Magnetic energy

- Steady d.c current $\rightarrow$ inductors appear as short circuits
- Alternating currents $\rightarrow$ effects of inductances on circuits and magnetic fields
- Assuming quasi-static conditions $\rightarrow$ currents vary very slowly in time
  $\rightarrow$ imply that dimensions of the circuits are very small in comparison to the wavelength
  $\rightarrow$ enough to ignore retardation and radiation effect

$cf\,)$
- Electrostatic energy : work to assemble charges
- Magnetostatic energy : work to send current into conducting loops
- Genarator is connected to the loop $\rightarrow$ current i increases from 0 to I_1

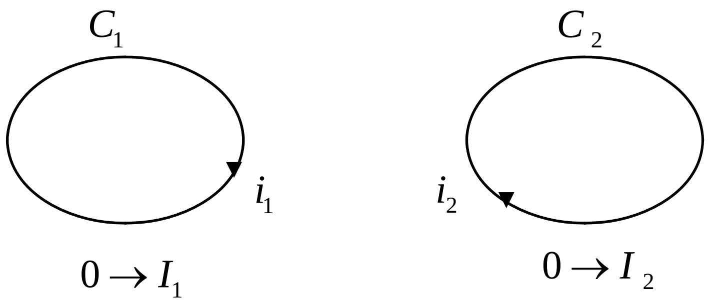
  $\Rightarrow$ current change $\Rightarrow$ emf will be induced in the loop to oppose the change of magnetic flux($i.e$ the change of current)

  $\Rightarrow$ work will be done to overcome this induced emf let $v_1 = L_1 \dfrac{di_1}{dt} \Rightarrow$ voltage across the inductance

Magnetic energy

❖ Work, $W_1 = \int v_1 i_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^{\,2}$

$\therefore W_1 = \frac{1}{2} L_1 I_1^{\,2} = \frac{1}{2} I_1 \Phi_1$: stored as magnetic energy



❖ Two circuit cases

① let $i_2 = 0, \quad i_1 = 0 \rightarrow I_1$

$\Rightarrow$ work W_1 in loop C_1, no work in loop $C_2 (\because i_2 = 0)$

② keep i_1 at $I_1, i_2 = 0 \rightarrow I_2$

induced emf that must be overcome by an external source

$v_{21} = L_{21} \frac{di_2}{dt}$ in order to keep i_1 constant at its value I_1

$W_{21} = \int v_{21} I_1 dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2$

Magnetic energy

In loop $C_2 \Rightarrow W_{22} = \frac{1}{2} L_2 I_2^{\;2}$

$\therefore$ total work done in raising the currents in loops C_1 and C_2 from zeros to I_1 and I_2

$$W_2 = \frac{1}{2} L_1 I_1^{\;2} + L_{21} I_1 I_2 + \frac{1}{2} L_2 I_2^{\;2} = \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} L_{jk} I_j I_k \quad \because L_{12} = L_{21}$$

- Generalization

N loops carrying $I_1, I_2, I_3 .... I_n$

$$W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k \ [J]$$

Energy stored in the magnetic field

for a current flowing in a single inductor $W_m = \frac{1}{2} L I^2$

- The work done to k-th loops in time dt

$$dW_k = v_k i_k dt = i_k d\phi_k \quad cf) v_k = \frac{d\phi_k}{dt}$$

$$dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k d\phi_k$$

$$\therefore W_m = \int dW_m = \sum_{k=1}^{N} I_k \phi_k \int_0^1 \alpha d\alpha = \frac{1}{2} \sum_{k=1}^{N} I_k \phi_k \quad cf) i_k = \alpha I_k, \phi_k = \alpha \Phi_k, \alpha = 0 \to 1$$

Magnetic energy in terms of field quantities

- A single current-carrying loop $\Rightarrow$ large number (N) of contiguous filamentary current elements of closed paths C_k $\Rightarrow$ each path with a current ΔI_k flowing in an infinitesimal cross-sectional area $\Delta a_k'$

