Class Handout: Chapter 5 Input-Output Stability

2006 Fall

I. $\mathcal{L}$ Stability

$$y = Hu$$

where $y : [0,\infty) \to \mathbb{R}^q$ and $u : [0,\infty) \to \mathbb{R}^m$ are functions, and H is a mapping (or operator) between two signals.

To measure the size of signals, let us introduce the norm function $\|\cdot\|$, with which it holds that

- $\|u\| = 0$ if and only if $u(t) \equiv 0$,
- $\|au\| = |a|\|u\|$,
- $\|u_1 + u_2\| \le \|u_1\| + \|u_2\|$.

For example,

$$\|u\|_{\mathcal{L}_\infty} := \sup_{t\ge 0} \|u(t)\|$$
$$\|u\|_{\mathcal{L}_p} := \left(\int_0^\infty \|u(t)\|^p dt\right)^{1/p}$$

(Here, the norm inside the integral is not necessarily equal to the p-norm, but usually it is.)

The set of all piecewise continuous functions whose $\mathcal{L}_p$ ($p \in [1,\infty]$) norm is finite, is denoted by $\mathcal{L}_p^m$. Sub-, or super-script may be omitted if clear from the context.

In fact, 'piecewise continuous' can be replaced by 'measurable'.

"Extended space":

$$\mathcal{L}_e^m = \{u : u_\tau \in \mathcal{L}^m, \forall \tau \in [0,\infty)\}$$

where u_τ is a truncation of u defined by

$$u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau, \\ 0, & t > \tau \end{cases}$$

The extended space is a linear space that contains the unextended space as a subset. It allows us to deal with unbounded growing signals (e.g., $u(t) = t$ belongs to $\mathcal{L}_{\infty,e}$).

"Causal" mapping H:

$$(Hu)_\tau = (Hu_\tau)_\tau, \qquad \forall \tau \ge 0.$$

Causality is an intrinsic property of dynamical systems represented by state models.

Definition 5.1

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if $\exists$ a class-$\mathcal{K}$ function α, and $\beta > 0$ s.t.

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \alpha(\|u_\tau\|_{\mathcal{L}}) + \beta$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

It is finite-gain $\mathcal{L}$ stable if $\exists$ nonnegative constants γ and β s.t.

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \gamma \|u_\tau\|_{\mathcal{L}} + \beta \tag{1}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

- β is a bias term, which possibly considers the initial condition.
- If (1) holds, then the system has an $\mathcal{L}$ gain less than or equal to γ. The smallest γ s.t. (1) holds is called the gain of the system.
- For causal, $\mathcal{L}$ stable systems, we have

$$u \in \mathcal{L}^m \Rightarrow Hu \in \mathcal{L}^q$$

and

$$\|Hu\|_{\mathcal{L}} \le \alpha(\|u\|_{\mathcal{L}}) + \beta, \qquad \forall u \in \mathcal{L}^m.$$

- $\mathcal{L}_\infty$ stability is the familiar notion of BIBO stability.

Example 5.1 Let

$$h(u) = a + b \tanh cu = a + b\frac{e^{cu} - e^{-cu}}{e^{cu} + e^{-cu}}$$

Then

$$|h(u)| \le a + bc|u|.$$

Hence, h is finite-gain $\mathcal{L}_\infty$ stable with $\gamma = bc$ and $\beta = a$.

If $a = 0$, h is $\mathcal{L}_p$ stable with zero bias, gain $\gamma = bc$, for each $p \in [1, \infty]$ since

$$\int_0^\infty |h(u(t))|^p dt \le (bc)^p \int_0^\infty |u(t)|^p dt.$$

If $h(u) = u^2$, it is $\mathcal{L}_\infty$ stable with zero bias and $\alpha(r) = r^2$, but is not finite-gain $\mathcal{L}_\infty$ stable.

Example 5.2 Consider a causal convolution

$$y(t) = \int_0^t h(t - \sigma) u(\sigma) d\sigma$$

where $h(t) = 0$ for $t < 0$. Suppose that $h \in \mathcal{L}_{1,e}$; that is, for every $\tau \in [0, \infty)$,

$$\|h_\tau\|_{\mathcal{L}_1} = \int_0^\infty |h_\tau(\sigma)| d\sigma = \int_0^\tau |h(\sigma)| d\sigma < \infty.$$

If $u \in \mathcal{L}_{\infty,e}$ and $\tau \geq t$, then

$$
\begin{aligned}
|y(t)| &\leq \int_0^t |h(t-\sigma)| \, |u(\sigma)| d\sigma \\
&\leq \int_0^t |h(t-\sigma)| d\sigma \sup_{0 \leq \sigma \leq \tau} |u(\sigma)| = \int_0^t |h(s)| ds \sup_{0 \leq \sigma \leq \tau} |u(\sigma)|
\end{aligned}
$$

Consequently,

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h_\tau\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_\infty}, \qquad \forall \tau \in [0, \infty).$$

This does not mean $\mathcal{L}$ stability of the system, because $\|h_\tau\|_{\mathcal{L}_1}$ is not *uniformly* bounded with respect to τ. (For example, $h(t)$ may diverge.)

Assume that $h \in \mathcal{L}_1$; that is,

$$\|h\|_{\mathcal{L}_1} = \int_0^\infty |h(\sigma)| d\sigma < \infty.$$

Then, it holds that

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_\infty}, \qquad \forall \tau \in [0, \infty),$$

which means that the system is finite-gain $\mathcal{L}_\infty$ stable.

