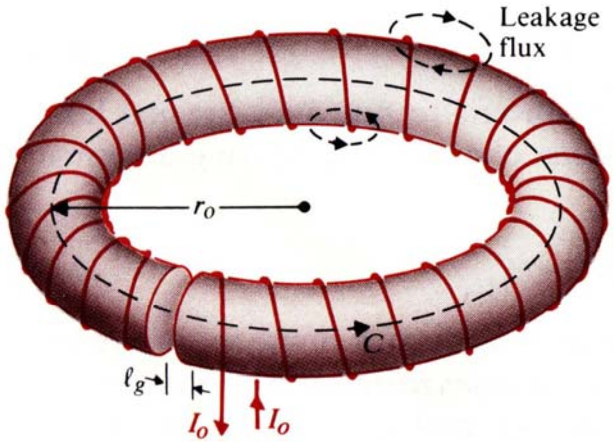
② permeability μ

③ radius r_0

④ circular cross-section radius a $(a \ll r_0)$

⑤ narrow air gap: l_g

⑥ steady current I_0



a) field $\overrightarrow{B_f}$ in the core   b) $\overrightarrow{H_f}$ in the core

c) $\overrightarrow{H_g}$ in the air gap

Ex 6-10) Coil on ferromagnetic toroid with air gap

sol) Ignore leakage and fringing effect,

$\vec{B}$ in the air gap and the core will be the same

$$a)\ \vec{B_f} = \vec{B_g} = B_\phi \hat{\phi} \rightarrow \oint \vec{H} \bullet \vec{dl} = NI_0,$$

choose the path along a circle with a mean radius r_0.

$$\therefore H_f = \hat{\phi}\frac{B_f}{\mu}\ ,\ H_g = \hat{\phi}\frac{B_f}{\mu_0},\qquad \frac{B_\phi}{\mu}(2\pi r_0 - l_g) + \frac{B_\phi}{\mu_0} l_g = NI_0$$

$$\therefore \vec{B_f} = B_\phi \hat{\phi} = \frac{\mu_0 \mu N I_0}{\mu_0 (2\pi r_0 - l_g) + \mu l_g} \hat{\phi}$$

$$b)\ \vec{H_f} = \frac{B_f}{\mu}\hat{\phi} = \frac{\mu_0 N I_0}{\mu_0 (2\pi r_0 - l_g) + \mu l_g}$$

$$c)\ \vec{H_g} = \frac{B_f}{\mu_0}\hat{\phi} = \frac{\mu N I_0}{\mu_0 (2\pi r_0 - l_g) + \mu l_g},\ H_g \gg H_f \quad \text{if} \quad \mu \gg 1$$

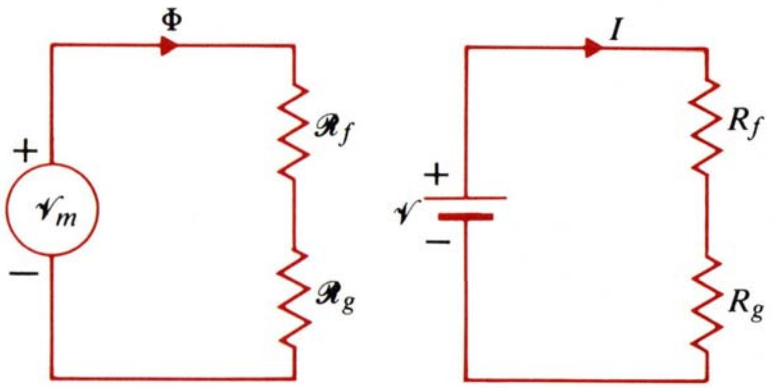
Magnetic Circuits

$\rightarrow$ If $r_0 \gg a$, $\vec{B}$ will be approximately constant

$$\therefore \Phi = \vec{B} \bullet \vec{S} = BS = \frac{NI_0}{\dfrac{(2\pi r_0 - l_g)}{\mu S} + \dfrac{l_g}{\mu_0 S}} = \frac{V_m}{\Re_f + \Re_g}$$

$$\text{where } \Re_f = \frac{(2\pi r_0 - l_g)}{\mu S} = \frac{l_f}{\mu S}, \quad \Re_g = \frac{l_g}{\mu_0 S},$$

$$cf)R = \rho\frac{l}{S} = \frac{l}{\sigma S}, \quad \text{unit : } \Phi\ [Weber],\ V_m\ [A \bullet turns],\ \Re_f\ [A/Wb = 1/H] \quad cf)H = Wb/A$$



(a) Magnetic circuit. (b) Electric circuit.

