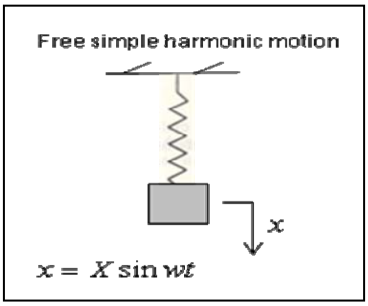
Wave propagation (overview)

Rayleigh`s method [Linear conservative system (no damping)]

- The equation of Energy for a vibrating system



- the original Eq. : $\ddot{x} + \frac{k}{m}x = 0$
- multiply by $\dot{x}$: $\ddot{x}\dot{x} + \frac{k}{m}x\dot{x} = 0$
- integration of the Eq. : $\frac{1}{2}(\dot{x})^2 + \frac{1}{2}\frac{k}{m}x^2 = c$ (c : integral constant)

$$\rightarrow \quad \underbrace{\frac{1}{2}m(\dot{x})^2}_{\text{Instantaneous kinetic Energy}} + \underbrace{\frac{1}{2}kx^2}_{\text{Instantaneous potential Energy}} = cm$$

- When the displacement x is a maximum X

  The velocity $\dot{x} = 0$ → all the dynamic energy in the system is potential

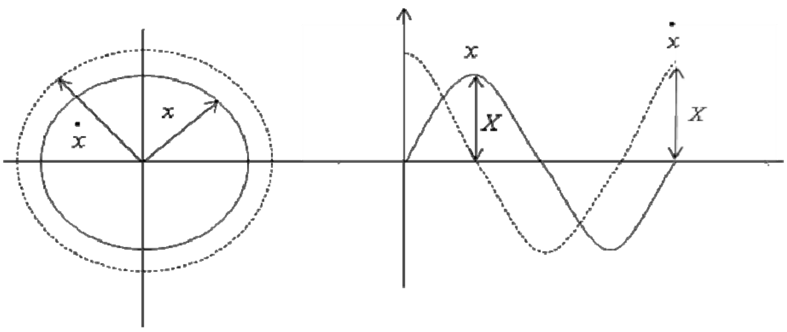
  $$\rightarrow \frac{1}{2}kX^2 = cm = P$$

- When the displacement x is zero, the velocity $\dot{x}$ will be a maximum $\dot{X}$

  → all the d.e. is kinetic

  $$\rightarrow \frac{1}{2}k\dot{X}^2 = cm$$

- Thus, $\frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2 = \frac{1}{2}kX^2 = \frac{1}{2}m(\dot{X})^2 = P$



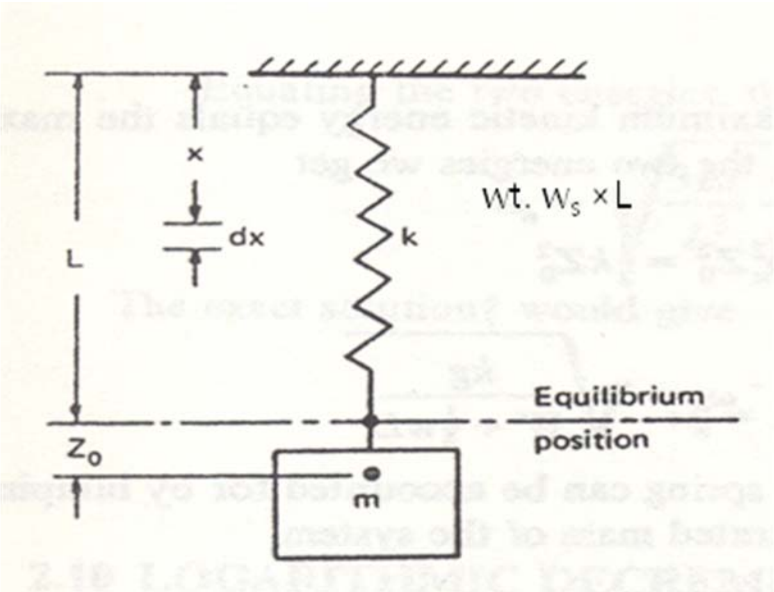
$$x = X\sin wt$$
$$\dot{x} = wX\cos wt (\dot{X} = wX)$$

- Rayleigh`s principles

- The fundamental natural frequency as calculated from the assumed shape of a dynamic deflection curve of a system will be equal to or higher than the system`s true natural frequency
  (upper bound nature ↔ Southwell-Dunkerley Method)

