Chap 7 Flow of a Real Fluid

viscosity, turbulence

7.1 Laminar and Turbulent Flow

- Laminar flow (層流):
  - 완만하고 규칙적인 흐름
  - 흐름의 층 사이에 대규모 혼합이 없음
- Turbulent flow (亂流):
  - 불규칙하고 무질서한 흐름
  - 흐름 내에 대규모 혼합이 존재함
- Reynolds experiment (Read text)

For the same fluid (same viscosity),

$V\uparrow$: laminar→turbulent (upper critical velocity)

$V\downarrow$: turbulent→laminar (lower critical velocity)

lower critical velocity < upper critical velocity

→ more engineering importance

Reynolds number, $\mathbf{R} = \dfrac{Vd\rho}{\mu} = \dfrac{Vd}{\nu} = \dfrac{\text{inertia force}}{\text{viscous force}}$

where d=characteristic length scale

(pipe flow: diameter, open channel: depth)

$\mathbf{R} > \mathbf{R}_c$: turbulent flow, $\mathbf{R} < \mathbf{R}_c$: laminar flow

where $\mathbf{R}_c$ = critical Reynolds number

7.2 Turbulent Flow and Eddy Viscosity

$$\underbrace{v'}_{\substack{\text{instantaneous}\\\text{velocity}}} = \underbrace{v}_{\substack{\text{time-mean}\\\text{velocity}}} + \underbrace{v_x + v_y}_{\substack{\text{turbulent fluctuations}\\\text{in } x \text{ and } y \text{ directions}}}$$

root-mean square (rms) of v_x or v_y

$$= \left(\overline{v_x^2}\right)^{1/2} \text{ or } \left(\overline{v_y^2}\right)^{1/2} = \text{turbulence intensity}$$

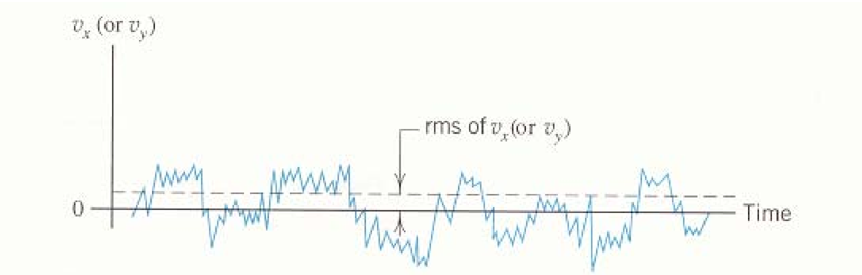
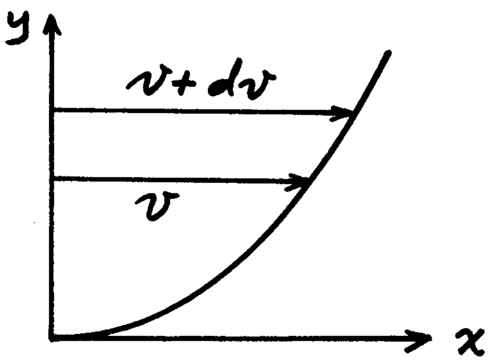


Fig. 7.4

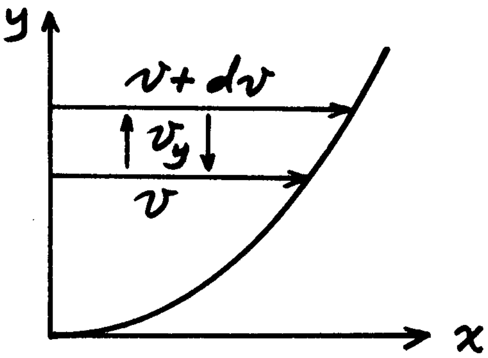
Laminar flow



$$\tau = \mu \frac{dv}{dy};$$ shear stress due to viscosity b/w molecules

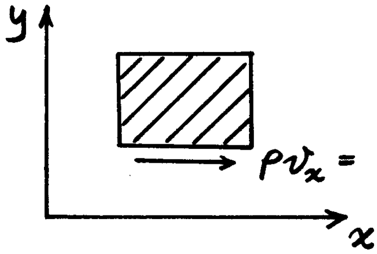
μ: fluid viscosity (property of fluid)

Turbulent flow



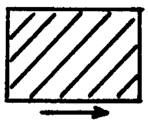
$$\tau = \varepsilon \frac{dv}{dy}\text{; shear stress due to momentum transfer}$$

ε — eddy viscosity (property of flow)



ρv_x = excess momentum per unit volume
($\because$ ρ = mass per unit volume)

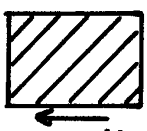
Time-averaged (or mean) shear stress due to momentum transfer:



$\vec{\tau} = \rho\overline{v_x v_y}$ > 0 : add momentum to the particle
→ accelerate particle

negative

< 0 : subtract momentum from the particle
→ decelerate particle



positive

$\tau = -\rho\overline{v_x v_y} \equiv$ Reynolds stress

Note: Positive shear stress retards the fluid motion.

