3. Chapter Turbulent Diffusion

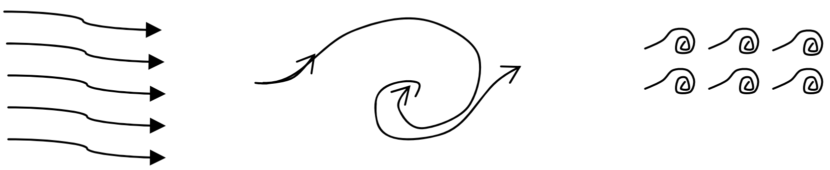
3.1 Introduction

- Mass introduced at a point will spread much faster in turbulent flow than in laminar flow.
- Velocities and pressures measured at a point in the fluid are unsteady and possess a random component.

◆ Turbulent flow: Irregularity, randomness
Diffusivity
High Reynolds number
3-D fluctuations
Dissipation of kinetic energy
Continuum phenomenon
Feature of flow

◆ Scale of turbulence

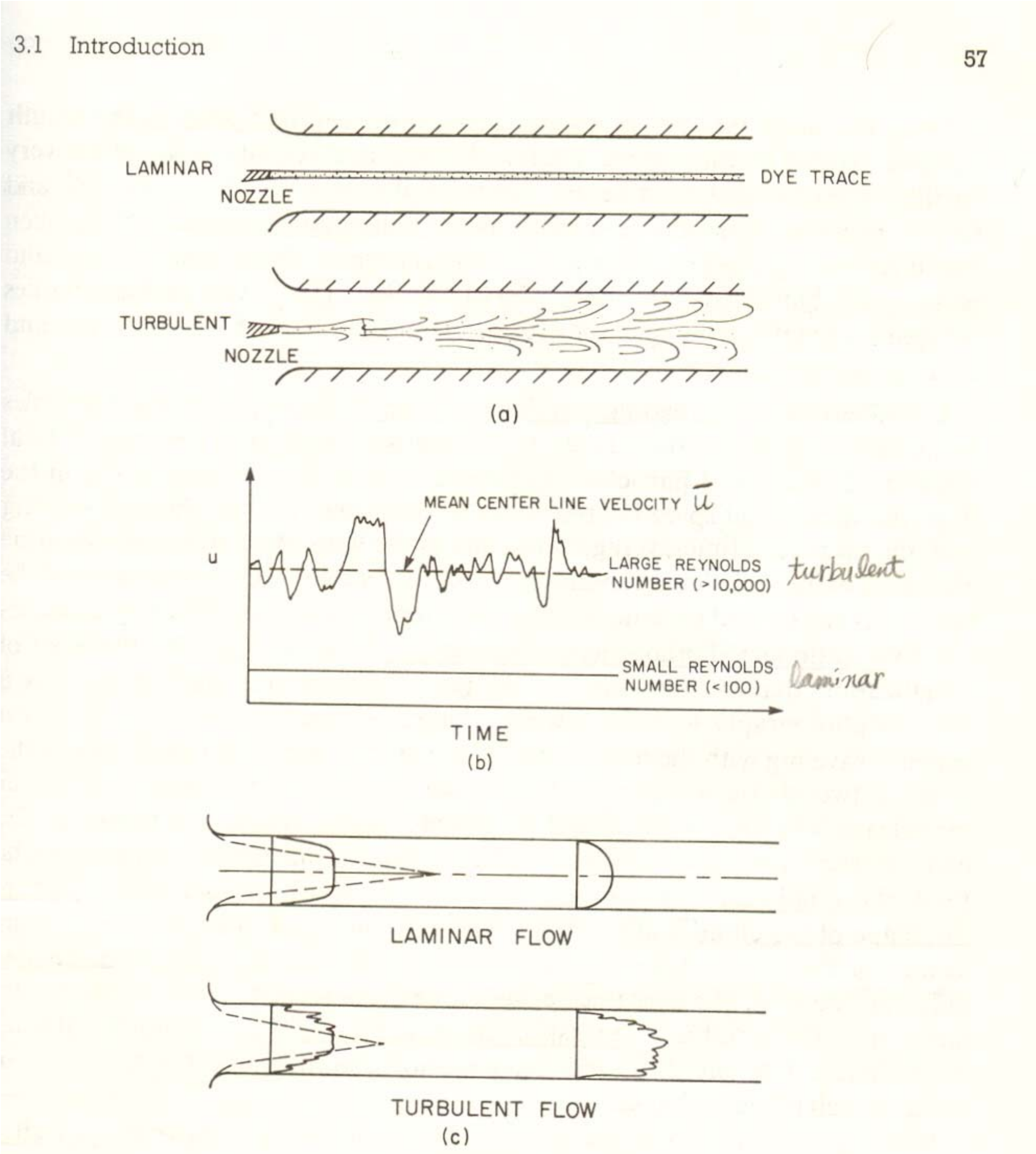
- Ranges of eddy sizes



mean flow	→	large eddy	→	small eddy	→	heat
	generation of turbulence		energy cascade		dissipation by viscosity	

In equilibrium, transfer rate = dissipation rate

◆ Reynolds experiment



◆ Kolmogorov's universal equilibrium theory of turbulence

- Behavior of the intermediate scale is governed only by the transfer of energy which, in turn, is exactly balanced by dissipation at the very small scales.

ε = time rate of energy dissipation per unit mass

$$[\varepsilon] = \left[\frac{energy}{time}\frac{1}{Mass}\right] = \left[\frac{F\,L}{t}\frac{1}{M}\right] = \left[\frac{ML^2T^{-2}}{T}\frac{1}{M}\right] = \left[L^2T^{-3}\right]$$