- Small departures from the shape of the true dynamic – deflection curve will not be critical in the determination of the system`s w_n
  (justifies the use of the static deflection curve)

Ex. 1. Natural frequency of the spring-mass system (spring w/ its weight)



- maximum KE($\frac{1}{2}m(\dot{X})^2$)

Spring: the displacement of spring at distance x

- $z = \frac{x}{L} Z_0 \cos w_n t$ (← assumed the extension of the spring is linear)
- Velocity of element dx, $z = -\frac{x}{L} w_n Z_0 \sin w_n t$
- Max. KE of the element (mass = $(\frac{w_s}{g})dx$)

d(KE)max = $\frac{w_s}{2g} dx \, (\, (\frac{x}{L} w_n Z_0)^2 \leftarrow [\frac{1}{2} \cdot \frac{w_s}{g} dx \cdot (\frac{x}{L} w_n Z_0)^2]$

- For the whole spring

$$(KE)_{\max} = \frac{w_s}{2g}\left(\frac{w_n Z_0}{L}\right)^2 \int_0^L x^2 dx$$
$$= \frac{1}{2}\frac{w_s L}{3g} w_n{}^2 Z_0{}^2$$

- mass :

$$(KE)_{\max} = \frac{1}{2}m(\dot{Z})^2 = \frac{1}{2}\frac{W_m}{g}(w_n Z_0)^2$$

- The total KE = $\frac{1}{2}\left(\frac{w_m + \frac{1}{3}w_s L}{g}\right)w_n{}^2 Z_0{}^2$

- maximum PE ($\frac{1}{2}kX^2$)

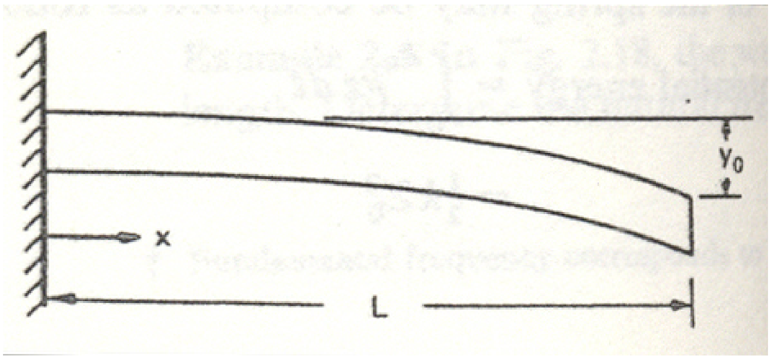
$$(PE)_{\max} = \frac{1}{2}kX^2 = \frac{1}{2}kZ_0{}^2$$

- $(KE)_{\max} = (PE)_{\max}$

$$\frac{1}{2}\left(\frac{W_m + \frac{1}{3}W_s L}{g}\right)w_n{}^2 Z_0{}^2 = \frac{1}{2}kZ_0{}^2$$

$$\therefore w_n = \sqrt{\frac{kg}{W_m + \frac{1}{3}w_s L}}$$

Ex. 2 Cantilever beam



$$z = y\cos w_n t$$

$$\dot{z} = -w_n y \sin w_n t$$

Assume the dynamic deflection curve of the beam as a static defection curve of a weightless beam w/ the concentrated load P acting at its end.

- $(PE)_{\max}$

$\rightarrow \quad y_0 = \dfrac{PL^3}{3EI} \quad \rightarrow \quad (k_{eq})$ at the free end is $(\dfrac{L^3}{3EI})$

$\rightarrow \quad (PE)_{\max} = \dfrac{1}{2}kX^2 = \dfrac{1}{2}ky_0^{\,2} = \dfrac{3EI}{2L^3}y_0^{\,2}$