Prandtl (1926): $v_x, v_y \propto \ell \dfrac{dv}{dy}$

where ℓ=mixing length (unknown function of y)

$$\therefore \tau = -\rho \overline{v_x v_y} = \rho \ell^2 \left(\frac{dv}{dy} \right)^2$$

$$\therefore \varepsilon = \rho \ell^2 \frac{dv}{dy}$$

Near a wall, $v_x, v_y \downarrow \ \rightarrow \ \ell \downarrow$

At the surface of a wall, $v_x, v_y = 0 \ \rightarrow \ \ell = 0$

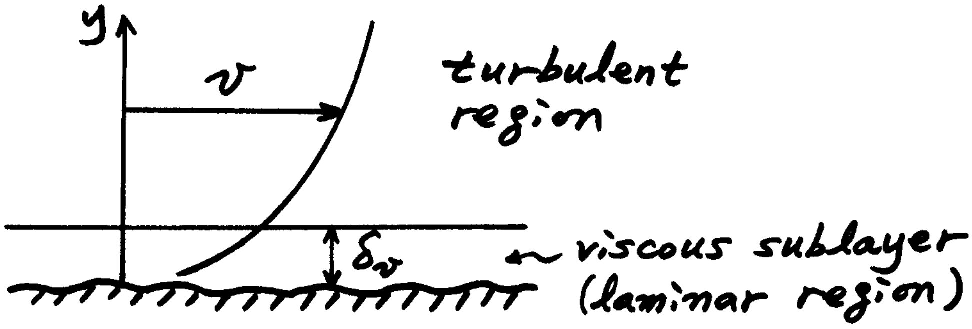
Assume $\ell = \kappa y$ (linear variation with y), which is in good agreement with experimental data, where κ =von Karman constant (≈ 0.4) and y=distance from the wall.

$$\therefore \tau = \rho \kappa^2 y^2 \left(\frac{dv}{dy} \right)^2$$

7.3 Fluid Flow Past Solid Boundaries

- For a real fluid flow, $v = 0$ at solid boundary (no-slip condition)

- Effect of surface roughness:
  - Laminar flow: Viscosity is dominant.
    Roughness has no effect on the flow.
  - Turbulent flow: Viscous sublayer is formed near the surface with thickness δ_v.



If roughness $\ll \delta_v$ $\rightarrow$ smooth boundary
$\rightarrow$ no effect of roughness on the flow
If roughness $\geq \delta_v / 3$ $\rightarrow$ rough boundary
$\rightarrow$ roughness has effect on the flow

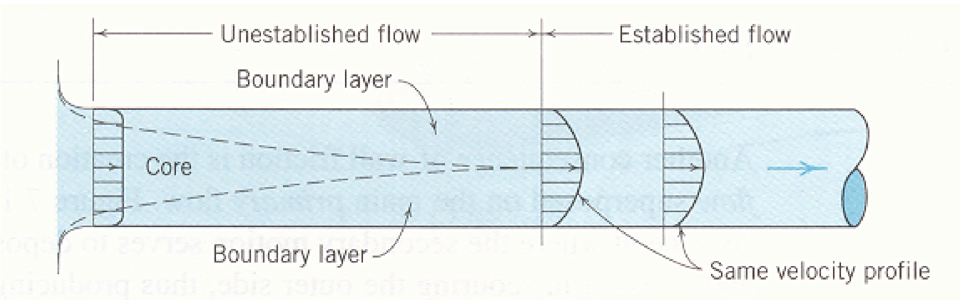
Note: For the same roughness, the boundary can be either smooth or rough depending on the property of the flow ($\because \delta_v$ depends on the flow characteristics). In general, $v \uparrow \Rightarrow \mathbf{R} \uparrow \Rightarrow \delta_v \downarrow$.