Magnetic Circuits	Electric Circuits
mmf, $\mathscr{V}_m (= NI)$	emf, $\mathscr{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathscr{R}$	resistance, R
permeability, μ	conductivity, σ

<Analogy between magnitude and electric circuit>

Magnetic Circuits

✓ Difficulties due to the followings

① leakage fluxes

cf) no leakage in current, $\sigma = 0$ in the air but $\mu_0 \neq 0$ in the air

② fringing effect

③ $\vec{B}, \vec{H}$ have a nonlinear relation

$i.e$) μ is not constant

✓ $\sum_j N_j I_j = \sum_k \Re_k \Phi_k$

$\Rightarrow$ Around a closed path in a magnetic circuit the algebraic sum of ampere-turns is equal to the algebraic sum of the products of the reluctance and fluxes

✓ $\nabla \bullet \vec{B} = 0 \Rightarrow \int \vec{B} \bullet \vec{ds} = 0 \quad \therefore \sum_j \Phi_j = 0$

$\Rightarrow$ Algebraic sum of all the magnetic fluxes flowing out of a junction in a magnetic circuit is zero

Boundary conditions for magnetostatic fields

✓ $\nabla \bullet \vec{B} = 0$ $\quad$ $cf)\nabla \times \vec{E} = 0$

$\nabla \times \vec{H} = \vec{J}$ $\quad$ $\nabla \bullet \vec{D} = \rho$

✓ Normal component

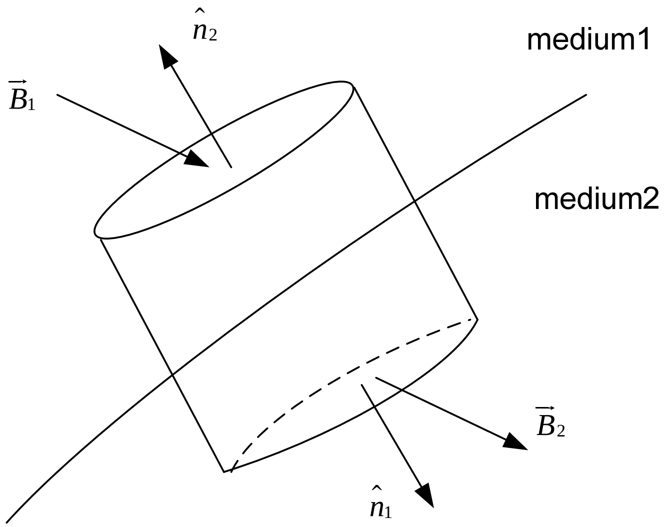
$\nabla \bullet \vec{B} = 0 \Rightarrow$ the normal component of $\vec{B}$ is continuous across an interface

$$cf)\int (\nabla \bullet \vec{B})dv = \oint_S \vec{B} \bullet \vec{ds} = (\vec{B_1} \bullet \widehat{n_2} + \vec{B_2} \bullet \widehat{n_1})\Delta S = \widehat{n_2} \bullet (\vec{B_1} - \vec{B_2})\Delta S = 0$$

$$\therefore \widehat{n_2} \bullet (\vec{B_1} - \vec{B_2})\Delta S = 0 \Rightarrow \boxed{B_{1n} = B_{2n}} \quad cf)D_{1n} = D_{2n} = \rho_s$$

for linear media, $\vec{B_1} = \mu_1 \vec{H_1}$ $\quad$ $\vec{B_2} = \mu_2 \vec{H_2}$

$$\therefore \boxed{\mu_1 H_{1n} = \mu_2 H_{2n}}$$



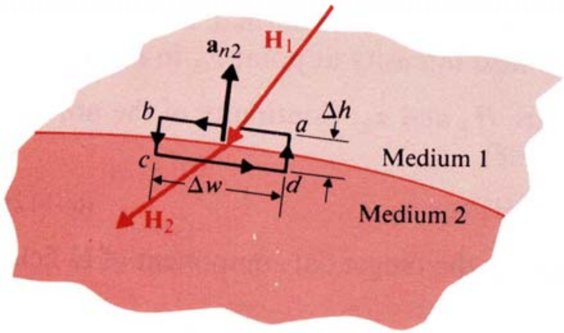
Boundary conditions for magnetostatic fields