- $(KE)_{\max}$

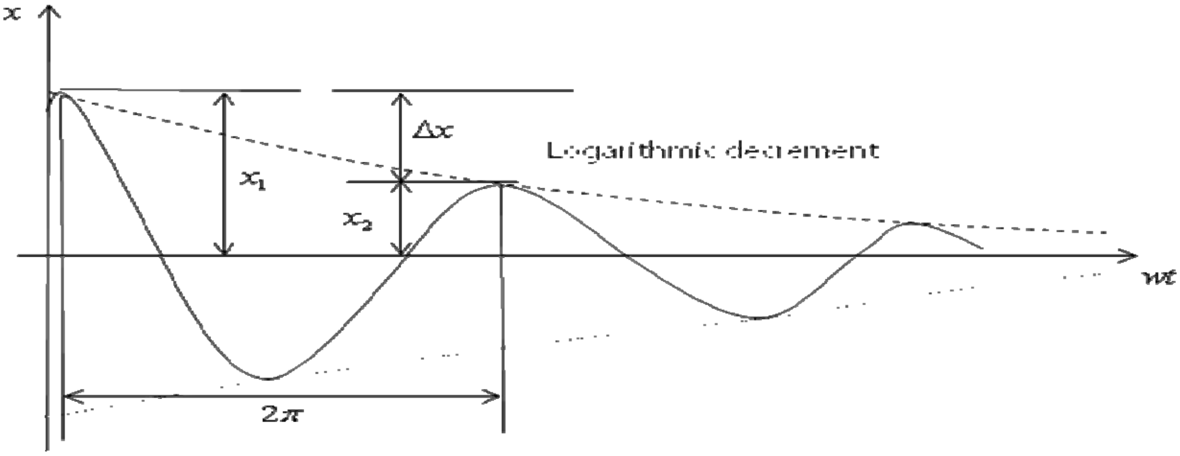
$$\rightarrow \frac{1}{2}m(\dot{X})^2 = \frac{1}{2}\frac{W}{g}\int_0^L (w_n y)^2 dx$$

$$= \frac{w}{2g}(\frac{w_n y_0}{2})^2 \int_0^L [3(\frac{x}{L})^2 - (\frac{x}{L})^3]^2 dx$$

$$= \frac{1}{2}(\frac{33wL}{140g})w_n^{\,2}y_0^{\,2}$$

- $w_n = 3.56\sqrt{\dfrac{gEI}{wL^4}}$ (cf. $(w_n)_{exact} = 3.515\sqrt{\dfrac{gEI}{wL^4}}$)

Logarithmic Decrement



- $\delta = \log_e \dfrac{x_1}{x_2}$, x_1 & x_2 : successive peak amplitudes

  And $z = Z_0 \exp(\dfrac{-\xi w_{nd} t}{\sqrt{1-\xi^2}}) \sin(w_{nd} t + \phi)$

  If x_1 is the amplitude at $w_{nd} t_1$, then x_2, at ($w_{nd} t_1 + 2\pi$)

$$\rightarrow \delta = \log_e \frac{\exp \frac{-\xi w_{nd} t_1}{\sqrt{1-\xi^2}}}{\exp \frac{-\xi (w_{nd} t_1 + 2\pi)}{\sqrt{1-\xi^2}}}$$

$$= \log_e \exp \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$= \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$\fallingdotseq 2\pi\xi$ (if $\xi << 1$)

※ quite often $\xi \le 0.1$ in practice

- In this case ($\xi << 1$)

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \cdots = \frac{x_{n-1}}{x_n} = e^{\delta} = e^{2\pi\xi}$$

$$\& \ \frac{x_0}{x_n} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \cdot \ \cdots \ \cdot \frac{x_{n-1}}{x_n} = (e^{\delta})^n$$

$$\rightarrow \delta = \frac{1}{n} \log_e \frac{x_0}{x_n}$$

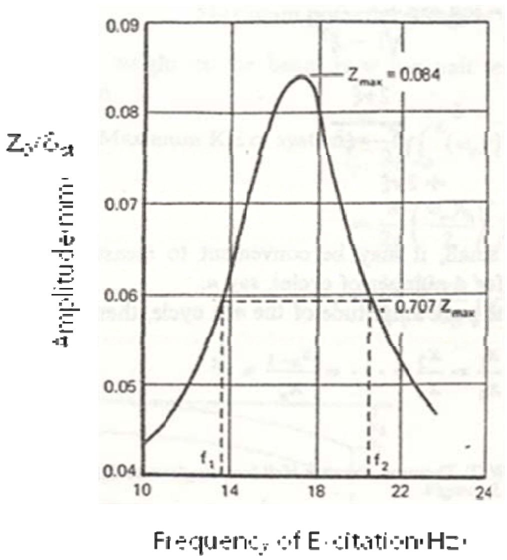
Determination of viscous Damping

- In a free vibration test,

$$\xi = \frac{\delta}{2\pi} = \frac{1}{2\pi}\log_e\frac{x_1}{x_2}$$

or $\xi = \frac{\delta}{2\pi} = \frac{1}{2n\pi}\log_e\frac{x_0}{x_n}$

- In a forced vibration test, [varying frequencies obtain a resonance curve]



$$\frac{Z_0}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = N \text{ (magnification factor)}$$