7.4 – 7.8 External Flows

(over a wing or flat plate, etc)

- External flows are less important in civil and environmental engineering, so it is not taught in this class.
- Internal flows: flows in ducts, channels, and pipes

7.9 Flow Establishment–Boundary Layers



unestablished flow ($\approx 20d$)

↓

complicated to analyze

↓

net effect = entrance head loss (Section 9.9)

7.10 Shear Stress and Head Loss

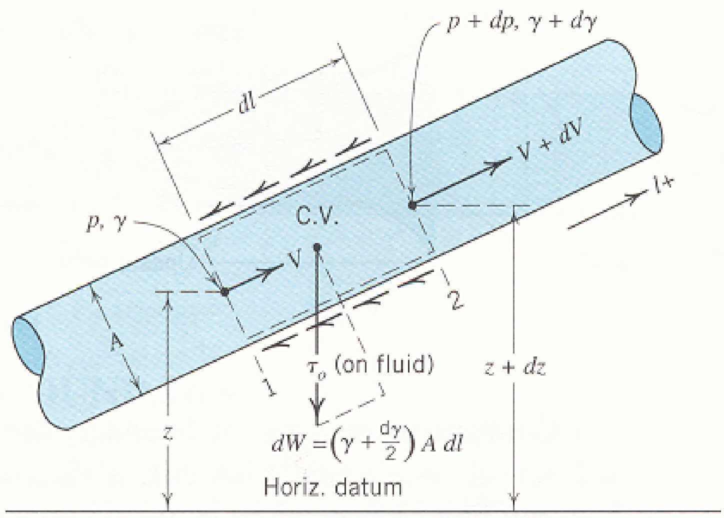


Fig. 7.21

A = constant cross-sectional area

P = perimeter of the pipe

$R_h = A/P$ = hydraulic radius

Impulse-momentum equation along the pipe (l):

$$\underbrace{pA-(p+dp)A}_{\text{pressure forces}}-\underbrace{\tau_o Pdl}_{\text{shear force}}-\underbrace{\left(\gamma+\frac{d\gamma}{2}\right)Adl\frac{dz}{dl}}_{\text{body force}}=(V+dV)^2A(\rho+d\rho)-V^2A\rho$$

Using $Q_1\rho_1 = VA\rho,\ \ Q_2\rho_2=(V+dV)A(\rho+d\rho),$

$Q_1\rho_1 = Q_2\rho_2 = Q\rho$ for steady flow

$$-Adp-\tau_o Pdl-\left(\gamma+\frac{d\gamma}{2}\right)Adz = Q\rho(V+dV-V)=\rho AVdV=\rho Ad\left(\frac{V^2}{2}\right)$$

$$\therefore\ \frac{dp}{\gamma}+d\left(\frac{V^2}{2g}\right)+dz=-\frac{\tau_o}{\gamma R_h}dl$$

$$\int_2^1 d\left(\frac{p}{\gamma}+\frac{V^2}{2g}+z\right)=\int_2^1 -\frac{\tau_o}{\gamma R_h}dl$$

$$\left(\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1\right)-\left(\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2\right)=\frac{\tau_o}{\gamma R_h}(l_2-l_1)=\Delta(EL)=h_{L_{1-2}}$$

head loss due to pipe friction

Shear stress in the fluid (τ):

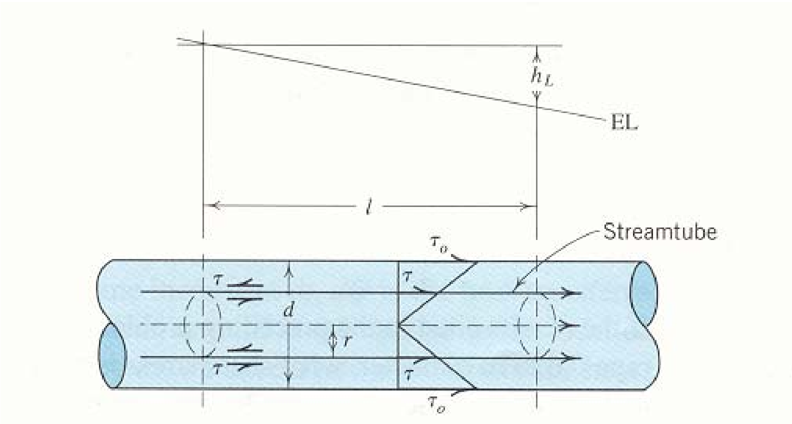


Fig. 7.22

For the streamtube of radius r,

$$h_L=\frac{\tau(l_2-l_1)}{\gamma R_h}=\frac{\tau l}{\gamma\frac{\pi r^2}{2\pi r}}=\frac{2\tau l}{\gamma r}$$

$$\therefore\ \tau=\left(\frac{\gamma h_L}{2l}\right)r$$

τ varies linearly with r from zero at the centerline to τ_o at the pipe wall.