✓ tangential component

$$\nabla \times \vec{H} = \vec{J} \Rightarrow \oint_C \vec{H} \bullet \vec{dl} = I$$

$$\oint_C \vec{H} \bullet \vec{dl} = \vec{H_1} \bullet \Delta w + \vec{H_2} \bullet (-\Delta w) = J_{sn} \Delta w$$ or $$\boxed{H_{1t} - H_{2t} = J_{sn}} \quad [A/m]$$

where J_{sn} : surface current density in the direction normal to the C



$$\boxed{\widehat{n_2} \times (\vec{H_1} - \vec{H_2}) = \vec{J_s}}$$

⇒ the tangential component of the $\vec{H}$ field is discontinuous across an interface where a free surface current exists

⇒ most of case $\vec{J_s} = 0$ except for perfect conductor.

(but volume current density $\vec{J}$ is defined)

Ex 6-12) Boundary conditions for magnetostatic field

① $B_{1n} = B_{2n}$

i.e) $\mu_2 H_2 \cos\alpha_2 = \mu_1 H_1 \cos\alpha_1$

② not PEC, no surface current density J_s

$$H_{1t} = H_{2t} \Rightarrow H_1 \sin\alpha_1 = H_2 \sin\alpha_2$$

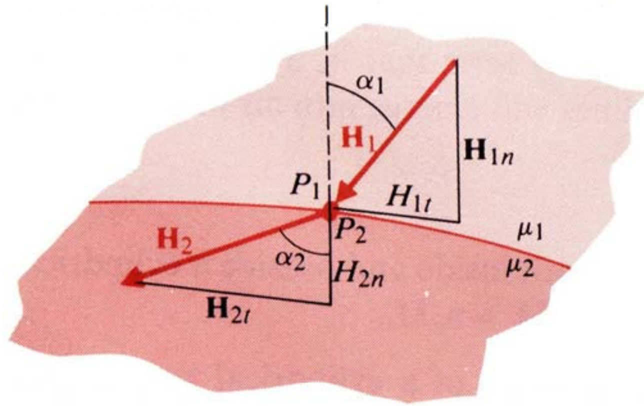
$$\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\mu_2}{\mu_1} \Rightarrow \therefore \alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan\alpha_1\right)$$

$$H_2 = \sqrt{{H_{2t}}^2 + {H_{2n}}^2} = \sqrt{(H_2 \sin\alpha_2)^2 + (H_2\cos\alpha_2)^2}$$

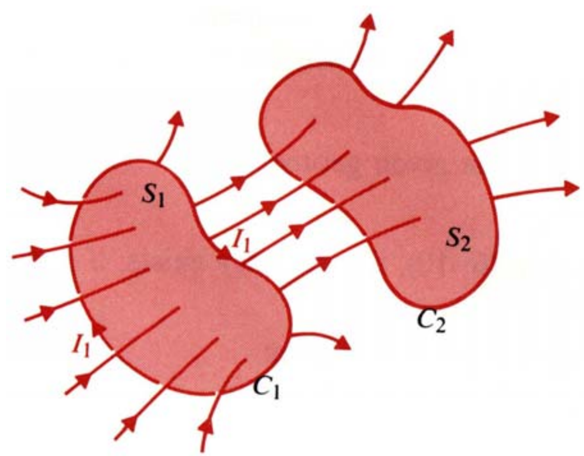
$$= H_1 \left\{(\sin^2\alpha_1) + \left(\frac{\mu_2}{\mu_1}\cos\alpha_1\right)^2\right\}^{\frac{1}{2}}$$

if $\mu_2 \gg \mu_1 \rightarrow \alpha_2 \triangleq \frac{\pi}{2}$

if $\mu_2 \ll \mu_1 \rightarrow \alpha_2 \triangleq 0$



Inductance and inductors



① $\Phi_{12} = \int_{S_2} \vec{B_1} \bullet \vec{ds_2}$

(exists even through I_1 is a steady D.C current)

cf) Faraday's law:

time varying $I_1 \Rightarrow$ time varying $\Phi_{12} \Rightarrow$ emf

② Biot-Savart's law : $B_1 \propto I_1,\ \Phi_{12} \propto B_1 \Rightarrow \Phi_{12} \propto I_1$