If $r = \frac{w}{w_n} = 1$(at resonance)

$\frac{Z_0}{\delta_{st}} = \frac{1}{2\xi}(N_{max})$, $(\because \xi << 1)$ n practice

(Fig 2.20)

- When the magnification factor of motion is $\frac{1}{\sqrt{2}}(\frac{1}{2\xi})$, the frequency ratio, r is,

$$\frac{1/\sqrt{2}}{2\xi} = \frac{1}{\sqrt{(1-r^2)^2+4\xi^2 r^2}}$$

$$\rightarrow r_2^{\,2} - r_1^{\,2} = 4\xi\sqrt{1-\xi^2} \fallingdotseq 4\xi \quad (if\ \xi << 1)$$

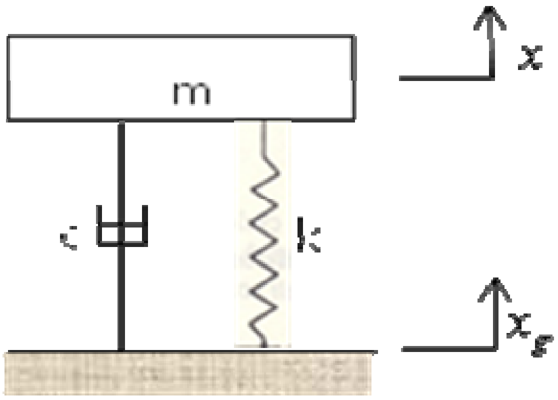
And $r_2^{\,2} - r_1^{\,2} = \frac{f_2^{\,2} - f_1^{\,2}}{f_n^{\,2}} = (\frac{f_2+f_1}{f_n})(\frac{f_2-f_1}{f_n})$

$\fallingdotseq 2\cdot(\frac{f_2-f_1}{f_n})$ - refer to Fig 2.20

$\rightarrow 4\xi = 2\cdot\frac{(f_2-f_1)}{f_n} \qquad \rightarrow \xi = \frac{1}{2}\cdot(\frac{f_2-f_1}{f_n})$

※ called 'bandwidth' method

Ground Acceleration



- $\sum F = ma$

$-c(\dot{x}-\dot{x}_g)-k(x-x_g)=m\ddot{x}$

$\rightarrow m\ddot{x}+c(\dot{x}-\dot{x}_g)+k(x-x_g)=0$ ....①

- let $x_d = x - x_{g1}$

Then, $\dot{x}_d = \dot{x}-\dot{x}_g$

$\ddot{x}_d = \ddot{x}-\ddot{x}_g \rightarrow \ddot{x}_d = \ddot{x}_d-\ddot{x}_g$ ....②

- Sub. Eq. ② into ①

$m(\ddot{x}_d+\ddot{x}_g)+c\dot{x}_d+kx_d=0$

$\rightarrow \boxed{m\ddot{x}_d+c\dot{x}_d+kx_d=-m\ddot{x}_g}$ ← Govern Eq. ....③

- If $x_g = x_g(t) = X_g \sin wt$

$\rightarrow \ddot{x}_g(t) = -w^2 X_g \sin wt$

$\rightarrow m\ddot{x}_d+c\dot{x}_d+kx_d = mw^2 X_g \sin wt$

- Solution :

$$x_d = X_d \sin(wt-\phi)$$

$$X_d = \frac{mw^2 X_g}{\sqrt{(k-mw^2)^2+(cw)^2}}, \quad \phi = \tan^{-1}\frac{cw}{k-mw^2}$$