  $\Rightarrow$ linking with magnetic flux Φ_k

$$\Phi_k = \int_{S_k} \vec{B} \bullet \hat{n} ds_k = \oint_{C_k} \vec{A} \bullet \overrightarrow{dl_k'} \quad (S_k\text{: surface boundary by } C_k)$$

$$W_m = \frac{1}{2}\sum_{k=1}^{N} \Delta I_k \Phi_k = \frac{1}{2}\sum_{k=1}^{N} \Delta I_k \oint_{C_k} \vec{A} \bullet \overrightarrow{dl_k'}$$

where $\Delta I_k \overrightarrow{dl_k'} = J(\Delta a_k')\overrightarrow{dl_k'} = \vec{J}\Delta v_k'$ when $N \to \infty$, $\Delta v_k' \to dv' \Rightarrow \sum_{k=1}^{N} \oint_{C_k} \Delta I_k \overrightarrow{dl_k'} \to \int_{V'} \vec{J} dv'$

$$\boxed{W_m = \frac{1}{2}\int_{V'} \vec{A} \bullet \vec{J} dv'} \qquad cf)\ W_e = \frac{1}{2}\int_{V'} \rho V dv'$$

- How to express W_m in terms of $\vec{B}$ and $\vec{H}$

  $\vec{A} \bullet \vec{J} = \vec{A} \bullet (\nabla \times \vec{H})$

  Using vector identity and $\nabla \times \vec{A} = \vec{B}$, $\quad \nabla \bullet (\vec{A} \times \vec{H}) = \vec{H} \bullet (\nabla \times \vec{A}) - \vec{A} \bullet (\nabla \times \vec{H})$

  $\therefore \vec{A} \bullet \vec{J} = \vec{H} \bullet \vec{B} - \nabla \bullet (\vec{A} \times \vec{H})$

Magnetic energy in terms of field quantities

- $W_m = \frac{1}{2}\int_{V'} \vec{H}\bullet\vec{B}dv' - \frac{1}{2}\oint_{S'} (\vec{A}\times\vec{H})\bullet\hat{n}ds'$

$cf)\left|\vec{A}\right| \propto \frac{1}{R}$ and $)\left|\vec{H}\right| \propto \frac{1}{R^2}$ but $S' \propto R^2$ $\quad \therefore R\to\infty, \ \oint_{S'} (\vec{A}\times\vec{H})\bullet\hat{n}ds \to 0$

$\therefore W_m = \frac{1}{2}\int_{V'} \vec{H}\bullet\vec{B}dv' = \frac{1}{2}\int_{V'} \frac{B^2}{\mu}dv' = \frac{1}{2}\int_{V'} \mu H^2 dv' \qquad cf)W_e = \frac{1}{2}\int_{V'} \vec{E}\bullet\vec{D}dv'$

Assuming linear medium

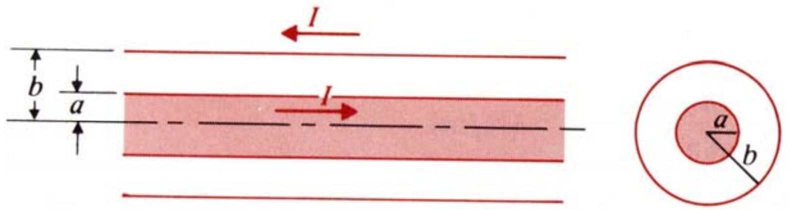
- Defining a magnetic energy density W_m

$W_m = \int_{V'} w_m dv'$ where $w_m = \frac{1}{2}\vec{H}\bullet\vec{B} = \frac{B^2}{2\mu} = \frac{1}{2}\mu H^2 \quad [J/m^3]$