In fact, this assumption implies finite-gain $\mathcal{L}_p$ stability for any $p \in [1, \infty]$. See the textbook pp. 199 and 200.

Definition 5.2 (Small-signal $\mathcal{L}$ stability)

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is small-signal (finite-gain) $\mathcal{L}$ stable if $\exists r > 0$ s.t. it is (finite-gain) $\mathcal{L}$ stable for all $u \in \mathcal{L}_e^m$ with $\sup_{0 \leq t \leq \tau} \|u(t)\| \leq r$.

Example 5.3 See the textbook.

II. $\mathcal{L}$ Stability of State Models

Consider

$$
\begin{aligned}
\dot{x} &= f(t, x, u), \qquad x(0) = x_0, \\
y &= h(t, x, u)
\end{aligned}
$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $f : [0, \infty) \times D \times D_u \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in (x, u), $h : [0, \infty) \times D \times D_u \to \mathbb{R}^q$ is piecewise continuous in t and continuous in (x, u), D is a domain containing $x = 0$, D_u is a domain containing $u = 0$.

* For each fixed $x_0 \in D$, the above system defines an operator H that assigns to each input signal $u(t)$ the corresponding output signal $y(t)$.

Assume that $x = 0$ of

$$\dot{x} = f(t, x, 0)$$

is an equilibrium.

Theorem 5.1

Let $r > 0$ and $r_u > 0$ s.t. $\{\|x\| \le r\} \subset D$ and $\{\|u\| \le r_u\} \subset D_u$.

- IF $x = 0$ of $\dot{x} = f(t, x, 0)$ is LES, and $\exists V(t, x)$ s.t.

$$c_1\|x\|^2 \le V(t, x) \le c_2\|x\|^2 \tag{2}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) \le -c_3\|x\|^2 \tag{3}$$

$$\left\| \frac{\partial V}{\partial x}(t, x) \right\| \le c_4\|x\| \tag{4}$$

for all $(t, x) \in [0, \infty) \times D$, and,

IF

$$\|f(t, x, u) - f(t, x, 0)\| \le L\|u\|$$

$$\|h(t, x, u)\| \le \eta_1\|x\| + \eta_2\|u\|$$

for all $(t, x, u) \in [0, \infty) \times D \times D_u$,

THEN, for each $\|x_0\| \le r\sqrt{c_1/c_2}$, the system is small-signal finite-gain $\mathcal{L}_p$ stable ($p \in [1, \infty]$). In particular, for each $u \in \mathcal{L}_{p,e}$ with $\sup_{0 \le t \le \tau} \|u(t)\| \le \min\{r_u, c_1c_3r/(c_2c_4L)\}$, the output $y(t)$ satisfies

$$\|y_\tau\|_{\mathcal{L}_p} \le \gamma\|u_\tau\|_{\mathcal{L}_p} + \beta \tag{5}$$

for all $\tau \in [0, \infty)$, with

$$\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}, \qquad \beta = \eta_1\|x_0\|\sqrt{\frac{c_2}{c_1}}\rho$$

where

$$\rho = \begin{cases} 1, & \text{if } p = \infty, \\ \left(\frac{2c_2}{c_3 p}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases}.$$

- IF the origin is GES and $D = R^n$, $D_u = R^m$, THEN the above holds for any x_0 and $u(t)$.

Proof. Derivative of V along $\dot{x} = f(t, x, u)$ is

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) + \frac{\partial V}{\partial x}[f(t, x, u) - f(t, x, 0)] \\ &\le -c_3\|x\|^2 + c_4L\|x\|\|u\|\end{aligned}$$

(At time point, the proof for global case is actually done already. Can you see?)

(For numerical analysis, we continue...) Take $W(t) = \sqrt{V(t, x(t))}$.

If $V(t, x(t)) \neq 0$,

$$\begin{aligned}
\dot{W} = \frac{\dot{V}}{2\sqrt{V}} &\leq \frac{1}{2\sqrt{V}} \left(-c_3\|x\|^2 + c_4 L\|x\|\|u\|\right) \\
&\leq \frac{1}{2\sqrt{V}} \left(-\frac{c_3}{c_2} V(t, x) + c_4 L \frac{\sqrt{V}}{\sqrt{c_1}}\|u\|\right) \\
&= -\frac{c_3}{2c_2} W + \frac{c_4 L}{2\sqrt{c_1}}\|u\|
\end{aligned}$$

If $V(t, x(t)) = 0$ (i.e., $x(t) = 0$), it can be shown that

Exercise 5.6 and Exercise 3.24 are employed here.

$$D^+W(t) \leq \frac{c_4 L}{2\sqrt{c_1}}\|u(t)\|.$$

Hence, for all $V(t, x(t))$, we have

$$D^+W(t) \leq -\frac{1}{2}\frac{c_3}{c_2} W + \frac{c_4 L}{2\sqrt{c_1}}\|u(t)\|.$$

By the comparison lemma,

$$W(t) \leq e^{-t\frac{c_3}{2c_2}} W(0) + \frac{c_4 L}{2\sqrt{c_1}} \int_0^t e^{-(t-\tau)\frac{c_3}{2c_2}} \|u(\tau)\| d\tau.$$