7.11 The First Law of Thermodynamics and Shear Stress Effects

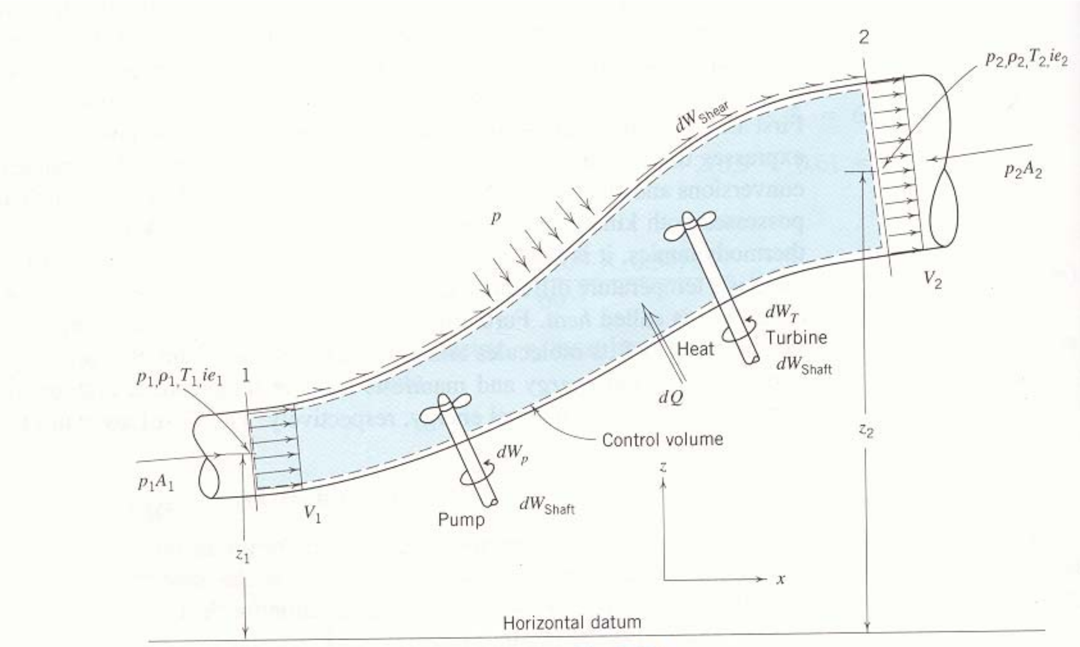


Fig. 7.23

$$dQ_H + dW = dE$$

where dQ_H =heat transferred across system boundary by temperature difference, which was neglected in derivation of work-energy equation.

$$\frac{dQ_H}{dt} + \frac{dW}{dt} = \frac{dE}{dt}$$

Examine term by term:

$$\frac{dQ_H}{dt} = \dot{m}q_H = Q\rho q_H$$

where q_H =added heat per unit mass of fluid.

$$\frac{dW}{dt} = Q\gamma\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma} + E_p - E_t\right) = Q\rho g\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma} + E_p - E_t\right)$$

Note: no work done by shear stress because

$v = 0$ at the pipe wall for real fluid

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\left(\iiint_{CV} \rho\left(gz + \frac{V^2}{2} + ie\right) d\forall\right)$$

$$+ \iint_{CS_{out}} \rho\left(gz + \frac{V^2}{2} + ie\right)\vec{v}\cdot\vec{n}dA + \iint_{CS_{in}} \rho\left(gz + \frac{V^2}{2} + ie\right)\vec{v}\cdot\vec{n}dA$$

$$= Q\rho\left[\left(gz + \frac{V^2}{2} + ie\right)_2 - \left(gz + \frac{V^2}{2} + ie\right)_1\right]$$

$$\therefore\ Q\rho q_H + Q\rho g\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma} + E_p - E_t\right)$$

$$= Q\rho\left(gz_2 + \frac{V_2^2}{2} + ie_2 - gz_1 - \frac{V_1^2}{2} - ie_1\right)$$

$$\left(\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1\right) + E_p = \left(\frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2\right) + E_t + \frac{1}{g}\left(ie_2 - ie_1 - q_H\right)$$

If no pump or turbine,

$$\left(\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2\right) = \underbrace{\frac{1}{g}\left(ie_2 - ie_1 - q_H\right)}_{=h_{L_{1-2}}}$$

$$h_{L_{1-2}} = \frac{\tau_o(l_2 - l_1)}{\gamma R_h} = \frac{1}{g}(ie_2 - ie_1 - q_H)$$

Note:

1. head loss = energy converted to heat (q_H) and internal energy (ie).
2. In hydraulic engineering, it is not necessary to evaluate q_H and ie separately, and it is more convenient to use the concept of head loss.
3. $ie_2 - ie_1 = c(T_2 - T_1)$ where c = specific heat (=4180 J/kg·K for water)