③ Defining mutual inductance between loops C_1 and C_2 as proportionality constant

$\Phi_{12} = L_{12} I_1,\ L_{12}$ with henry$[H]$ unit

note) C_2 has N turns, the flux linkage $\Lambda_{12} = N_2\ \Phi_{12}$

then $\Lambda_{12} = N_{12}\ I_1$ or $L_{12} = \dfrac{\Lambda_{12}}{I_1}\ [H \equiv Wb / A]$

Inductance and inductors

④ Definition:

the mutual inductance between two circuits is the magnetic flux linkage with one circuit per unit current in the other

cf) Assumption : linear media. (μ does not change with I_1)

⑤ General definition for L_{12},

$$L_{12} = \frac{d\Lambda_{12}}{dI_1}$$

⑥ Self-inductance for linear medium, $L_{12} = \frac{\Lambda_{11}}{I_1}$ and general definition, $L_{12} = \frac{d\Lambda_{11}}{dI_1}$

cf) Self-inductance depends on

- ☞ geometrical shape of conductor
- ☞ physical arrangement of conductor
- ☞ permeability (not linear medium)

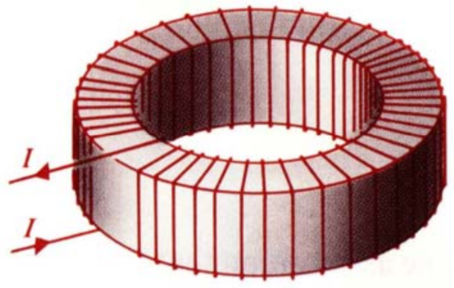
Inductance and inductors

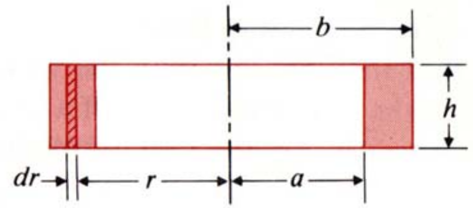
⑦ procedure to determine the self-inductance of an inductor

1. choose a coordinate system
2. assume a current I
3. Find $\vec{B}$ from I by Ampere's circuital law or by Biot-Savart's law
4. Find the flux linking Φ with each turn
5. Find the flux linkage Λ with N turns
6. Find L by taking the ratio $L = \frac{\Lambda}{I}$

Ex 6-14) A closely wound toroidal coil

Toroidal frame of retangular cross section permeability μ_0. self-inductance?





sol)

① cylindrical coordinate

② Assuming current I

③ for $a < r < b,\quad \vec{B} = \hat{\phi}B_\phi,\ \ \vec{dl} = \hat{\phi}rd\phi$

$$\Rightarrow \oint_C \vec{B}\bullet\vec{dl} = \int_0^{2\pi} B_\phi r d\phi = 2\pi r B_\phi$$

④ $2\pi r B_\phi = \mu_0 NI \qquad \therefore B_\phi = \dfrac{\mu_0 NI}{2\pi r}$

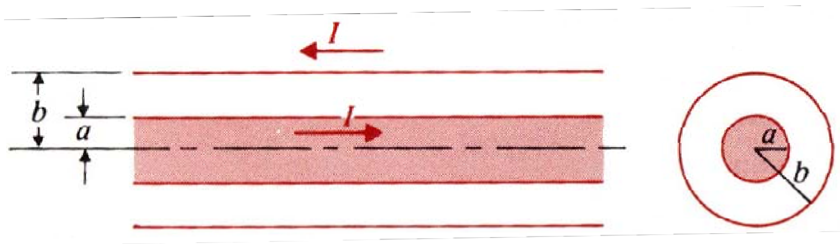
⑤ $\Phi = \int_S \vec{B}\bullet\vec{ds} = \int_S \left(\hat{\phi}\dfrac{\mu_0 NI}{2\pi r}\right)\bullet\left(\hat{\phi}hdr\right) = \dfrac{\mu_0 NIh}{2\pi}\int_a^b \dfrac{dr}{r} = \dfrac{\mu_0 NIh}{2\pi}\ln\dfrac{b}{a}$

