Wave propagation in an Elastic Medium

Longitudinal Vibration of Rods

\[ \sum F_x = -\sigma_x \cdot A + (\sigma_x + \frac{\partial \sigma_x}{\partial x} dx) A \]

applying Newton’s 2nd law,

\[ -\sigma_x \cdot A + \sigma_x \cdot A + \frac{\partial \sigma_x}{\partial x} dx \cdot A = dx \cdot A \cdot \frac{\gamma}{g} \cdot \frac{\partial^2 u}{\partial t^2} \]

\[ \sum \sum \sum \sum \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} \]

Thus, \[ \frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} \]

\[ \therefore \frac{\gamma}{g} = \rho (mass \ density) \]

\[ = C_r^2 \frac{\partial^2 u}{\partial x^2} \]

where, \[ C_r = \sqrt{\frac{E}{\rho}} \]: Longitudinal wave velocity of rod

\( u \): displacement function in \( x \) direction

\( A \): Cross-sectional area

\( E \): Young’s modulus

\( \gamma \): Unit weight

\( g \): gravitational acceleration

\( \rho \): mass density

\( C_r \): Longitudinal wave velocity of rod

\( V \): wave velocity

\( m \): mass

\( a \): acceleration

\( \partial \sigma_x / \partial x = \frac{\gamma}{g} \frac{\partial^2 u}{\partial t^2} \)

\( \partial \sigma_x / \partial x = E \frac{\partial u}{\partial x} \)

\( \therefore \varepsilon_x = \frac{\partial u}{\partial x} \)

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- Impulse-momentum principle

The vector impulse of the resultant force on a particle, in any time interval, is equal in magnitude and direction to the vector change in momentum of the particle.

\[
\int_{t_1}^{t_2} \mathbf{F} \, dt = m \mathbf{v}_2 - m \mathbf{v}_1
\]

- Calculation of longitudinal wave velocity of rod

\[
F_i = \sigma _t \cdot A \\
\Rightarrow \quad \text{Longitudinal momentum} = mv
\]

\[
= \rho \cdot C_r \cdot t \cdot A \cdot \nu
\]

\[
\text{Longitudinal impulse}
\]

\[
= \int_0^t F_x \, dt \\
= F_x t \\
= \sigma _x \cdot A \cdot t \\
= E \cdot \varepsilon \cdot A \cdot t \\
= E \cdot \frac{v}{C_r} \cdot A \cdot t \quad (\because \varepsilon = \frac{\Delta L}{L}) \quad \text{...Eq.①}
\]

\[
\rightarrow \rho \cdot C_r \cdot t \cdot A \cdot \nu = E \cdot \frac{v}{C_r} \cdot A \cdot t
\]

\[
\therefore C_r^2 = \frac{E}{\rho}
\]
Remarks

- When a wave travel in a material substance, it travels in one direction with a certain velocity \( C_r \), while every particle of the substance oscillates about its equilibrium position (i.e., it vibrates).
- Wave velocities depend upon the elastic properties of the substance through which it travels.

\[
\text{Ex. } C_r = \frac{E}{\rho}, \quad C_f = \frac{B}{\rho}
\]

[ \( B \) : Bulk modulus, \( C_f \) : wave velocity of the liquid confined in a tube ]

- Particle velocity (\( v \)) depends on the intensity of stress or strain induced, while \( C_r \) is only a function of the material properties.

From Eq. (1) of page 2/13

\[
\sigma_x = E \cdot \varepsilon_x = E \cdot \frac{v}{C_r} \\
\rightarrow v = \frac{\sigma_x \cdot C_r}{E} \quad \text{i.e., stress dependent}
\]

- When compressive stress applied, both \( C_r \) & \( v \) are in the same direction (\( \therefore \) compressive \( \sigma_x \rightarrow \) Positive ), and for tensile stress, opposite direction.
Solution of Wave Equation

\[
\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \quad \text{...①}
\]

\[d'Alembert's \ Solution\]