$note)\ W_m = \frac{1}{2}LI^2 \qquad \therefore \quad L = \frac{2W_m}{I^2}$

Ex 6-20) The inductance of an air coaxial cable

By using stored magnetic energy, determine the inductance per unit length of an air coaxial cable



sol) the magnetic energy per unit length

- In the inner conductor :

$$W'_{m1} = \frac{1}{2\mu_0}\int_0^a B_{\phi 1}^2 2\pi r dr = \frac{1}{2\mu_0}\int_0^a \left(\frac{\mu_0 r I}{2\pi a^2}\right)^2 \bullet 2\pi r dr$$

$$= \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi} \quad [J/m]$$

- In the region between two conductors :

$$W'_{m2} = \frac{1}{2\mu_0}\int_a^b B_{\phi 2}^2 2\pi r dr = \frac{1}{2\mu_0}\int_a^b \left(\frac{\mu_0 I}{2\pi r}\right)^2 \bullet 2\pi r dr$$

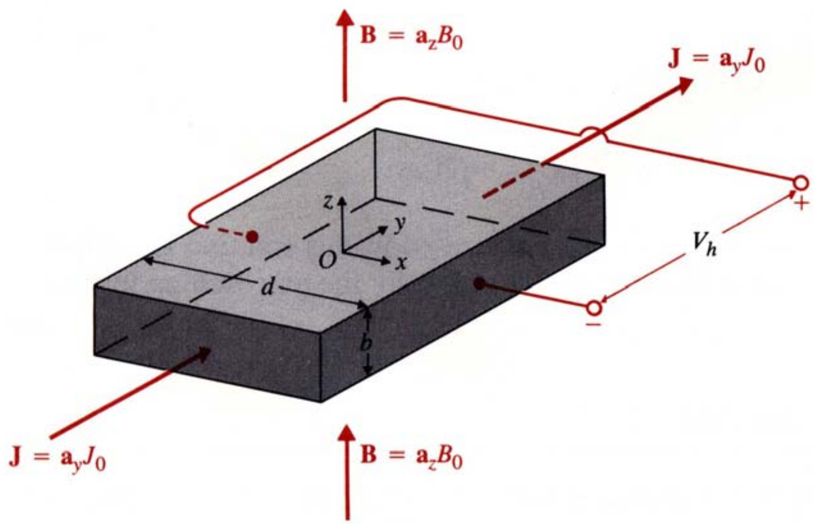
$$= \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a} \quad [J/m]$$

$$\therefore L' = \frac{2W'_m}{I^2} = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a}$$

Magnetic forces and torques

- Lorentz force equation
  - $\vec{F}=q(\vec{E}+\vec{u}\times\vec{B}) \Rightarrow$ Magnetic force : experimental law $\vec{F_m}=q\vec{u}\times\vec{B}$

- Hall effect



  - Assuming n-type semiconductor or conducting material

    $\Rightarrow$ electrons : charge carriers $\rightarrow$ move into $-\hat{y}$ direction

    $\Rightarrow$ magnetic forces act on carriers in the positive x direction

    $\Rightarrow$ Transverse electric field to reach the equilibrium

    $\Rightarrow$ No net force in the steady state

Magnetic forces and torques

- $\overrightarrow{E_h} + \vec{u} \times \vec{B} = 0 \quad \therefore \overrightarrow{E_h} = -\vec{u} \times \vec{B} \Rightarrow$ Hall effect,

  $\overrightarrow{E_h}$: Hall field

  $\overrightarrow{E_h} = -(-\hat{y}u_0) \times \hat{z}B_0 = \hat{x}u_0B_0$

  $V_h = -\int_0^d E_h dx = u_0B_0d$ for electron carriers,

  V_h : Hall voltage

  $\frac{E_x}{J_yB_z} = \frac{1}{Nq}$: Hall coefficient, $\frac{E_x}{J_yB_z} = \frac{u_0B_0}{Nqu_0B_0} = \frac{1}{Nq}$

  $\Rightarrow$ can be used to measure B field and type of charge carriers or carrier density

Forces and torques on current-carrying conductors

- consider an element of conductor $\vec{dl}$ with a cross-sectional area S
- N charge carriers/unit volume
- Assuming electrons

$$d\vec{F}_m = -NeS\left|\vec{dl}\right|\vec{u}\times\vec{B} = -NeS\left|\vec{u}\right|\vec{dl}\times\vec{B} = I\vec{dl}\times\vec{B}$$