7.12 Velocity Distribution and Its Significance

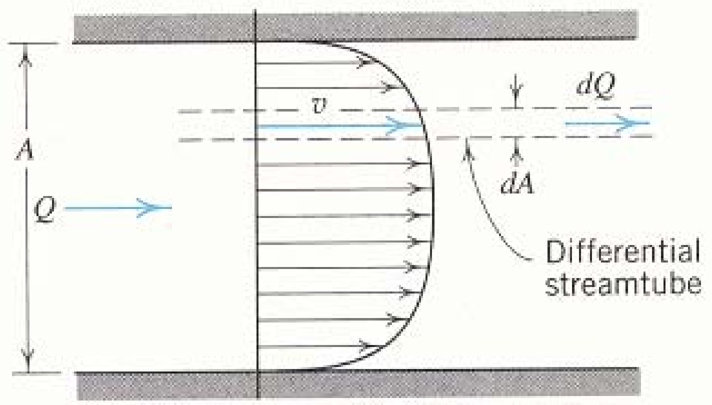


Fig. 7.24

Total flowrate $Q = \iint_A v dA$

Mean velocity $V = \frac{1}{A}\iint_A v dA = \frac{Q}{A}$

Total flux of K.E. $= \frac{\rho}{2}\iint_A v^3 dA = \alpha Q \frac{1}{2}\rho V^2 = Q\rho\left(\alpha \frac{V^2}{2}\right)$

Total momentum flux = $\rho\iint_A v^2 dA = \beta Q \rho V$

where α, β = correction factors:

$$\alpha = \frac{\frac{\rho}{2}\iint v^3 dA}{Q\rho \frac{V^2}{2}} = \frac{1}{V^2}\frac{\iint v^3 dA}{\iint v dA}$$

$$\beta = \frac{\rho\iint v^2 dA}{Q\rho V} = \frac{1}{V}\frac{\iint v^2 dA}{\iint v dA}$$

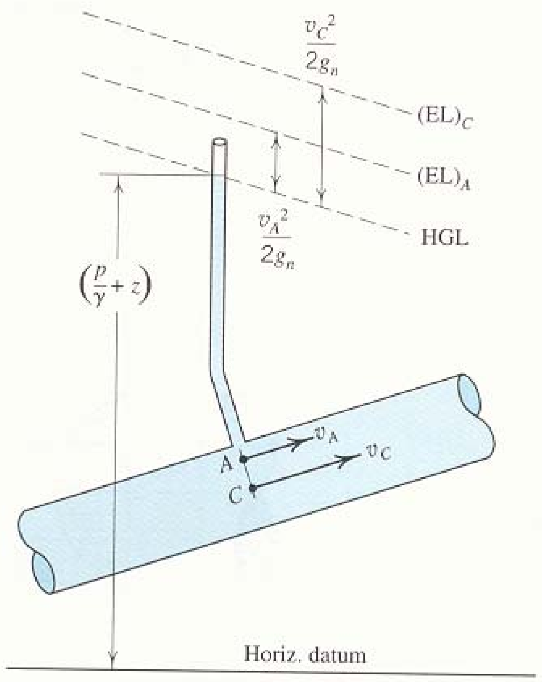


Fig. 7.25

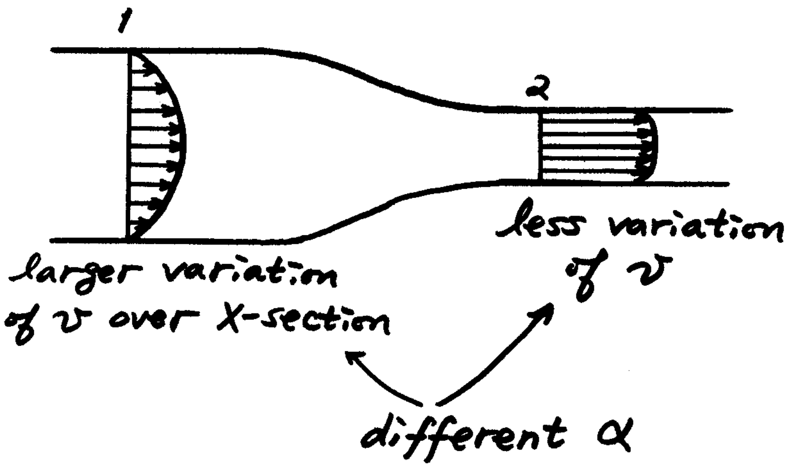
• EL's and HGL are parallel but not horizontal because of head loss.