We then obtain (since $\sqrt{c_1}\|x(t)\| \leq W(t) \leq \sqrt{c_2}\|x(t)\|$)

$$\begin{aligned}
\|x(t)\| &\leq \sqrt{\frac{c_2}{c_1}}\|x_0\| e^{-t\frac{c_3}{2c_2}} + \frac{c_4 L}{2c_1} \int_0^t e^{-(t-\tau)\frac{c_3}{2c_2}} \|u(\tau)\| d\tau \\
&= \sqrt{\frac{c_2}{c_1}}\|x_0\| e^{-t\frac{c_3}{2c_2}} + \frac{c_4 L}{2c_1}\frac{2c_2}{c_3}\left(1 - e^{-t\frac{c_3}{2c_2}}\right)\left(\sup_{0\leq\sigma\leq t}\|u(\sigma)\|\right)
\end{aligned}$$

(We now check if $x(t) \in B_r$ for all $t \geq 0$ so that the whole analysis is valid.) Then, since

$$\|x_0\| \leq r\sqrt{\frac{c_1}{c_2}}, \qquad \sup_{0\leq\sigma\leq t}\|u(\sigma)\| \leq \frac{c_1 c_3 r}{c_2 c_4 L},$$

we have

$$\|x(t)\| \leq r e^{-t\frac{c_3}{2c_2}} + \left(1 - e^{-t\frac{c_3}{2c_2}}\right) r = r.$$

(We now obtain (5).) From the assumption, we have

$$\|y(t)\| \leq k_1 e^{-at} + k_2 \int_0^t e^{-a(t-\tau)}\|u(\tau)\| d\tau + k_3\|u(t)\|$$

where

$$k_1 = \sqrt{\frac{c_2}{c_1}}\|x_0\|\eta_1, \qquad k_2 = \frac{c_4 L \eta_1}{2c_1}, \qquad k_3 = \eta_2, \qquad a = \frac{c_3}{2c_2}.$$

Set

$$y_1(t) = k_1 e^{-at}, \qquad y_2(t) = k_2 \int_0^t e^{-a(t-\tau)}\|u(\tau)\| d\tau, \qquad y_3(t) = k_3\|u(t)\|.$$

Then, for any $p \in [1, \infty]$, we have

$$\|y_{2,\tau}\|_{\mathcal{L}_p} \leq \frac{k_2}{a}\|u_\tau\|_{\mathcal{L}_p}$$

since $\|h\|_{\mathcal{L}_1} = 1/a$ and from Example 5.2, and

$$\|y_{3,\tau}\|_{\mathcal{L}_p} \leq k_3 \|u_\tau\|_{\mathcal{L}_p}.$$

For the term y_1,

$$\|y_{1,\tau}\|_{\mathcal{L}_p} \leq k_1 \rho$$

where

$$\rho = \begin{cases} 1, & \text{if } p = \infty, \\ \left(\frac{1}{ap}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases}.$$

Thus, by the triangular inequality,

$$\gamma = k_3 + \frac{k_2}{a}, \qquad \beta = k_1 \rho.$$

On the other hand, the global case follows easily.

Corollary 5.1

IF the origin of $\dot{x} = f(t, x, 0)$ is GES(LES) and

$$\|f(t, x, u) - f(t, x, 0)\| \leq L\|u\|$$
$$\|h(t, x, u)\| \leq \eta_1 \|x\| + \eta_2 \|u\|$$

THEN the original system is (small-signal) finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$.

Corollary 5.2

A LTI system is finite-gain $\mathcal{L}_p$ stable if A is Hurwitz.

(In particular,

$$\gamma = \|D\|_2 + \frac{2\lambda_{\max}^2(P)\|B\|_2\|C\|_2}{\lambda_{\min}(P)}$$
$$\beta = \|C\|_2 \|x_0\| \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\rho$$
$$\rho = \begin{cases} 1, & p = \infty, \\ \left(\frac{2\lambda_{\max}(P)}{p}\right)^{\frac{1}{p}}, & p \in [1, \infty). \end{cases}$$

where $A^T P + PA = -I$.)

Example 5.4

$$\dot{x} = -x - x^3 + u, \qquad x(0) = x_0,$$
$$y = \tanh x + u$$

With $V(x) = x^2/2$, we can show that the system is finite-gain $\mathcal{L}_p$ stable.

Let us consider LUAS case restricting to $\mathcal{L}_\infty$ stability.

Theorem 5.2 (local version)

Let $r > 0$ and $r_u > 0$ s.t. $\{\|x\| \le r\} \subset D$ and $\{\|u\| \le r_u\} \subset D_u$.

IF $x = 0$ of $\dot{x} = f(t, x, 0)$ is LUAS, and $\exists V(t, x)$ s.t.

$$\begin{aligned} \alpha_1(\|x\|) \le V(t,x) &\le \alpha_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) &\le -\alpha_3(\|x\|) \\ \left\| \frac{\partial V}{\partial x}(t,x) \right\| &\le \alpha_4(\|x\|) \end{aligned}$$

for all $(t, x) \in [0, \infty) \times D$, and,

IF

$$\|f(t,x,u) - f(t,x,0)\| \le \alpha_5(\|u\|)$$

$$\|h(t,x,u)\| \le \alpha_6(\|x\|) + \alpha_7(\|u\|) + \eta$$

for all $(t, x, u) \in [0, \infty) \times D \times D_u$,

THEN, for each $\|x_0\| \le \alpha_2^{-1}(\alpha_1(r))$, the system is small-signal $\mathcal{L}_\infty$ stable.

Proof.