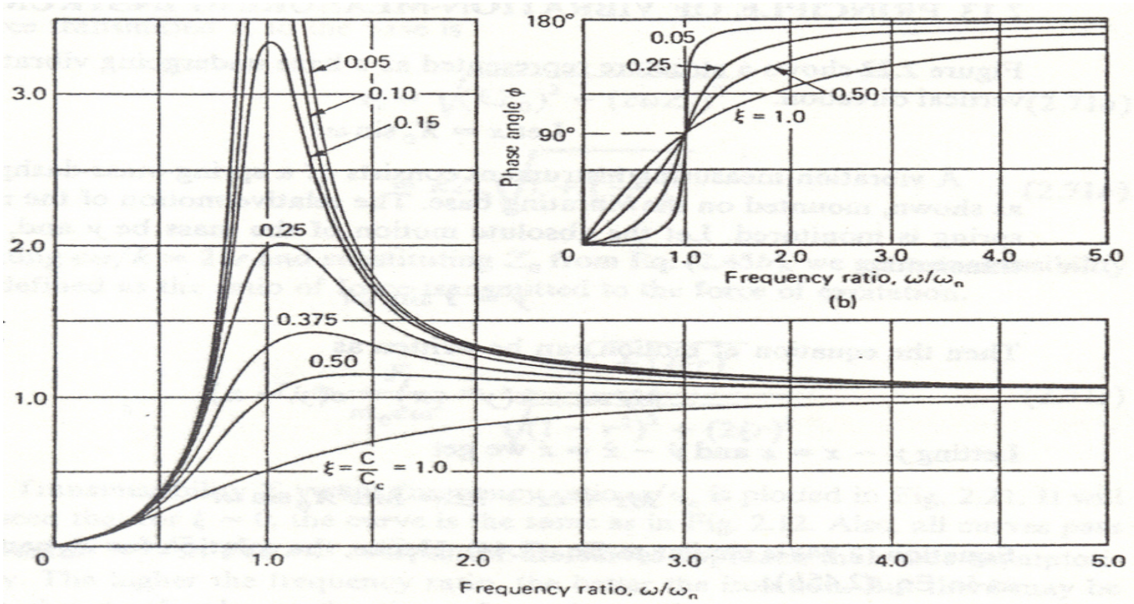
$$X_d / X_g = \frac{(w/w_n)^2}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \quad \text{....④}, \quad \phi = \tan^{-1}\frac{2\xi r}{1-r^2}$$

The Governing Eq. of the system of vibration measuring instrument is same with Eq. ③

- Displacement pickup

  → For large value of $w/w_n (=r)$

  $X_d / X_g = 1$ for all values of damping (refer to Fig. 2.23)



  → i.e. the resulting relative motion $X_d = X_g \, (X_{base})$

  Picked up by the instrument

- Acceleration pickup

  → Rearrange Eq. ④

$$\frac{X_d}{w^2 X_b} = \frac{1}{w_n^{\ 2}\sqrt{(1-(w/w_n)^2)^2 + [2\xi(w/w_n)^2]^2}} = \frac{1}{w_n^{\ 2}\sqrt{D}}$$

  For $\xi = 0.69$ → $\sqrt{D} = 1$ when w/w_n is small

w/w_n	0.0	0.1	0.2	0.3	0.4
$\sqrt{D}$	1.000	0.9995	0.9989	1.0000	1.0053

  → i.e. $X_d \propto w^2 X_b$ (max. acceleration of the vibrating base)

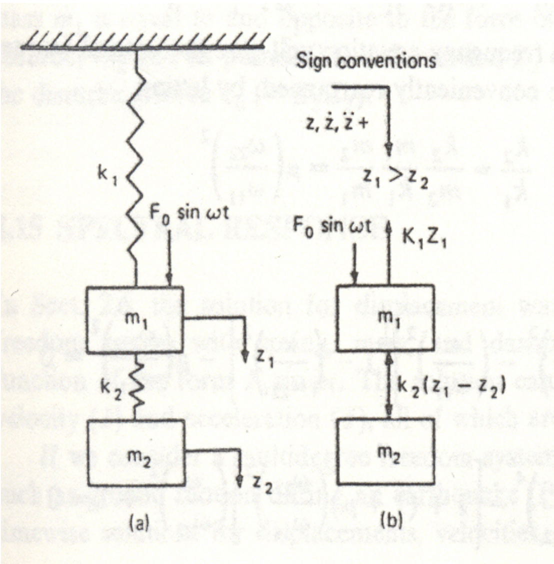
$$X_d = \frac{1}{w_n^{\ 2}} \cdot w^2 X_b$$

$$x = X_b \sin wt$$

$$\dot{x} = wX_b \cos wt$$

$$\ddot{x} = -w^2 X_b \sin wt \ \rightarrow \ (\ddot{x})_{\max} = \left| w^2 X_b \right|$$

System w/ Two-degrees of Freedom



- mass 1

$$m_1 \ddot{z}_1 = F_0 \sin wt - k_1 z_1 - k_2(z_1 - z_2) \quad \text{....①}$$

- mass 2

$$m_2 \ddot{z}_2 = k_2(z_1 - z_2) \quad \text{.... ①'}$$

- assume(∵ vibrating frequency = exciting frequency)

$$z_1 = Z_1 \sin wt$$
$$z_2 = Z_2 \sin wt$$

- Sub. These into Eq. ①, ①'

$$(-m_1 w^2 + k_1 + k_2)Z_1 - k_2 Z_2 = F_0 \quad \text{....②}$$
$$(-m_2 w^2 + k_2)Z_2 - k_2 Z_1 = 0 \quad \text{....②'}$$