⑥ $\Lambda = \dfrac{\mu_0 N^2 Ih}{2\pi}\ln\dfrac{b}{a}$

⑦ $L = \dfrac{\Lambda}{I} = \dfrac{\mu_0 N^2 h}{2\pi}\ln\dfrac{b}{a}$ (Assuming no linkage)

cf) self-inductance $\propto N^2$

Ex 6-16) A coaxial transmission line



sol) cylindrical symmetry $\vec{B} = B_\phi \hat{\phi}$

a) inside the innner conductor $(0 \le r \le a)$

- $\vec{B_1} = B_{\phi 1}\hat{\phi} = \hat{\phi}\dfrac{\mu_0 r I}{2\pi a^2}$

b) $a \le r \le b,\ \vec{B_2} = B_{\phi 2}\hat{\phi} = \hat{\phi}\dfrac{\mu_0 I}{2\pi r}$

Linkage between $d\Phi'$ and current in the annular ring

$cf)\Lambda = N\Phi \quad N$: the number of turns $\Rightarrow$ Ampere turn

$d\Lambda' = dN' d\Phi'$ where, dN': the fraction of the total current in the annular ring

$$d\Phi' = \int_r^a B_{\phi 1} dr + \int_a^b B_{\phi 2} dr = \frac{\mu_0 I}{4\pi a^2}(a^2 - r^2) + \frac{\mu_0 I}{2\pi}\ln\frac{b}{a}$$

$$d\Lambda' = \frac{2rdr}{a^2} d\Phi'$$

$$\therefore \Lambda' = \int_{r=0}^{r=a} d\Lambda' = \frac{\mu_0 I}{\pi a^2}\left[\frac{1}{2a^2}\int_0^a (a^2 - r^2) r dr + \ln\frac{b}{a}\int_0^a r dr\ \right] = \frac{\mu_0 I}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right)$$

Ex 6-16) A coaxial transmission line

$$\therefore L' = \frac{\Lambda'}{I} = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a}$$

Internal inductance : the flux linkage internal to the Solid inner conductor	External inductance : Linkage of the flux that exists between The inner and the outer conductor

cf) In high frequency $\Rightarrow$ skin effect (current shift to skin)

Extreme case $\Rightarrow$ the internal self-inductance is reduced to zero

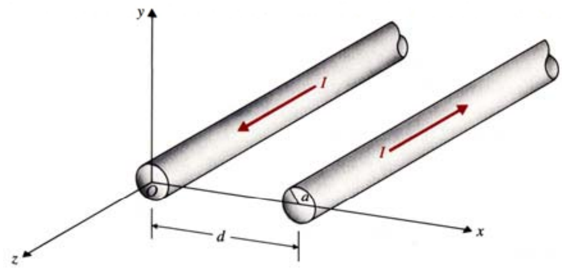
cf) Internal inductance

☞ Magnetic energy in an inductor $= \frac{1}{2}LI^2$

☞ $\frac{1}{2}LI^2 = \int_V \frac{\mu}{2}H^2 dv = \int_0^a \frac{\mu}{2}\left(\frac{Ir}{2\pi a^2}\right)^2 2\pi r dr = \frac{\mu I^2}{4\pi a^4}\frac{a^4}{4}$ $\quad \therefore L = \frac{\mu_0}{8\pi}$

Ex 6-17) A two-wire transmission line

Calculate the internal and external inductances per unit length of a transmission line



sol) ➢ The internal self-inductance per unit length of each wire : $\dfrac{\mu_0}{8\pi}$

$\therefore$ for two wires, $L_i' = 2\times\dfrac{\mu_0}{8\pi} = \dfrac{\mu_0}{4\pi}\ [H/m]$

➢ The external self-inductance per unit length

a) calculate the magnetic flux linking

- $\vec{B}_1$ contributed by conductor1 on the $z-x$ plane, $\vec{B}_1 = \dfrac{\mu_0 I}{2\pi x}\hat{y}$
- $\vec{B}_2$ contributed by conductor2 on the $x-z$ plane, $\vec{B}_2 = \dfrac{\mu_0 I}{2\pi(d-x)}\hat{y}$