Derivative of V along $\dot{x} = f(t, x, u)$ is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)] \\ &\le -\alpha_3(\|x\|) + \alpha_4(\|x\|)\alpha_5(\|u\|) \\ &\le -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \alpha_4(r)\alpha_5\left(\sup_{0\le t\le\tau} \|u(t)\|\right) \end{aligned}$$

where $0 < \theta < 1$. Set

$$\mu = \alpha_3^{-1}\left(\frac{\alpha_4(r)\alpha_5\left(\sup_{0\le t\le\tau}\|u(t)\|\right)}{\theta}\right).$$

We consider only such $u(t)$ that $\sup_{0\le t\le\tau}\|u(t)\|$ is small enough for $\mu < \alpha_2^{-1}(\alpha_1(r))$. Then,

$$\dot{V} \le -(1-\theta)\alpha_3(\|x\|), \qquad \forall \|x\| \ge \mu.$$

From Theorem 4.18,

$$\|x(t)\| \le \beta(\|x_0\|, t) + \gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right)$$

for all $0 \le t \le \tau$. Hence,

$$\begin{aligned} \|y(t)\| &\le \alpha_6\left(\beta(\|x_0\|, t) + \gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right)\right) + \alpha_7(\|u(t)\|) + \eta \\ &\le \alpha_6(2\beta(\|x_0\|, t)) + \alpha_6\left(2\gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right)\right) + \alpha_7(\|u(t)\|) + \eta \end{aligned}$$

Therefore, we have

$$\|y_\tau\|_{\mathcal{L}_\infty} \le \gamma_0(\|u_\tau\|_{\mathcal{L}_\infty}) + \beta_0$$
$$\gamma_0 = \alpha_6 \circ 2\gamma + \alpha_7, \qquad \beta_0 = \alpha_6(2\beta(\|x_0\|, 0)) + \eta$$

Theorem 5.3 (global version)
IF $D = \mathbb{R}^n$, $D_u = \mathbb{R}^m$, the system $\dot{x} = f(t, x, u)$ is ISS, and

$$\|h(t, x, u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta,$$

THEN, the system is $\mathcal{L}_\infty$ stable.

Proof. Trivial. Isn't it?

* Think about why the global asymptotic stable case needs so strong property (ISS)? Consider

$$\dot{x} = -\frac{x}{1 + x^2} + u,$$

which is GAS (but not GES), and is not $\mathcal{L}_\infty$ stable (also not ISS).

III. $\mathcal{L}_2$ Gain

"$\mathcal{L}_2$ stability plays a special role in systems analysis. It is natural to work with square-integrable signals, which can be viewed as finite-energy signals. In many control problems, the system is represented as an input-output map, from a disturbance input u to a controlled output y, which is required to be small."

Here we study how to calculate the $\mathcal{L}_2$ gain for TI systems.

Theorem 5.4 Consider

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B + D$. Then, the $\mathcal{L}_2$ gain is

$$\sup_{\omega\in\mathbb{R}} \|G(j\omega)\|_2 = \sup_{\omega\in\mathbb{R}} \sqrt{\lambda_{\max}[G^T(-j\omega)G(j\omega)]}.$$

Proof. Due to linearity, we set $x(0) = 0$. From Fourier transform theory, for a causal signal $y \in \mathcal{L}_2$,

$$Y(j\omega) = \int_0^\infty y(t)e^{-j\omega t}dt, \qquad Y(j\omega) = G(j\omega)U(j\omega).$$

By Parseval's theorem,

$$\begin{aligned}
\|y\|_{\mathcal{L}_2}^2 &= \int_0^\infty y^T(t)y(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)Y(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)G^T(-j\omega)G(j\omega)U(j\omega)d\omega \\
&\le \left(\sup_{\omega\in\mathbb{R}} \|G(j\omega)\|_2\right)^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)U(j\omega)d\omega \\
&= \left(\sup_{\omega\in\mathbb{R}} \|G(j\omega)\|_2\right)^2 \|u\|_{\mathcal{L}_2}^2
\end{aligned}$$

which shows that the $\mathcal{L}_2$ gain is less than or equal to $\sup_{\omega\in\mathbb{R}} \|G(j\omega)\|_2$. See Appendix C.10 to show that the $\mathcal{L}_2$ gain is, in fact, equal to $\sup_{\omega\in\mathbb{R}} \|G(j\omega)\|_2$.

Consider

$$\dot{x} = f(x) + G(x)u, \qquad x(0) = x_0,$$
$$y = h(x)$$

where f is locally Lipschitz, G and h are continuous, and $f(0) = 0$ and $h(0) = 0$.

Theorem 5.5

IF $\exists$ C^1 positive semidefinite function $V(x)$ and $\gamma > 0$ s.t.

$$\mathcal{H}(V, f, G, h, \gamma) := \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \leq 0,$$

THEN, for each $x_0 \in \mathbb{R}^n$, the system is finite-gain $\mathcal{L}_2$ stable and the gain $\leq \gamma$.

* Hamilton-Jacobi inequality

* Compare with Theorem 5.1—no exponential stability, but needs a solution to HJ eq.