- Let the natural circular frequencies of the systems 1 & 2 be

$$w_{11} = \sqrt{\frac{k_1}{m_1}}, \quad w_{22} = \sqrt{\frac{k_2}{m_2}}$$

& $Z_0 = \dfrac{F_0}{k_1}$ (i.e. the static deflection of the mass m_1 due to F_0)

- Dividing Eq. ② & ②` by $k_1 \& k_2$ respectively,

$$[1 + \frac{k_1}{k_2} - (\frac{w}{w_{11}})^2]Z_1 - \frac{k_2}{k_1} Z_2 = Z_0 \quad \text{....③}$$

$$-Z_1 + [1 - (\frac{w}{w_{22}})]Z_2 = 0 \quad \text{....③'}$$

- From Eq. ③', we get

$$Z_2 = \frac{Z_1}{1-(w/w_{22})^2}$$

- Sub. Into Eq. ③

$$[1+\frac{k_2}{k_1}-(\frac{w}{w_{11}})^2]Z_1 - \frac{k_2}{k_1}\frac{Z_1}{[1-(\frac{w}{w_{22}})^2]} = Z_0$$

- Therefore,

$$\frac{Z_1}{Z_0} = \frac{1-(\frac{w}{w_{22}})^2}{[1-(\frac{w}{w_{22}})^2][1+\frac{k_2}{k_1}-(\frac{w}{w_{11}})^2]-\frac{k_2}{k_1}} \quad \ldots④$$

Similarly, we get

$$\frac{Z_2}{Z_0} = \frac{1}{[1-(\frac{w}{w_{22}})^2][1+\frac{k_2}{k_1}-(\frac{w}{w_{11}})^2]-\frac{k_2}{k_1}} \quad \ldots④'$$

- From Eq. ④

If $w_{22} = w \quad \rightarrow \quad Z_1 = 0$

then $\rightarrow \; Z_2 = -Z_0\frac{k_1}{k_2} = -\frac{F_0}{k_2}$

※ the system 2 is used as a vibration absorber for the main system 1

- the natural frequencies of the system

(consider the free vibration → R.H.S of ③ = 0)

From Eq. ③ & ③'

$$\frac{Z_1}{Z_2} = \frac{k_2/k_1}{[1+k_2/k_1-(w/w_{11})^2]} = 1-(\frac{w}{w_{22}})^2$$

or $$[1+\frac{k_2}{k_1}-(\frac{w}{w_{11}})^2][1-(\frac{w}{w_{22}})^2]-\frac{k_1}{k_2}=0$$ [Frequency equation]

or, since $\frac{k_2}{k_1} = \frac{k_2}{m_2}\frac{m_1}{k_1}\frac{m_2}{m_1} = \mu(\frac{w_{22}}{w_{11}})^2$ where, $\mu = \frac{m_2}{m_1}$

$$[\frac{w_{22}}{w_{11}}]^2(\frac{w}{w_{22}})^4 - [1+(1+\mu)(\frac{w_{22}}{w_{11}})^2](\frac{w}{w_{22}})^2 + 1 = 0$$

Spectral Response

- Response quantities : the quantities that are examined as a result of a forcing function (may or may not be a mathematical function)

Ex. Displacement, velocity, acceleration, stress, strain…..

- maximum response of a multidegree freedom system

$$X = \sqrt{(A_{11}X_1)^2 + (A_{22}X_2)^2 + \cdots + (A_{nn}X_n)^2}$$

where, $A_{11}, A_{22,} \cdots$: the mode participation factor ($A_{11} > A_{22} > \cdots$)

X_1 : the maximum response of the 1st mode

Mode participation factor depends on

1) type of system
2) boundary condition
3) type of response quantity

- e.g., Longitudinal vibration of Prismatic Bar

$$u = \frac{-4Pl}{\pi^2 AE} \sum_{n=1,3,5..}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{l} \cos w_n t$$

$$\sigma = \frac{4P}{\pi A} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{l} \cos w_n t$$

- Response spectrum (Fig.2.25) : max. response of SDF system vs. natural period (of frequency) under different damping $\leftarrow$ ($N = f(w_n, c)$)