• Straight and parallel streamlines
  → hydrostatic pressure distribution
  → $p/\gamma + z = \text{const.}$ over cross-section

• EL is different for different locations over the cross-section ($\because$ v is different)

• mean EL = $\dfrac{p}{\gamma} + z + \dfrac{\alpha V^2}{2g}$

Flow through a contraction:



Exact $h_{L_{1-2}} = \left(\alpha_1 \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 \right) - \left(\alpha_2 \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 \right)$

Conventional $h_{L_{1-2}} = \left(\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 \right) - \left(\frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 \right)$

Exact – Conventional = $(\alpha_1 - 1)\frac{V_1^2}{2g} - (\alpha_2 - 1)\frac{V_2^2}{2g}$

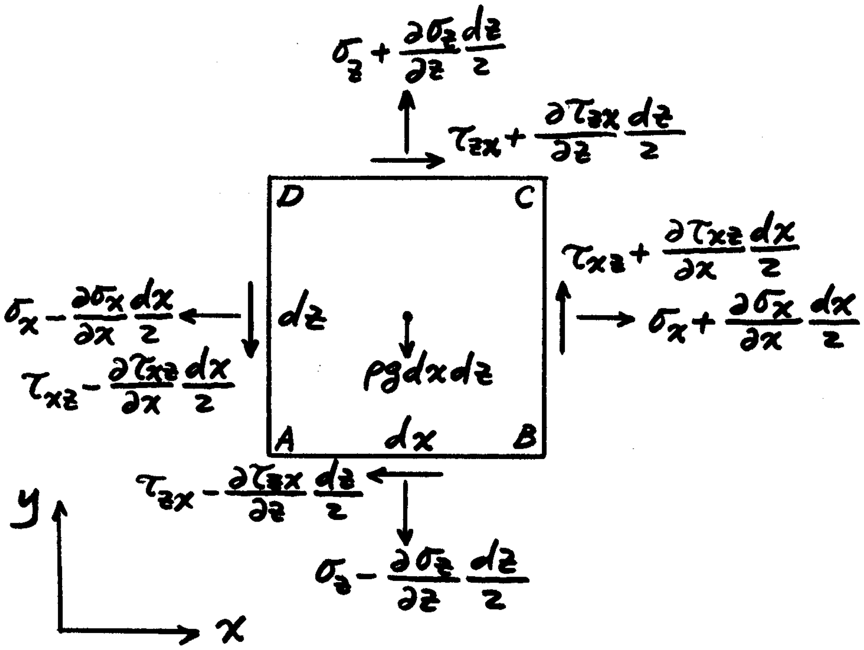
$\therefore$ Conventional $h_{L_{1-2}}$ = Exact $h_{L_{1-2}} + (\alpha_2 - 1)\frac{V_2^2}{2g} - (\alpha_1 - 1)\frac{V_1^2}{2g}$

$$\left.\begin{array}{l}\text{7.13 Separation} \\ \text{7.14 Secondary Flow}\end{array}\right\} \rightarrow \text{local energy dissipation} \rightarrow \text{head loss}$$

7.15 Derivation of Navier-Stokes Equations

→ Euler equation + viscosity

2-D, unsteady, incompressible flow

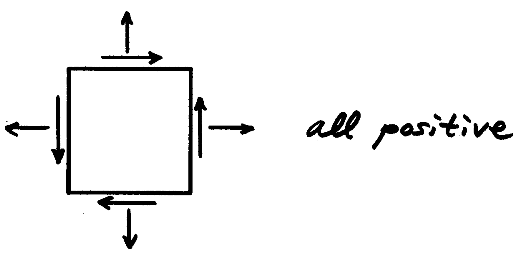


σ_x, σ_z = normal stresses (positive outward normal to the surface)

τ_{xz}, τ_{zx} = shear stresses

(1st subscript: plane, 2nd subscript: direction)

Sign convention: Stresses are positive if positive direction at positive plane or negative direction at negative plane.