Proof.

$$\begin{aligned}
\dot{V} = L_f V(x) + L_G V(x) u &= -\frac{1}{2}\gamma^2 \|u - \frac{1}{\gamma^2}(L_G V)^T(x)\|_2^2 + L_f V(x) \\
&\quad + \frac{1}{2\gamma^2} L_G V(x) (L_G V)^T(x) + \frac{1}{2}\gamma^2 \|u\|_2^2 \\
&\leq \frac{1}{2}\gamma^2 \|u\|_2^2 - \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\gamma^2 \|u - \frac{1}{\gamma^2}(L_G V)^T(x)\|_2^2 \\
&\leq \frac{1}{2}\gamma^2 \|u\|_2^2 - \frac{1}{2}\|y\|_2^2
\end{aligned}$$

Thus,

$$V(x(\tau)) - V(x(0)) \leq \frac{1}{2}\gamma^2 \int_0^\tau \|u(t)\|_2^2 dt - \frac{1}{2}\int_0^\tau \|y(t)\|_2^2 dt.$$

Since $V(x) \geq 0$, we have

$$\int_0^\tau \|y(t)\|_2^2 \leq \gamma^2 \int_0^\tau \|u(t)\|_2^2 dt + 2V(x_0).$$

Then,

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{2V(x_0)}.$$

Example 5.8

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a x_1^3 - k x_2 + u, \qquad a > 0, k > 0, \\
y &= x_2
\end{aligned}$$

Let $V(x) = \alpha(a x_1^4/4 + x_2^2/2)$. Then, $L_f V = -\alpha k x_2^2$, $L_G V = \alpha x_2$, $h(x) = x_2$, so that

$$\mathcal{H} = \left(-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2} \right) x_2^2.$$

That is, if

$$\gamma^2 \geq \frac{\alpha^2}{2\alpha k - 1}$$

then the system is $\mathcal{L}_2$ stable. The right-hand side has a minimum $1/k^2$ for $\alpha = 1/k$, so with $\gamma = 1/k$, we conclude that the system is finite-gain $\mathcal{L}_2$ stable with the gain less than or equal to $1/k$.

Example 5.9 Consider the system with the property

$$L_fW(x) \le -kh^T(x)h(x), \qquad k>0$$
$$L_GW(x) = h^T(x)$$

where $W(x)$ is a C^1 positive semidefinite function.

Let $V(x) = \alpha W(x)$. Then,

$$\mathcal{H} = \left(-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2}\right) h^T(x)h(x).$$

By Example 5.8, we conclude that the system is finite-gain $\mathcal{L}_2$ stable with the gain $\le 1/k$.

Example 5.10 Consider the system with the property

$$L_fW(x) \le 0$$
$$L_GW(x) = h^T(x)$$

where $W(x)$ is a C^1 positive semidefinite function.

Let an output feedback control

$$u = -ky + v, \qquad k > 0.$$

Then, the closed-loop system becomes

$$\dot{x} = f(x) - kG(x)G^T(x)\left(\frac{\partial W}{\partial x}\right)^T + G(x)v$$
$$y = h(x)$$

This system is, in fact, the case of Example 5.9. (Verify!)

So, this system is finite-gain $\mathcal{L}_2$ stable with the gain $\le 1/k$ from v to y. Note that the gain can be arbitrarily assigned with the feedback.

Example 5.11 Consider

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Suppose $\exists P \ge 0$ s.t.

$$PA + A^TP + \frac{1}{\gamma^2}PBB^TP + C^TC = 0$$

(the Riccati equation), with some $\gamma > 0$.

Then, $V(x) = \frac{1}{2}x^TPx$ satisfies the HJ equation for this system, i.e., $\mathcal{H} = 0$. Thus, the system is finite-gain $\mathcal{L}_2$ stable with the gain $\le \gamma$. (In fact, the Riccati equation has a solution $P \ge 0$ *if* and only if the system's $\mathcal{L}_2$ gain is less than or equal to γ.)

• From the proof of Theorem 5.5, we note that, if the assumptions hold only on a finite domain D, then we obtain the same conclusion as long as the solution $x(t)$ stays in D. (**Corollary 5.4**.) This possibly restricts the class of input signal, which also depends on the initial condition x_0.

• If $\dot{x} = f(x)$ is AS and if $\|x_0\|$ and $\sup_{0\le t\le\tau} \|u(t)\|$ are sufficiently small (relative to D), then the solution $x(t)$ remains in the neighborhood of the origin. This leads to the following Lemma.

Consider

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, \qquad x(0) = x_0, \\ y &= h(x) \end{aligned}$$

where f is C^1, G and h are continuous on a set $D \subset \mathbb{R}^n$, and $f(0) = 0$ and $h(0) = 0$.

Lemma 5.1

IF the origin of $\dot{x} = f(x)$ is AS, and $\exists$ C^1 positive semidefinite function $V(x)$ and $\gamma > 0$ s.t.

$$\mathcal{H}(V, f, G, h, \gamma) := \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} G(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h^T(x)h(x) \le 0$$

on a domain D,

THEN $\exists k_1 > 0$ s.t. for each $\|x_0\| \le k_1$, the system is small-signal finite-gain $\mathcal{L}_2$ stable with the gain $\le \gamma$.

Proof. We can apply Corollary 5.4, if we show that $x(t)$ stays in a neighborhood of the origin.

By the converse theorem, $\exists$ a C^1 function $W(x)$ and $r_0 > 0$ s.t.

$$\begin{gathered} \alpha_1(\|x\|) \le W(x) \le \alpha_2(\|x\|) \\ \frac{\partial W}{\partial x} f(x) \le -\alpha_3(\|x\|) \end{gathered}$$

for all $x \in B_{r_0}$, and without loss of generality, we assume that $B_{r_0} \subset D$.

Let k and L be an upper bound of $\|\partial W/\partial x\|$ and $\|G(x)\|$, respectively. Then,

$$\begin{aligned} \dot{W}(x, u) &\le \frac{\partial W}{\partial x} f(x) + \frac{\partial W}{\partial x} G(x)u \le -\alpha_3(\|x\|) + kL\|u\| \\ &\le -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + kL \sup_{0\le t\le\tau} \|u(t)\| \\ &\le -(1-\theta)\alpha_3(\|x\|), \qquad \forall \|x\| \ge \alpha_3^{-1}\left(kL \sup_{0\le t\le\tau}\|u(t)\|/\theta\right) \end{aligned}$$

where $0 < \theta < 1$.