ν = kinematic viscosity $= \dfrac{\mu}{\rho} = \left[\dfrac{ML^{-1}T^{-1}}{ML^{-3}}\right] = \left[L^2T^{-1}\right]$

Kolmogorov scales :

→ length scale $\propto \varepsilon,\ \nu$

i) length $= \left(\dfrac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} = \left[\dfrac{L^6T^{-3}}{L^2T^{-3}}\right]^{\frac{1}{4}} = [L]$

ii) time $= \left(\dfrac{\nu}{\varepsilon}\right)^{\frac{1}{2}} = \left[\dfrac{L^2T^{-1}}{L^2T^{-3}}\right]^{\frac{1}{2}} = [T]$

iii) velocity $= \dfrac{\left(\dfrac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}}{\left(\dfrac{\nu}{\varepsilon}\right)^{\frac{1}{2}}} = (\nu\varepsilon)^{\frac{1}{4}} = \left[(L^2T^{-1})(L^2T^{-3})\right]^{\frac{1}{4}} = \left[LT^{-1}\right]$

For open ocean

$$\varepsilon = 0.01 cm^2/\sec^3 \;\; ; \;\; \nu = 0.01 cm^2/s \;\; (20°C)$$

dissipation length scale ≈ 0.1cm

time scale ≈ 1sec

velocity scale ≈ 0.1cm/sec

◆ Spreading of a slug of tracer in a high Reynolds number flow

(1) Small scale fluctuations, which are different for each cloud, distort the shape of the cloud and produce steep concentration differences over short distance.

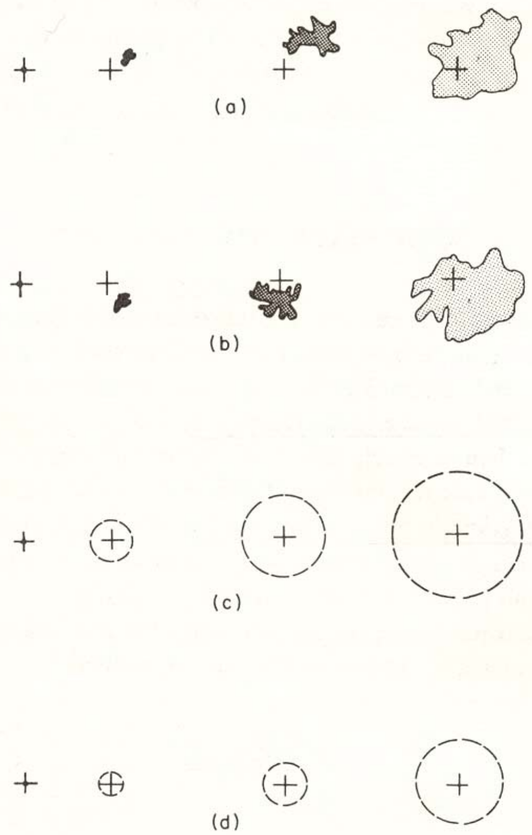
(2) Large scale fluctuations transport the entire cloud.