Newton's 2nd law:

$$\sum \vec{F} = m\vec{a} = \frac{d}{dt}\left(m\vec{v}\right)$$

Surface forces + Body forces:

$$\sum \vec{F} = \sum F_x \vec{e}_x + \sum F_z \vec{e}_z$$

where $\vec{e}_x, \vec{e}_z$ = unit vectors in x and z directions

$$\sum F_x = \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdz \qquad (7.56)$$

$$\sum F_z = \left(\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} - \rho g\right) dxdz \qquad (7.57)$$

Reynolds transport theorem:

$$\frac{d}{dt}\left(m\vec{v}\right) = \frac{\partial}{\partial t}\left(\iiint_{CV} \rho \vec{v} d\mathcal{V}\right) + \iint_{CS} \vec{v}\rho\vec{v}\cdot\vec{n}dA$$

where

$$\frac{\partial}{\partial t}\left(\iiint_{CV} \rho \vec{v} d\mathcal{V}\right) = \frac{\partial}{\partial t}\left(\iiint_{CV} \rho\left(u\vec{e}_x + w\vec{e}_z\right) d\mathcal{V}\right)$$

$$= \frac{\partial}{\partial t}\left(u\vec{e}_x + w\vec{e}_z\right)\rho dxdz = \left(\frac{\partial u}{\partial t}\vec{e}_x + \frac{\partial w}{\partial t}\vec{e}_z\right)\rho dxdz$$

$$\iint_{CS} \vec{v}\rho\vec{v}\cdot\vec{n}dA = \iint_{AB} + \iint_{BC} + \iint_{CD} + \iint_{DA} \vec{v}\rho\vec{v}\cdot\vec{n}dA$$

For example, for the plane AB,

$$\vec{v} = \left(u - \frac{\partial u}{\partial z}\frac{dz}{2} \right)\vec{e}_x + \left(w - \frac{\partial w}{\partial z}\frac{dz}{2} \right)\vec{e}_z\,, \quad \vec{n} = -\vec{e}_z\,,$$

$$\vec{v} \cdot \vec{n} = -w + \frac{\partial w}{\partial z}\frac{dz}{2}$$

$$\iint_{AB} \vec{v}\rho\vec{v} \cdot \vec{n} dA = \left(u - \frac{\partial u}{\partial z}\frac{dz}{2} \right)\left(-w + \frac{\partial w}{\partial z}\frac{dz}{2} \right)\rho dx\vec{e}_x$$

$$+\left(w - \frac{\partial w}{\partial z}\frac{dz}{2} \right)\left(-w + \frac{\partial w}{\partial z}\frac{dz}{2} \right)\rho dx\vec{e}_z$$

$$= \left(-uw + u\frac{\partial w}{\partial z}\frac{dz}{2} + w\frac{\partial u}{\partial z}\frac{dz}{2} - \frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{dz^2}{4} \right)\rho dx\vec{e}_x$$

$$+\left(-w^2 + w\frac{\partial w}{\partial z}\frac{dz}{2} + w\frac{\partial w}{\partial z}\frac{dz}{2} - \frac{\partial w}{\partial z}\frac{\partial w}{\partial z}\frac{dz^2}{4} \right)\rho dx\vec{e}_z$$

$$\therefore \iint_{CS} \vec{v}\rho\vec{v} \cdot \vec{n} dA = \left(2u\frac{\partial u}{\partial x} + u\frac{\partial w}{\partial z} + w\frac{\partial u}{\partial z} \right)\rho dxdz\vec{e}_x$$

$$+\left(2w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial x} + w\frac{\partial u}{\partial x} \right)\rho dxdz\vec{e}_z$$

$$= \left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right)\rho dxdz\vec{e}_x + \left(w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial x} \right)\rho dxdz\vec{e}_z$$

$$\therefore \frac{d}{dt}\left(m\vec{v} \right) = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right)\rho dxdz\vec{e}_x$$

$$+\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \right)\rho dxdz\vec{e}_z$$

$$x: \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zx}}{\partial z}$$

$$z: \quad \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} - \rho g$$

Stokes hypothesis: stresses = $f(p, u, w, \mu)$

$$\left.\begin{aligned} \sigma_x &= -p + 2\mu\frac{\partial u}{\partial x} \\ \sigma_z &= -p + 2\mu\frac{\partial w}{\partial z} \end{aligned}\right\} \rightarrow p = -\frac{\sigma_x + \sigma_z}{2}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - g}$$

These are Navier-Stokes equations for 2-D flow.

For steady flow of inviscid fluid:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \end{aligned}\right\} \text{Euler equations}$$

3-D Navier-Stokes equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

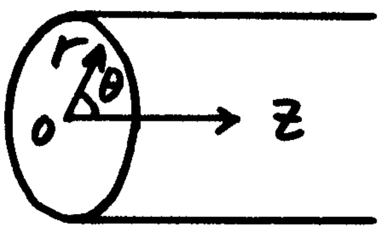
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-g$$

Continuity: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

4 equations for 4 unknowns (p, u, v, w)