This implies that (you may recall Theorem 4.18), $\exists k_1, k_2 > 0$ s.t., for every $\|x_0\| < k_1$ and for inputs such that $\sup_{0 \le t \le \tau} \|u(t)\| \le k_2$, the solution $x(t) \in B_{r_0}$. Hence, the conclusion follows from Corollary 5.4.

Lemma 5.2

IF $\exists$ C^1 positive *semidefinite* function $V(x)$ and $\gamma > 0$ s.t.

$$\mathcal{H}(V, f, G, h, \gamma) := \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \le 0$$

on D, and 'no solution of $\dot{x} = f(x)$ can stay identically in $S = \{x \in D : h(x) = 0\}$ other than the trivial solution $x(t) \equiv 0$',

THEN the origin of $\dot{x} = f(x)$ is AS, and $\exists k_1 > 0$ s.t. for each $\|x_0\| \le k_1$, the system is small-signal finite-gain $\mathcal{L}_2$ stable with the gain $\le \gamma$.

- In this lemma the Lyapunov function $V(x)$ came from the HJ inequality.
- How to interpret the phrase: 'no solution of $\dot{x} = f(x)$ can stay identically in $S = \{x \in D : h(x) = 0\}$ other than the trivial solution $x(t) \equiv 0$'? (Not clear in the textbook. Discuss it in the proof.) I would say 'there is no *locally invariant set* in S except the origin'. By a locally invariant set L in S, I mean that, if $x(0) = x_0 \in L$, there exist nonnegative constants δ_1 and δ_2, not both zero, s.t. $x(t) \in L$ for $t \in [-\delta_1, \delta_2]$.

Locally invariant set $\subset$ Invariant set

Proof. The proof is done by Lemma 5.1 if we show that the origin of $\dot{x} = f(x)$ is AS.

From the HJ inequality,

$$\frac{\partial V}{\partial x} f(x) \le -\frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T - \frac{1}{2} h^T(x) h(x) \le -\frac{1}{2} h^T(x) h(x),$$

which seemingly means that the origin is stable, and by employing the LaSalle's theorem, we have the conclusion. (In particular, the set $\{x \in D : \frac{\partial V}{\partial x} f(x) = 0\} \subset S$. In addition, since there is no locally invariant set in S except the origin, the only invariant set in S is the origin. So, the LaSalle's theorem can be applied.) This is true if $V(x)$ is positive definite.

However, since $V(x)$ is assumed just positive semidefinite, we now prove that, from the assumption, the function $V(x)$ is in fact positive definite.

Let $r > 0$ be s.t. $B_r \subset D$ (strict inclusion). Then, for each $x \in B_r$, $\exists \delta > 0$ s.t. the solution from x, $\phi(t; x)$, stays in D for $t \in [0, \delta]$. Integrating the above inequality over $[0, \delta]$, we obtain

$$V(\phi(\delta; x)) - V(x) \le -\frac{1}{2} \int_0^\delta \|h(\phi(t; x))\|_2^2 dt.$$

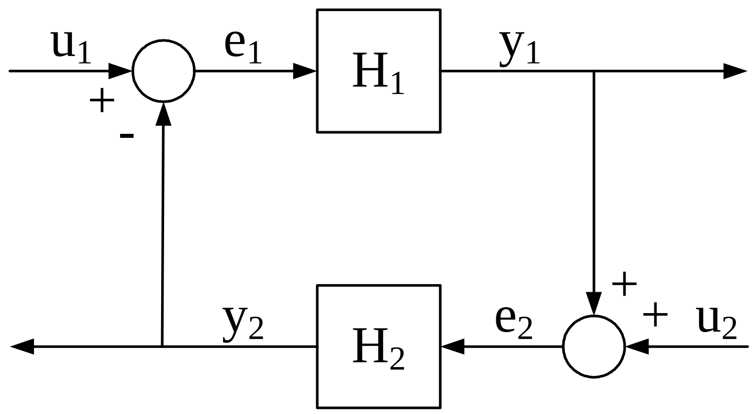
Using $V(\phi(\delta; x)) \ge 0$, we have

$$V(x) \ge \frac{1}{2} \int_0^\delta \|h(\phi(t; x))\|_2^2 dt.$$

Now suppose that $\exists x \neq 0$ s.t. $V(x) = 0$ (not positive definite). Then, the above implies that $h(\phi(t; x)) = 0$ for $t \in [0, \delta]$. This means that $x \neq 0$ is an element of locally invariant set. However, since the only element of the locally invariant set is the origin, $x = 0$ (contradiction).

Example 5.12 DIY.

IV. Feedback Systems: The Small-Gain Theorem



From the textbook,

• The formalism of input-output stability is particularly useful in studying stability of interconnected systems, since the gain of a system allows us to track how the norm of a signal increases or decreases as it passes through the system.

• See Figure 5.1. $H_1 : \mathcal{L}_e^m \to \mathcal{L}_e^q$ and $H_2 : \mathcal{L}_e^q \to \mathcal{L}_e^m$. Suppose both system are finite-gain $\mathcal{L}$ stable; that is,

$$\|y_{1,\tau}\|_{\mathcal{L}} \leq \gamma_1 \|e_{1,\tau}\|_{\mathcal{L}} + \beta_1, \qquad \forall e_1 \in \mathcal{L}_e^m, \forall \tau \in [0, \infty)$$
$$\|y_{2,\tau}\|_{\mathcal{L}} \leq \gamma_2 \|e_{2,\tau}\|_{\mathcal{L}} + \beta_2, \qquad \forall e_2 \in \mathcal{L}_e^q, \forall \tau \in [0, \infty)$$

(finite-gain for simplicity).