◆ 'Ensemble average' = mean over many trials

- average the random motions over a long period time

- average out the effects of the largest eddies

◆ Statistical estimate of the size of an individual cloud


(a)
(b)
(c)
(d)

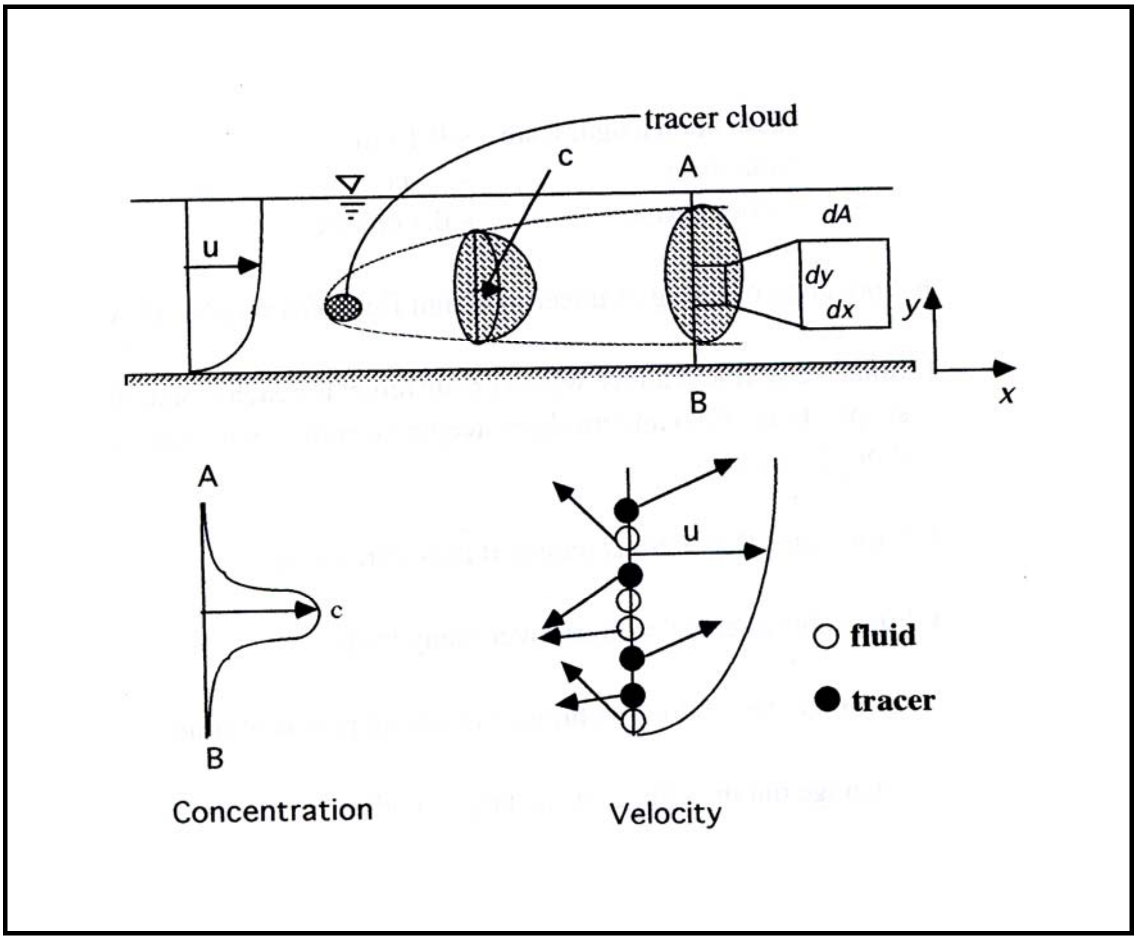
3.2 Unified View of Diffusion and Dispersion

- Similarity among the various types of diffusion and dispersion are shown.

- Diffusion and dispersion are actually advective transport mechanisms.

3.2.1 Molecular Diffusion

◆ 2-D open-channel flow



To write the mass balance equation, we need to know how many fluid molecules and how many tracer molecules pass through and the direction and spread of each molecule.