- by the chain rule (if \( u \) is a function possessing a second derivative)

\[
\frac{\partial f(x-ct)}{\partial t} = -cf'(x-ct), \quad \frac{\partial f(x-ct)}{\partial x} = f'(x-ct)
\]

\[
\frac{\partial^2 f(x-ct)}{\partial t^2} = c^2 f''(x-ct), \quad \frac{\partial^2 f(x-ct)}{\partial x^2} = f''(x-ct)
\]

\[\rightarrow \text{ thus, } u = f(x-ct) \text{ satisfies Eq. ①}\]

more generally,

\[u = f(x-ct) + g(x+ct) \quad \text{...②}\]

Eq. ② is a complete solution of Eq. ①, i.e., any solution of ① can be expressed in the form ②.

Ex. Suppose that the initial displacement of the string(rod, or anything satisfying Eq. ①) at any point \( x \) is given by \( \phi(x) \), and that the initial velocity by \( \theta(x) \), then (i.e., IC given)

\[u(x,0) = \phi(x) = [f(x-ct) + g(x+ct)]_{t=0} = f(x) + g(x) \quad \text{...③}\]

\[\left. \frac{\partial u}{\partial t} \right|_{t=0} = \theta(x) = [-cf'(x-ct) + cg(x+ct)]_{t=0} = -cf'(x) + cg'(x) \quad \text{...④}\]
Dividing Eq. ④ by $c$, and then integrating w.r.t. $x$

$$-f(x) + g(x) = \frac{1}{c} \int_{x_0}^{x} \theta(x) dx$$

Combining this with Eq. ③, [ and introducing dummy variable, $s$]

$$f(x) = \frac{1}{2} [\phi(x) - \frac{1}{c} \int_{x_0}^{x} \theta(s) ds], \quad g(x) = \frac{1}{2} [\phi(x) + \frac{1}{c} \int_{x_0}^{x} \theta(s) ds]$$

Now, $u = u(x,t) = f(x - ct) + g(x + ct)$

$$= \left[ \frac{\phi(x - ct)}{2} - \frac{1}{2c} \int_{x_0}^{x-ct} \theta(s) ds \right] + \left[ \frac{\phi(x + ct)}{2} + \frac{1}{2c} \int_{x_0}^{x+ct} \theta(s) ds \right]$$

$$= \frac{\phi(x - ct) + \phi(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \theta(s) ds$$

**Seperation of Variables**

[ for the undamped torsionally vibrating shaft of finite length ]

$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2}$$

- Assume that $\theta(x,t) = X(x)T(t)$

then,

$$\frac{\partial^2 \theta}{\partial x^2} = X''T, \quad \frac{\partial^2 \theta}{\partial t^2} = X \ddot{T}$$

$$\rightarrow X \dddot{T} = a^2 X''T$$

$$\rightarrow \frac{\dddot{T}}{T} = a^2 \frac{X''}{X} = u \text{ (constant)}$$

$$\rightarrow \dddot{T} = uT \quad \text{and} \quad X'' = \frac{u}{a^2} X$$
- Consider real values of $u$
  
  $u > 0$, $u = 0$, $u < 0$

If $u > 0$, (we can write) $u = \lambda^2$

\[ \tilde{T} = \lambda^2 T \rightarrow T = Ae^{\lambda t} + Be^{-\lambda t} \]

\[ X'' = \frac{\lambda^2}{a^2} X \rightarrow X = Ce^{\lambda x/a} + De^{-\lambda x/a} \]

$\rightarrow \theta(x,t) = X(x)T(t) = (Ce^{\lambda x/a} + De^{-\lambda x/a})(Ae^{\lambda t} + Be^{-\lambda t})$

(However, this cannot describe the vibrating system because it is not periodic.)

If $u = 0$

\[ T = 0 \rightarrow T = At + B \]

\[ X'' = 0 \rightarrow X = Cx + D \]

$\rightarrow \theta(x,t) = X(x)T(t) = (Cx + D)(At + B)$

(This Eq. is not periodic either.)