• Suppose that the feedback system is *well defined* in the sense that for every pair of inputs $u_1 \in \mathcal{L}_e^m$ and $u_2 \in \mathcal{L}_e^q$, $\exists$ unique outputs $e_1, y_2 \in \mathcal{L}_e^m$ and $e_2, y_1 \in \mathcal{L}_e^q$.

• Define $u = [u_1; u_2]$, $y = [y_1; y_2]$, $e = [e_1; e_2]$. We ask if the feedback system, viewed as a mapping from u to y (or to e since they are equivalent in the sense of Exercise 5.21, your homework), is finite-gain $\mathcal{L}$ stable.

• If the external input and output of interest is just u_1 and y_1, why should we consider $\mathcal{L}$ stability from both (u_1, u_2) to both (y_1, y_2)? The reason is that, by considering all the pairs of inputs and outputs, any possibly hidden internal mode comes out. (See Exercise 5.20.)

Theorem 5.6 (Small-gain Theorem)

IF $\gamma_1\gamma_2 < 1$,

THEN the feedback system is finite-gain $\mathcal{L}$ stable.

Proof.

$$e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau, \qquad e_{2\tau} = u_{2\tau} + (H_1 e_1)_\tau$$

Then,

$$\begin{aligned}
\|e_{1\tau}\|_{\mathcal{L}} &\le \|u_1\tau\|_{\mathcal{L}} + \|(H_2e_2)_\tau\|_{\mathcal{L}} \le \|u_1\tau\|_{\mathcal{L}} + \gamma_2\|e_{2\tau}\|_{\mathcal{L}} + \beta_2 \\
&\le \|u_1\tau\|_{\mathcal{L}} + \gamma_2(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1\|e_{1\tau}\|_{\mathcal{L}} + \beta_1) + \beta_2 \\
&= \gamma_1\gamma_2\|e_{1\tau}\|_{\mathcal{L}} + (\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1)
\end{aligned}$$

Since $\gamma_1\gamma_2 < 1$,

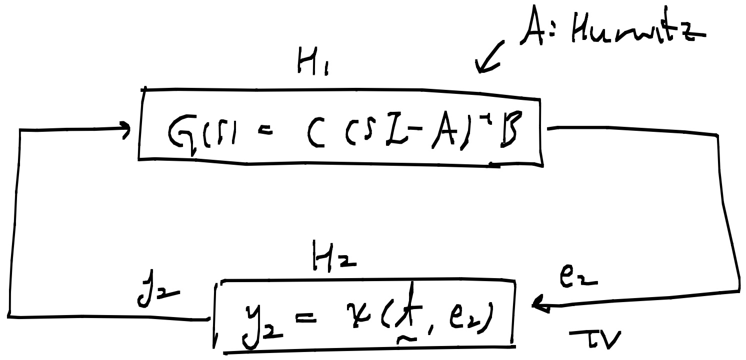
$$\|e_{1\tau}\|_{\mathcal{L}} \le \frac{1}{1-\gamma_1\gamma_2}(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1)$$

for all $\tau \in [0, \infty)$. Similarly,

$$\|e_{2\tau}\|_{\mathcal{L}} \le \frac{1}{1-\gamma_1\gamma_2}(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1\|u_{1\tau}\|_{\mathcal{L}} + \beta_1 + \gamma_1\beta_2)$$

for all $\tau \in [0, \infty)$. The proof completes since $\|e\|_{\mathcal{L}} \le \|e_1\|_{\mathcal{L}} + \|e_2\|_{\mathcal{L}}$.

Example 5.17



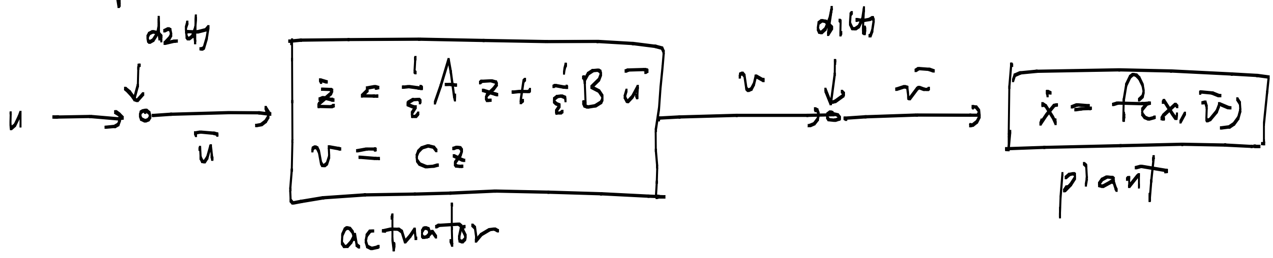
$$\|\gamma(t, e_2)\|_2 \le \gamma_2 \|e_2\|_2$$

H_1: finite-gain $\mathcal{L}_2$ gain : $\gamma_1 = \sup \|G(j\omega)\|_2$

H_2: " : γ_2

$\therefore$ $\gamma_1 \gamma_2 < 1$ $\Rightarrow$ feedback sys. is finite-gain $\mathcal{L}_2$ stable

Example 5.14



A: Hurwitz, ε: small positive

$-CA^{-1}B = I$

$d_1, d_2 \in \mathcal{L}$

$\uparrow$

e.g., $\mathcal{L}_p$ ($p \in [0, \infty]$)

Goal: attenuate the effect of $d_1(t)$, $d_2(t)$ on $x(t)$

i.e.,

$$\begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \xrightarrow[\substack{\mathcal{L} \text{ stable} \\ \text{gain} < \delta}]{} x(t)$$

* Why can we ignore the fast actuator?