$\Rightarrow$ Magnetic force on a complete(closed) circuit of contour C carrying a current I in a magnetic field $\vec{B}$

$$\vec{F}_m = I\oint_C \vec{dl}\times\vec{B} \quad [N]$$

cf) two circuits carrying currents I_1 and I_2

$\Rightarrow \vec{B}_{21}$ caused by the current I_2 in C_2, the force $\vec{F}_{21}$ on circuit C_1

$$\vec{F}_{21} = I_1\oint_{C_1}\vec{dl}_1\times\vec{B}_{21}, \quad \text{where } \vec{B}_{21} = \frac{\mu_0 I_2}{4\pi}\oint_{C_2}\frac{\vec{dl}_2\times\widehat{R_{21}}}{R_{21}^2}$$

$$\vec{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{\vec{dl}_1\times(\vec{dl}_2\times\widehat{R_{21}})}{R_{21}^2} \quad [N]$$: Ampere's law of force

Forces and torques on current-carrying conductors

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_2} \oint_{C_1} \frac{\vec{dl}_2 \times (\vec{dl}_1 \times \widehat{R}_{12})}{R_{12}^2}$$

❖ Question $\vec{F}_{12} = -\vec{F}_{21}$ or not ? since $\vec{dl}_2 \times (\vec{dl}_1 \times \widehat{R}_{12}) \neq -\vec{dl}_1 \times (\vec{dl}_2 \times \widehat{R}_{21})$

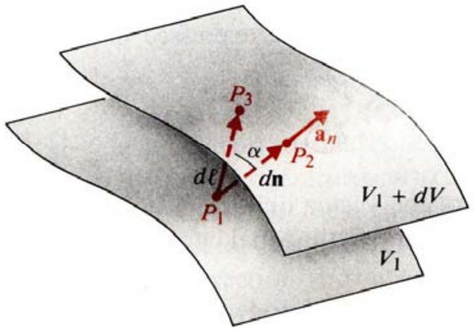
$$\frac{\vec{dl}_1 \times (\vec{dl}_2 \times \widehat{R}_{21})}{R_{21}^2} = \frac{\vec{dl}_2 (\vec{dl}_1 \bullet \widehat{R}_{21})}{R_{21}^2} - \frac{\widehat{R}_{21} (\vec{dl}_2 \bullet \vec{dl}_1)}{R_{21}^2}$$

$$\oint_{C_1} \oint_{C_2} \frac{\vec{dl}_2 (\vec{dl}_1 \bullet \widehat{R}_{21})}{R_{21}^2} = \oint_{C_2} \vec{dl}_2 \oint_{C_1} \frac{\vec{dl}_1 \bullet \widehat{R}_{21}}{R_{21}^2} = \oint_{C_2} \vec{dl}_2 \oint_{C_1} \vec{dl}_1 \bullet (-\nabla_1 \frac{1}{R_{21}})$$

$$cf)\nabla_1 \frac{1}{R_{21}} = -\frac{\widehat{R}_{21}}{R_{21}^2} \quad i.e)\nabla'(\frac{1}{R}) = \frac{\hat{R}}{R^2}, \quad \nabla(\frac{1}{R}) = -\frac{\hat{R}}{R^2}, \; R_{21} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$cf)\nabla V \triangleq \hat{n} \frac{dV}{dn}$$

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\hat{n} \bullet \hat{l} = (\nabla V) \bullet \hat{l} \qquad \because dV = (\nabla V) \bullet \hat{l} dl = \nabla V \bullet \vec{dl}$$



$$\therefore \vec{dl} \bullet (-\nabla \frac{1}{R_{21}}) = -d(\frac{1}{R_{21}}) \text{ and } \oint_{C_1} d(\frac{1}{R_{21}}) = 0$$

$$\therefore \vec{F}_{21} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{\widehat{R}_{21} (\vec{dl}_1 \bullet \vec{dl}_2)}{R_{21}^2} = -\vec{F}_{12}$$