→ molecular approach → statistical manner → FPM

◆ Continuum approach

- Assume fluid carries tracer through at a rate depending on the concentration, c, and the fluid velocity, u.
- However, the fluid u, cannot completely represent the tracer movement because the velocity, u, does not account for the movement of the molecules which have directions and speeds different from u.
- Molecular diffusion accounts for the difference between the true molecular motion and the manner chosen to represent the motion.(i.e., by u)

$$\Delta u = u_m - u$$

Fick' law

- transport called molecular diffusion is proportional to the concentration gradient.

$$j_x = \Delta u \, c \propto \frac{\partial c}{\partial x}$$

$$j_x = -D_m \frac{\partial c}{\partial x}$$

D_m = constant of proportionality = molecular diffusivity

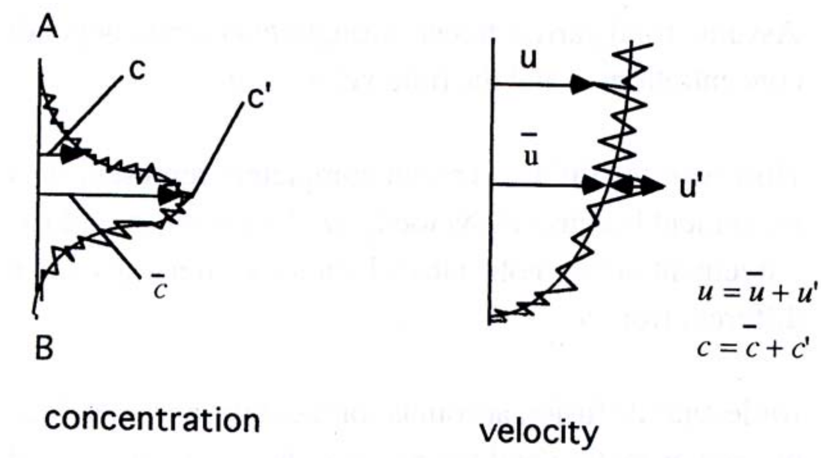
$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D_m \frac{\partial^2 c}{\partial x^2} + D_m \frac{\partial^2 c}{\partial y^2} \qquad (3.1)$$

① $\dfrac{\partial c}{\partial t}$ = time rate of change of concentration at a point

② $u\dfrac{\partial c}{\partial x}$ = advection of tracer with the fluid

③ $D_m\dfrac{\partial^2 c}{\partial x^2}$, $D_m\dfrac{\partial^2 c}{\partial y^2}$ = molecular diffusion