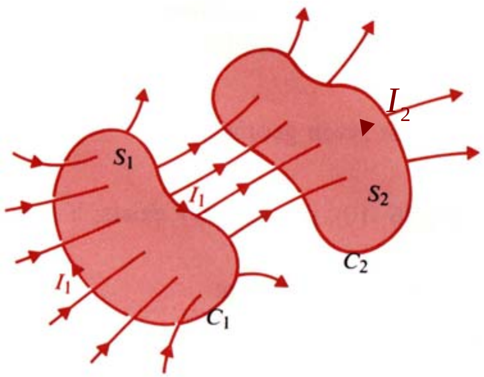
b) the flux linkage per unit length

$$\Phi' = \int_a^{d-a}(B_{y1}+B_{y2})dx = \frac{\mu_0 I}{\pi}\ln\left(\frac{d-a}{a}\right) \cong \frac{\mu_0 I}{\pi}\ln\left(\frac{d}{a}\right)\ [Wb/m] \quad \therefore L_e' = \frac{\Phi'}{I} = \frac{\mu_0 I}{\pi}\ln\left(\frac{d}{a}\right)\ [H/m]$$

$\therefore$ Total self-inductance per unit length

$$L' = L_i' + L_e' = \frac{\mu_0}{\pi}\left(\frac{1}{4}+\ln\frac{b}{a}\right)\ [H/m]$$

Inductance and inductors



- ✓ Reciprocity and mutual inductance
  - $L_{12} = L_{21}\,?$

$$proof)\ \Phi_{12} = \int_{S_2} \vec{B_1} \cdot \vec{ds_2} = L_{12} I_1, \quad \Lambda_{12} = N_2 \Phi_{12} \quad \therefore L_{12} = \frac{\Lambda_{12}}{I_1}$$

$$i.e)\ \boxed{L_{12} = \frac{N_2}{I_1} \int_{S_2} \vec{B_1} \cdot \vec{ds_2}} \text{ and } \vec{B_1} = \nabla \times \vec{A_1}$$

$$\therefore L_{12} = \frac{N_2}{I_1} \int_{S_2} (\nabla \times \vec{A_1}) \cdot \vec{ds_2} = \frac{N_2}{I_1} \oint_{C_2} \vec{A_1} \cdot \vec{dl_2} \quad \Rightarrow \quad \vec{A_1} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \frac{\vec{dl_1}}{R}$$

$$\therefore \boxed{L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\vec{dl_1} \cdot \vec{dl_2}}{R}} \text{ or } \boxed{L_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\vec{dl_1} \cdot \vec{dl_2}}{R}}$$

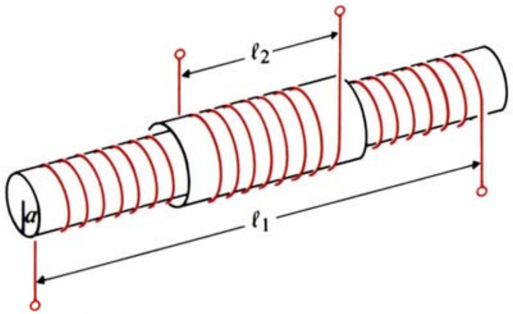
: Neumann formula for mutual inductance

N_1 and N_2 are absorbed in the contour integrals

- $L_{12} = L_{21} \left(\because \frac{\vec{dl_1} \cdot \vec{dl_2}}{R} = \frac{\vec{dl_2} \cdot \vec{dl_1}}{R} \right.$ where R is distance between $\vec{dl_1}$ and $\left. \vec{dl_2} \right)$

Ex 6-18) A solenoid with two windings

- two coils of N_1 and N_2 turns around a straight cylindrical core of radius a and permeability μ



- I_1 in the inner coil
  - $\Phi_{12} = \mu\left(\frac{N_1}{l_1}\right)(\pi a^2)I_1$, $cf)B = \mu n I_1,\ n = \frac{N_1}{l_1}$
  - $\Lambda_{12} = N_2\Phi_{12} = \frac{\mu}{l_1}N_1N_2\pi a^2 I_1 \qquad \therefore L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\mu}{l_1}N_1N_2\pi a^2$