Ex 6-21) Force between two parallel current-carrying wires

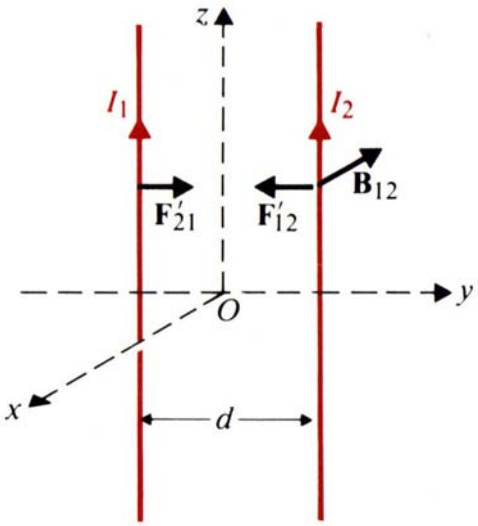
Determine the force per unit length between two infinitely long parallel conducting wire carrying I_1 and I_2 in the same direction

sol) $\overrightarrow{F'_{12}} = I_2(\hat{z} \times \overrightarrow{B_{12}}),$

$\overrightarrow{F'_{12}}$: force per unit length

$\overrightarrow{B_{12}}$: magnetic flux density at wire2 by the current I_1

$\Rightarrow$ constant over wire2 and $-\hat{x}$ direction



$$\overrightarrow{B_{12}} = -\hat{x}\frac{\mu_0 I_1}{2\pi d}$$

$$\therefore \overrightarrow{F'_{12}} = -\hat{y}\frac{\mu_0 I_1 I_2}{2\pi d} \Rightarrow \text{ attraction force}$$

$$\overrightarrow{F_{21}} = \hat{y}\frac{\mu_0 I_1 I_2}{2\pi d} \Rightarrow \text{ attraction force}$$

Torque

- A small current loop with radius b
- $\vec{B}$ field is decomposed into $B_\perp$ and $B_\parallel$

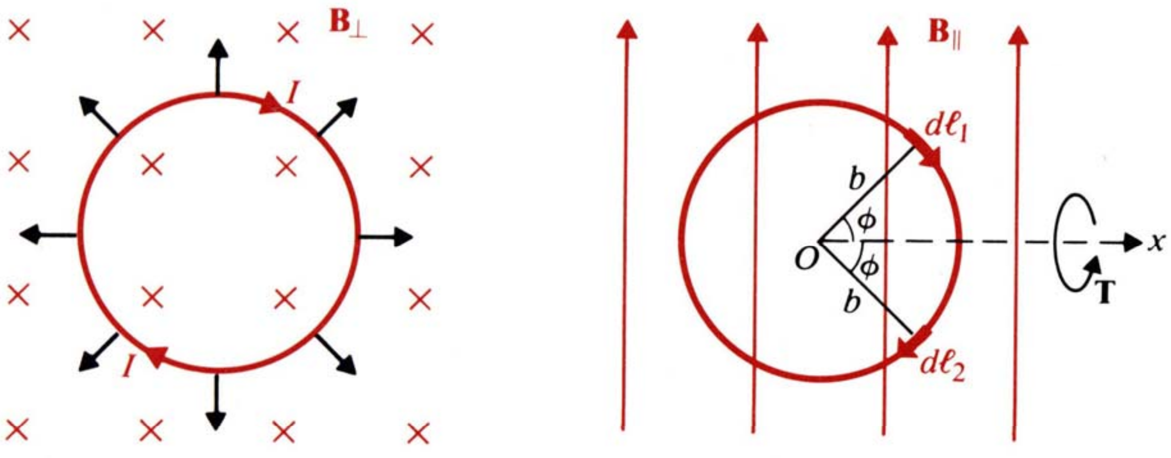
  $i.e)\vec{B} = \vec{B_\perp} + \vec{B_\parallel}$