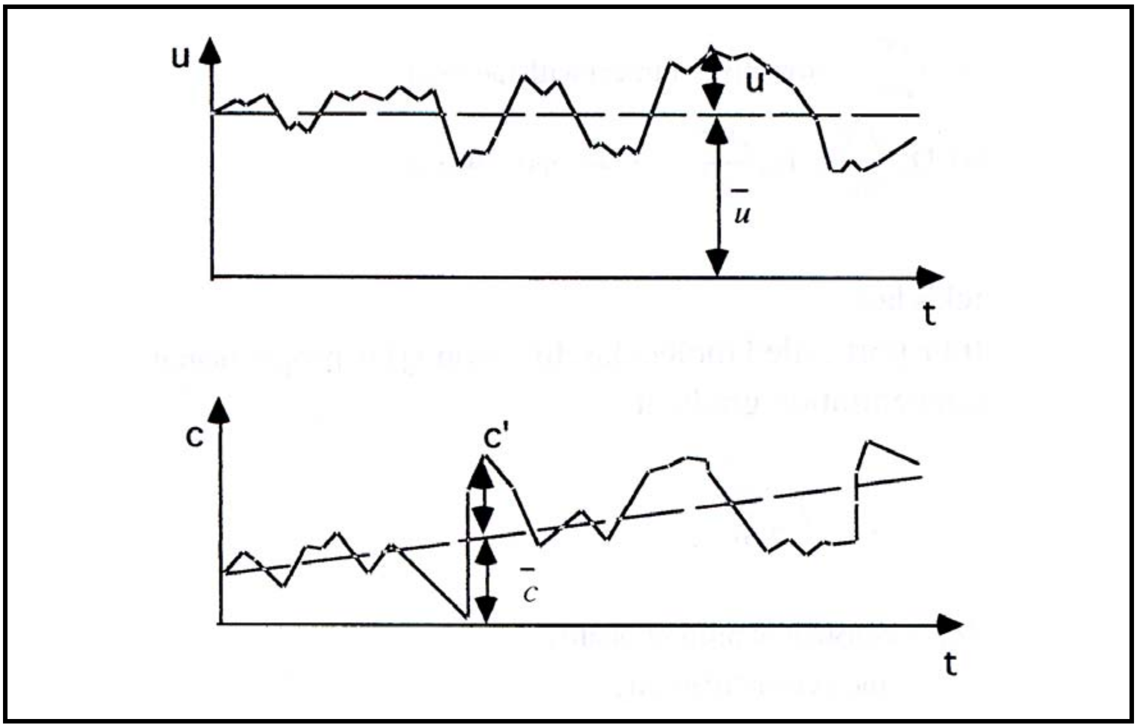
3.2.2 Turbulent Diffusion



$\bar{u}$, $\bar{c}$ = time-averaged values of u and c

$$\bar{u} \equiv \frac{1}{T}\int_0^T u\,dt$$

$$\overline{u'} \equiv 0$$



◆ Mass balance equation

For 2-D flow, advection-diffusion equation

$$\frac{\partial c}{\partial t} + \frac{\partial uc}{\partial x} + \frac{\partial vc}{\partial y} = D_m \frac{\partial^2 c}{\partial x^2} + D_m \frac{\partial^2 c}{\partial y^2} \tag{3.2}$$

$u = \bar{u} + u'$

$c = \bar{c} + c'$ (assume only fluctuation in y-direction) (a)

$v = v'$

Substitute (a) into (3.2), then Eq. (3.2) becomes

$$\frac{\partial(\bar{c}+c')}{\partial t}+\frac{\partial\left(\bar{u}+u'\right)(\bar{c}+c')}{\partial x}+\frac{\partial v'(\bar{c}+c')}{\partial y}=D_m\frac{\partial^2(\bar{c}+c')}{\partial x^2}$$

$$+D_m\frac{\partial^2(\bar{c}+c')}{\partial y^2}$$

$$\frac{\partial\bar{c}}{\partial t}+\frac{\partial}{\partial x}\bar{u}\,\bar{c}=D_m\frac{\partial^2\bar{c}}{\partial x^2}+D_m\frac{\partial^2\bar{c}}{\partial y^2}$$

$$-\frac{\partial c'}{\partial t}+\frac{\partial}{\partial x}(\bar{u}\,c')-\frac{\partial}{\partial x}(u'\bar{c})-\frac{\partial}{\partial x}(u'c')\ -\frac{\partial}{\partial y}(v'\bar{c})-\frac{\partial}{\partial y}(v'c')$$

$$+D_m\frac{\partial^2 c'}{\partial x^2}+D_m\frac{\partial^2 c'}{\partial y^2}$$

Integrate (average) w.r.t. time

$$\overline{\frac{\partial\bar{c}}{\partial t}}+\overline{\frac{\partial\left(\bar{u}\,\bar{c}\right)}{\partial x}}=\overline{D_m\frac{\partial^2\bar{c}}{\partial x^2}}+\overline{D_m\frac{\partial^2\bar{c}}{\partial y^2}}$$

$$-\cancel{\overline{\frac{\partial c'}{\partial t}}}+\cancel{\overline{\frac{\partial(\bar{u}\,c')}{\partial x}}}-\cancel{\overline{\frac{\partial(u'\bar{c})}{\partial x}}}-\overline{\frac{\partial u'c'}{\partial x}}-\cancel{\overline{\frac{\partial v'\bar{c}}{\partial y}}}-\overline{\frac{\partial v'c'}{\partial y}}$$

$$+\cancel{D_m\overline{\frac{\partial^2 c'}{\partial x^2}}}+\cancel{D_m\overline{\frac{\partial^2 c'}{\partial y^2}}}$$