Let $\varepsilon = 0$. $\quad z = -A^{-1}B\,(u + d_2)$

$\Rightarrow \quad v = -CA^{-1}B\,(u + d_2) = u + d_2$

$\Rightarrow \quad \dot{x} = f(x,\, u + \underbrace{d_1 + d_2}_{\triangleq\, d(t)})$

Controller $u = \gamma(x)$ only considers x (not z).

<u>Ass</u>: $\gamma(x)$ is designed s.t.

$\dot{x} = f(x,\, \gamma(x) + d)$

$\|x\|_{\mathcal{L}} \le \lambda \|d\|_{\mathcal{L}} + \beta$, $\quad \lambda < \infty$

what about the actuator dynamics?

* Robustness to unmodeled actuator dynamics

$\dot{x} = f(x,\, Cz + d_1)$

$\varepsilon \dot{z} = Az + B\,(\gamma(x) + d_2)$

<u>Ass</u>: $d_2(t)$ is differentiable & $\dot{d}_2 \in \mathcal{L}$

Trick begins by $\quad \eta \triangleq z + A^{-1}B\,[\gamma(x) + d_2]$

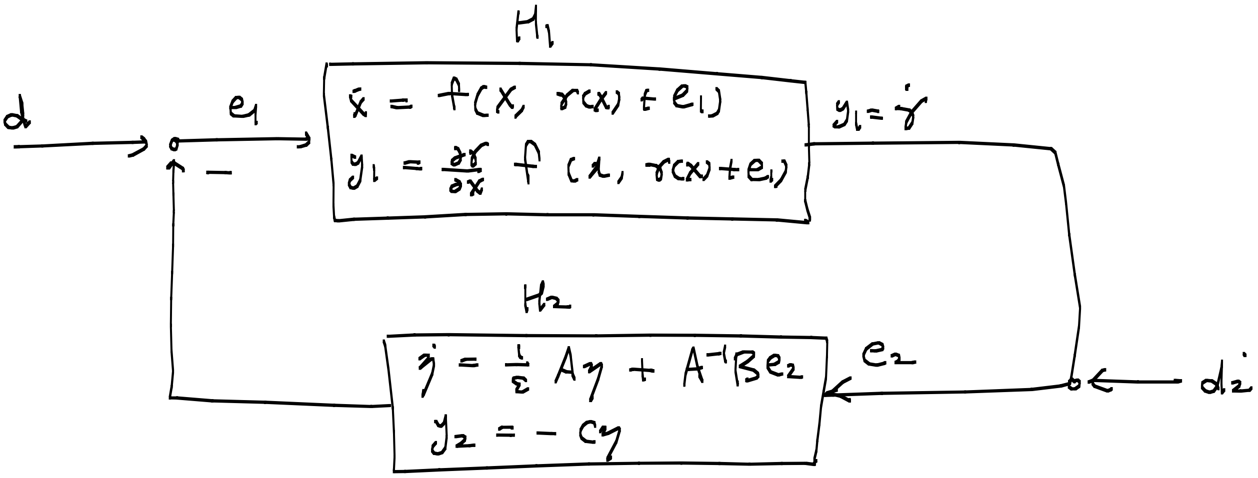
$\dot{x} = f(x,\, C(\eta - A^{-1}B\,(\gamma(x) + d_2)) + d_1)$

$= f(x,\, C\eta + \gamma(x) + d_1 + d_2)$

$\varepsilon\dot{\eta} = A(\eta - A^{-1}B\,(\gamma + d_2)) + B(\gamma + d_2) + \varepsilon A^{-1}B\,(\dot{\gamma} + \dot{d}_2)$

$= A\eta + \varepsilon A^{-1}B\,(\dot{\gamma} + \dot{d}_2)$

$\left(\dot{\gamma} = \frac{\partial \gamma}{\partial x} f(x,\, \gamma + d + C\eta)\right)$



Asr: $\left\| \frac{\partial r}{\partial x}(x) f(x, r(x)+e_1)\right\| \le c_1\|x\| + c_2\|e_1\|$

H_1: $\|y_1\|_{\mathcal{L}} \le \underbrace{(c_1\lambda + c_2)}_{=\lambda_1} \|e_1\|_{\mathcal{L}} + \underbrace{c_1\rho}_{=\rho_1}$

H_2: Hurwitz LTI (Corollary 5.2)

$$\|y_2\|_{\mathcal{L}} \le \frac{2\lambda_{max}^2(P)\|B\|_2\|C\|_2}{\lambda_{min}(P)} \|e_2\|_{\mathcal{L}} + \rho_2$$

$$= \varepsilon\lambda_f \|e_2\|_{\mathcal{L}} + \rho_2$$

where $P(A/\varepsilon) + (A/\varepsilon)^T P = -I$

$\Rightarrow P = \varepsilon\bar P, \quad \bar P A + A^T\bar P = -I$

$\therefore$ if $\underbrace{\lambda_1 \varepsilon \lambda_f < 1}$ then o.k

$\uparrow$ for sufficiently small ε.