  $\vec{B_\perp}$ is perpendicular to the plane of loop , $\vec{B_\parallel}$ is parallel to the plane of loop

  then $I\vec{dl} \times \vec{B_\perp}$: expansion or extraction of the loop

  $I\vec{dl} \times \vec{B_l}$: rotational force

net force
To move the
Loop : zero



(a) (b)

Torque

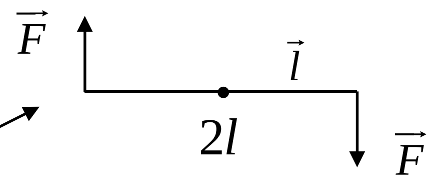
- Net force to move the loop : zero

  but net force to rotate the loop $\neq$ zero $\rightarrow$ Torque

  $\Rightarrow$ In tne direction to align the magnetic field due to the loop current with

  the external $\vec{B}_{\parallel}$ field

- $d\vec{F}_2 = -d\vec{F}_1$



$$\therefore d\vec{T} = \hat{x}(dF)2b\sin\phi = \boxed{2\vec{l}\times\vec{F}} = \hat{x}(IdlB_{\parallel}\sin\phi)2b\sin\phi = \hat{x}2Ib^2B_{\parallel}\sin^2\phi d\phi$$

where $dF = \left|d\vec{F}_1\right| = \left|d\vec{F}_2\right|, \quad \left|\vec{dl}\right| = \left|\vec{dl}_1\right| = \left|\vec{dl}_2\right| = bd\phi$

$$\therefore \vec{T} = \int d\vec{T} = \hat{x}2Ib^2B_{\parallel}\int_0^{\pi}\sin^2\phi d\phi = \hat{x}I(\pi b^2)B_{\parallel}$$

$$\boxed{\vec{m} = \hat{n}I(\pi b^2) = \hat{n}IS}$$

$$\boxed{\therefore \vec{T} = \vec{m}\times\vec{B}} \quad cf)\,\vec{m}\times\vec{B} = \vec{m}\times(\vec{B}_{\perp} + \vec{B}_{\parallel}) = \vec{m}\times\vec{B}_{\parallel}$$

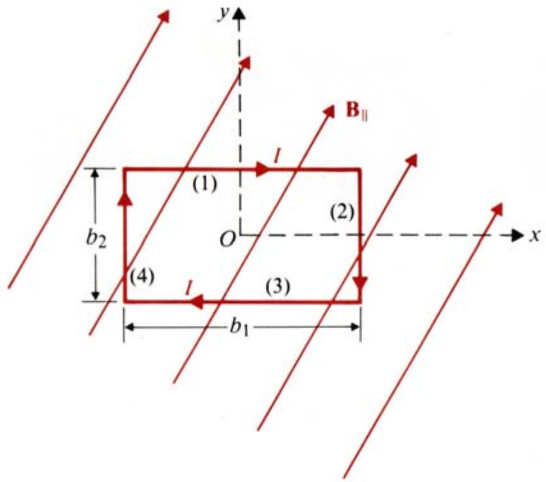
Valid only for uniform $\vec{B}$

Ex 6-22) A rectangular loop in a uniform magnetic field

Rectangular current loop

$\rightarrow$ Determine the force and forque on the loop

- $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$
- $\vec{B_\perp} = \hat{z}B_z$
- $\vec{B_\parallel} = \hat{x}B_x + \hat{y}B_y$



① force due to perpendicular $\vec{B_\perp}$

- side1, $\hat{x}Ib_1 \times \hat{z}B_z = -\hat{y}Ib_1B_z$ · side3, $-\hat{x}Ib_1 \times \hat{z}B_z = \hat{y}Ib_1B_z$
- side2, $-\hat{y}Ib_2 \times \hat{z}B_z = -\hat{x}Ib_2B_z$ · side4, $\hat{y}Ib_2 \times \hat{z}B_z = \hat{x}Ib_2B_z$