If $u < 0$, we can write $u = -\lambda^2$

\[ \tilde{T} = -\lambda^2 T \rightarrow T = A\cos\lambda t + B\sin\lambda t \]

\[ X'' = -\frac{\lambda^2}{a^2} X \rightarrow X = C\cos\frac{\lambda}{a} x + D\sin\frac{\lambda}{a} x \]

$\rightarrow \theta(x,t) = X(x)T(t) = (C\cos\frac{\lambda}{a} x + D\sin\frac{\lambda}{a} x)(A\cos\lambda t + B\sin\lambda t) \quad \text{Eq. \ ①}$

* Periodic : repeating itself every time $t$ Increases by $\frac{2\pi}{\lambda}$

$\rightarrow \text{period} = \frac{2\pi}{\lambda}$, frequency $= \frac{\lambda}{2\pi}$

$\lambda$ : circular(natural) frequency
- Now, find values of $\lambda$ and the constants A,B,C,D from B.C and/or I.C

These are 3 cases:

1. Both ends fixed  
2. Both ends free  
3. One end fixed, one end free

1. Both ends fixed  
\[ \theta(0, t) = \theta(l, t) = 0 \text{ for all } t \]

\[ \theta(0,t) = 0 = C(A \cos \lambda t + B \sin \lambda t) \]

If $A = B = 0$, satisfied, but leads to trivial solution  
\[ \therefore \theta(x, t) = 0 \text{ at all times} \]

\[ \rightarrow \text{ Let } C = 0, \text{ then from Eq. 1} \]

\[ \theta(x, t) = D \sin \frac{\lambda l}{a} x(A \cos \lambda t + B \sin \lambda t) \quad \ldots 2 \]

\[ \theta(l,t) = 0 = D \sin \frac{\lambda l}{a} l(A \cos \lambda t + B \sin \lambda t) \]

$A \neq 0, B \neq 0$ (set already)

If $D=0 \rightarrow$ leads to the trivial case again ($\because C = 0$ already)

\[ \therefore \sin \frac{\lambda l}{a} = 0, \text{ or } \frac{\lambda l}{a} = n\pi \]

\[ \rightarrow \lambda_n = \frac{n\pi a}{l}, \text{ n = 1, 2, 3... [ remember that } a : \text{ wave velocity } ] \]

\[ \rightarrow \theta_n(x,t) = \sin \frac{\lambda_n}{a} x(A_n \cos \lambda_n t + B_n \sin \lambda_n t) \]

\[ = \sin \frac{n\pi x}{a} \left( A_n \cos \frac{n\pi at}{l} + B_n \sin \frac{n\pi at}{l} \right) \]

\[ \rightarrow \theta(x, t) = \sum_{n=1}^{\infty} \theta_n(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left( A_n \cos \frac{n\pi at}{l} + B_n \sin \frac{n\pi at}{l} \right) \quad \ldots \text{3} \]
- Initial Conditions: \( \theta(x,0) = f(x), \quad \frac{\partial \theta}{\partial t} \bigg|_{t=0} = g(x) \)

\[ \theta(x,0) \equiv f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \quad \text{(from Eq. 3 after t=0 substituted)} \]

\[ A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \quad \text{[Fourier series: Euler coefficients in the half-range sine expansion of } f(x) \text{ over } (0,l) \text{ ]} \]

and,

\[ \frac{\partial \theta}{\partial t} = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[ -A_n \sin \frac{n\pi a}{l} + B_n \cos \frac{n\pi a}{l} \right] \frac{n\pi a}{l} \]

\[ \frac{\partial \theta}{\partial t} \bigg|_{t=0} = g(x) = \sum_{n=1}^{\infty} \left( \frac{n\pi a}{l} B_n \right) \sin \frac{n\pi x}{l} \]

\[ \Rightarrow \frac{n\pi a}{l} B_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx \]

or \( B_n = \frac{2}{n\pi a} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx \)
- End Conditions for free end & for fixed end
  (linear Eq. → superposition valid)