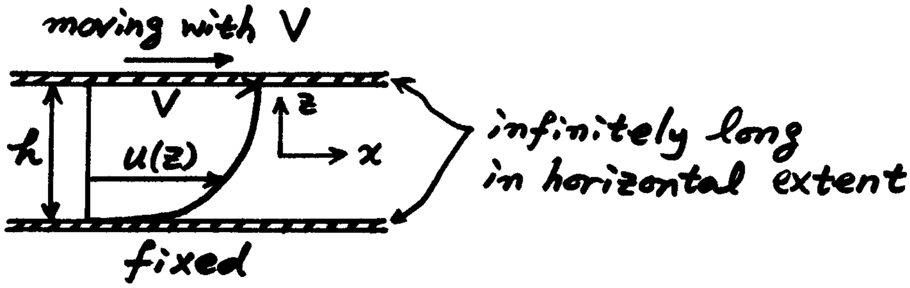
Navier-Stokes equations in cylindrical coordinates for axi-symmetric flows (r, z) → Eq. (7.64) in text (gravitational force is neglected)

Continuity: $\frac{\partial u_z}{\partial z}+\frac{u_r}{r}+\frac{\partial u_r}{\partial r}=0$



7.16 Applications of Navier-Stokes Equations

(1) Couette flow



$$x: \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\therefore 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$z: \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - g$$

$$\therefore 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad \rightarrow \quad \frac{\partial p}{\partial z} = -\rho g \quad \rightarrow \quad p(x,z) = -\rho g z + C(x)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{\nu}\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial z} = \frac{1}{\nu}\frac{1}{\rho}\frac{\partial p}{\partial x} z + C_1$$

$$u(z) = \frac{1}{\nu}\frac{1}{\rho}\frac{\partial p}{\partial x}\frac{z^2}{2} + C_1 z + C_2$$

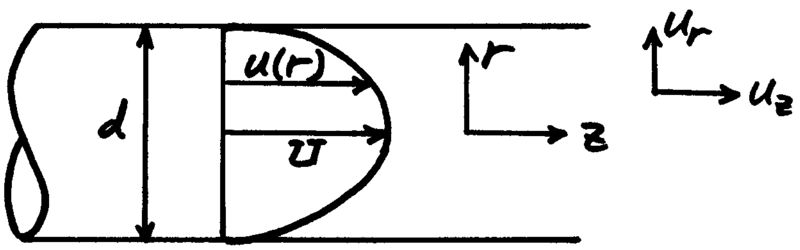
B.C.'s: $u(-h/2) = 0, \quad u(h/2) = V$

$$\therefore u(z) = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[z^2 - \left(\frac{h}{2}\right)^2\right] + V\left(\frac{1}{2} + \frac{z}{h}\right)$$

If $\frac{\partial p}{\partial x} = 0$, $u(z) = V\left(\frac{1}{2} + \frac{z}{h}\right)$: linear variation

If $V = 0$, $u(z) = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[z^2 - \left(\frac{h}{2}\right)^2\right]$: parabolic profile

(2) Hagen-Poiseuille flow



$$r: \frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right)$$

$$\therefore 0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \rightarrow \quad \frac{\partial p}{\partial r} = 0$$

$$z: \frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

$$\therefore 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) = -\frac{1}{\rho}\frac{dp}{dz} + \nu\left(\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr}\right)$$

$$\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}$$

$$r\frac{d^2u_z}{dr^2}+\frac{du_z}{dr}=\frac{1}{\mu}\frac{dp}{dz}r$$

$$\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=\frac{1}{\mu}\frac{dp}{dz}r$$

$$r\frac{du_z}{dr}=\frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2}+C_1$$

B.C.: $\frac{du_z}{dr}=0$ at $r=0$ $\rightarrow$ $C_1=0$

$$\therefore\frac{du_z}{dr}=\frac{1}{2\mu}\frac{dp}{dz}r$$

$$u_z=\frac{1}{2\mu}\frac{dp}{dz}\frac{r^2}{2}+C_2$$

B.C.: $u_z=0$ at $r=\frac{d}{2}$ $\rightarrow$ $C_2=-\frac{1}{4\mu}\frac{dp}{dz}\left(\frac{d}{2}\right)^2$

$\therefore u_z=\frac{1}{4\mu}\frac{dp}{dz}\left[r^2-\left(\frac{d}{2}\right)^2\right]$: parabolic profile

Time-averaged Navier-Stokes equations for turbulent flow:

$$\left.\begin{aligned} u &\rightarrow \bar{u} \\ w &\rightarrow \bar{w} \\ p &\rightarrow \bar{p} \end{aligned}\right\}\ \text{time - averaged (mean) velocity and pressure}$$

$$\nu\rightarrow\nu_T=\nu+\varepsilon$$