} no net force but contraction

Ex 6-22) A rectangular loop in a uniform magnetic field

② force due to parallel $\vec{B}_{\parallel}$

- side1, $\hat{x}Ib_1 \times (\hat{x}B_x + \hat{y}B_y) = \hat{z}Ib_1B_y$ · side3, $-\hat{x}Ib_1 \times (\hat{x}B_x + \hat{y}B_y) = -\hat{z}Ib_1B_y$

  : no net force but torque $\vec{T}_1 = \hat{x}b_2Ib_1B_y = \hat{x}Ib_1b_2B_y$ $cf)\, Ib_1b_2 = |\vec{m}|,\ \vec{m} = -Ib_1b_2\hat{z}$

- side2, $-\hat{y}Ib_2 \times (\hat{x}B_x + \hat{y}B_y) = \hat{z}Ib_2B_y$, · side4, $\hat{y}Ib_2 \times (\hat{x}B_x + \hat{y}B_y) = -\hat{z}Ib_1B_y$

  : no net force but torque $\vec{T}_2 = -\hat{y}b_1Ib_2B_x = -\hat{y}Ib_1b_2B_x$ $cf)\, Ib_1b_2 = |\vec{m}|$

$\therefore \vec{T} = \vec{T}_1 + \vec{T}_2 = \vec{m} \times \vec{B}$

Forces and torques in terms of stored magnetic energy

❖ Find magnetic forces and torques based on the principle of virtual displacement

① A system of circuits with constant magnetic flux linkage

② A system of circuits with constant currents

1. constant flux linkages

➢ Assuming that no changes in flux linkages result from a virtual differential displacement dl of one of current carrying circuits

$\Rightarrow$ no induced emf's $\Rightarrow$ the source will supply no energy to the system

$\Rightarrow$ mechanical work done by the system is at the expense of a decrease in the stored magnetic energy

$$cf)\text{emf}=L\frac{di}{dt} \text{ or } \frac{d\Phi}{dt}$$

$$\vec{F}_\Phi \bullet \vec{dl} = -dW_m = -(\nabla W_m)\bullet\vec{dl}$$

$$\therefore \vec{F}_\Phi = -\nabla W_m,\ (F_\Phi)_x = -\frac{\partial W_m}{\partial x}, (F_\Phi)_y = -\frac{\partial W_m}{\partial y}, (F_\Phi)_z = -\frac{\partial W_m}{\partial z}$$

if the circuit is constrained to rotate about an axis(z-axis) the mechanical work done by the system will be $(T_\Phi)_z d\phi$

$$(T_\Phi)_z = -\frac{\partial W_m}{\partial \phi}$$

Forces and torques in terms of stored magnetic energy

2. constant currents

- circuit connected to current sources $\Rightarrow$ counteract the induced emf's resulting from changes in flux linkages by a virtual displacement dl.

  the work done or energy supplied by the source

  $$dW_s = \sum_k I_k d\Phi_k$$

  this energy be equal to the sum of the mechanical work done by the system

  $dW = \vec{F}_I \bullet \vec{dl}$ and the increase in the stored magnetic energy dW_m

  $$dW_s = dW + dW_m, \quad dW_m = \frac{1}{2}\sum_k I_k d\Phi_k = \frac{1}{2} dW_s$$

  $$dW = \vec{F} \bullet \vec{dl} = dW_m = (\nabla W_m) \bullet \vec{dl}$$

$\therefore \vec{F}_I = \nabla W_m$, $(T_I)_z = \dfrac{\partial W_m}{\partial \phi}$ with a constraint. sign difference $\rightarrow$ different constraint

Forces and torques in terms of mutual inductance

- for two circuits,

$$W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + L_{12} I_1 I_2$$

under the conditions of constant current, assume one of circuits has a virtual displacement

$\Rightarrow$ change in W_m , $\quad \vec{F_I} = I_1 I_2 \nabla L_{12}$

Torque $\quad (T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi}$