※ In compression, wave travel & particle velocity
→ same direction

In tension → opposite direction
Experimental Determination of Dynamic Elastic Moduli

- Travel – time method:
  measure the time \( t \) for an elastic wave to travel a distance \( l \) along a rod.
  Since, \( C_r^2 = \frac{E}{\rho} \)
  \[ E = \rho C_r^2 = \frac{\gamma}{g} \left( \frac{l}{l} \right)^2 \]
  \[ G = \frac{\gamma}{g} \left( \frac{l}{l} \right)^2 \]

  Shear modulus for a torsional wave

- Resonant – column method:
  A column of material is excited either longitudinally or torsionally, and the wave velocity is determined from the frequency at resonance and from the dimensions of the specimen.
  [ End conditions: free – free or fixed – free ]

  - a free – free column,
    \[ \omega_n = 2\pi f_n = \frac{\pi C_r}{l} \text{, for } n = 1 \quad \left( \lambda_n = \frac{n\pi a}{l} \right) \]
    \[ C_r = 2f_n l \]
    \[ E = \rho (2 f_n l)^2 = \frac{\gamma}{g} (2 f_n l)^2 \]

  - a fixed – free with a mass at the free end.

    \[ C_r = \frac{2\pi f_n l}{\beta} \quad \text{where, } \beta \tan \beta = \frac{A \cdot l \cdot \gamma}{m \cdot g} = \frac{W_{rod}}{W_{mass}} \]

    \[ E = \rho \left( \frac{2\pi f_n l}{\beta} \right)^2 \] [ refer to Vibrations of Soils & Foundations] by Richart, et. al
Waves in an Elastic –Half Space

1. Compression wave (Primary wave, P wave, dilatational wave, irrotational wave)

\[ C_c = \sqrt{\frac{\lambda + 2G}{\rho}} \quad \left( > C_{rod} = \sqrt{\frac{E}{\rho}} \right) \quad \because \text{Confined laterally} \]

, \( \lambda \) & \( G \) : Lame’s Constants

\[
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \\
G = \frac{E}{2(1 + \nu)}
\]

- if \( \nu = 0.5 \), \( C_c \to \infty \)

In water-saturated soils, \( C_c \) is a compression wave velocity of water, not for soil \( \because \) water relatively incompressible

2. Shear wave (Secondary wave, S wave, distortional wave, equivoluminal wave)

\[ C_s = \sqrt{\frac{G}{\rho}} \quad \left( = C_{rod} = \sqrt{\frac{G}{\rho}} \right) \]

- In water – saturated soils, \( C_s \) represents the Soil properties only, since water has no shear strength \( \because G = 0 \)

Thus, in field experiments, shear wave is used in the determination of soil properties.
3. Rayleigh wave (R wave)

\[ C_R : \text{refer to Fig. 3.10} \rightarrow \text{practically the same with } C_S \]

The elastic wave which is confined to the neighborhood of the surface of a half space

![Graph showing values of \( v / v_t \) versus Poisson's ratio, \( \nu \), with P waves, S waves, and R waves indicated.]

4. Love wave (exists only in layered media)

A horizontally polarized shear wave trapped in a superficial layer and propagated by multiple total deflection (Ref: Kramer pp. 162 ~ 5.3.2)
Remarks

- The distribution of total input energy:
  R-wave (67%), S-wave (26%), P-wave (7%)

- Geometrical damping: (or Radiation damping)
  All of the waves encounter an increasingly larger volume of material as they travel outward

→ the energy density in each wave decrease with distance from the source
→ this decrease in energy density (i.e., decrease in displacement amplitude) is called geometrical damping

- Attenuation of the waves by geometric damping
  
  Body waves (P, S) ∝ \( \frac{1}{r} \)
  "Body waves" on the surface ∝ \( \frac{1}{r^2} \)
  R-wave ∝ \( \frac{1}{\sqrt{r}} \) → i.e., decay the slowest

∴ R-wave is of primary concern for foundations on or near the surface of earth
(∵ 67% & \( \frac{1}{\sqrt{r}} \))