◆ Reynolds rules of averages (Schlichting; p460, 371)

$$\bar{\bar{f}}=\bar{f}$$

$$\overline{f+g}=\bar{f}+\bar{g}$$

$$\overline{f\cdot\bar{g}}=\bar{f}\cdot\bar{g}$$

$$\overline{\frac{\partial f}{\partial s}}=\frac{\partial\bar{f}}{\partial s}$$

$$\int\overline{f ds}=\int\bar{f}ds$$

◆ Definition of turbulence

$$u = \overline{u} + u', \; v = \overline{v} + v'$$

$$\overline{u}' = \overline{v}' = 0$$

Drop all zero terms using Reynolds rules of averages

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} = D_m \frac{\partial^2 \overline{c}}{\partial y^2} + D_m \frac{\partial^2 \overline{c}}{\partial y^2} + \underbrace{\frac{\partial(-\overline{u'c'})}{\partial x} + \frac{\partial(-\overline{v'c'})}{\partial y}}_{\substack{\text{advective transport} \\ \text{due to } u',v',\text{and } c'}}$$

◆ It is assumed and confirmed experimentally that transport associated with the turbulent fluctuations is proportional to the concentration gradient.

$$\overline{u'c} = -\varepsilon_x \frac{\partial \overline{c}}{\partial x}$$

$$\overline{v'c} = -\varepsilon_y \frac{\partial \overline{c}}{\partial y}$$

ε_x, ε_y = turbulent diffusion coefficient

$$\frac{\partial}{\partial x}\left(-\overline{u'c}\right) = \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial \overline{c}}{\partial x}\right)$$

$$\frac{\partial}{\partial y}\left(-\overline{v'c}\right) = \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial \overline{c}}{\partial y}\right)$$

Assuming that ε_x and ε_y are constant, the mass balance equation for turbulent flow is given as

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} = (D_m + \varepsilon_x)\frac{\partial^2 \overline{c}}{\partial x^2} + (D_m + \varepsilon_y)\frac{\partial^2 \overline{c}}{\partial y^2} \tag{3.3}$$

Drop overbars, neglect molecular diffusion

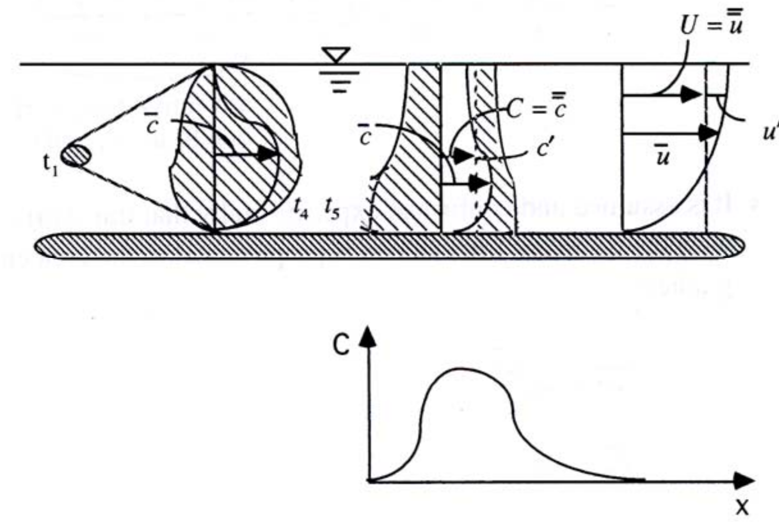
$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = \varepsilon_x \frac{\partial^2 c}{\partial x^2} + \varepsilon_y \frac{\partial^2 c}{\partial y^2} \tag{3.4}$$

For 3-D flow :

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}(\varepsilon_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y}(\varepsilon_y \frac{\partial c}{\partial y}) + \frac{\partial}{\partial z}(\varepsilon_z \frac{\partial c}{\partial z}) \tag{3.5}$$

☞ Remember $\varepsilon_x \frac{\partial c}{\partial x}, \ \varepsilon_y \frac{\partial c}{\partial y}, \ \varepsilon_z \frac{\partial c}{\partial z}$ and are actually <u>advective transport</u>.

3.2.3 Longitudinal Dispersion



After the tracer is essentially completely mixed laterally, the primary variation of concentration is in just longitudinal direction.

→ one-dimensional equation

Integrate (average) Eq. (3.3) over the corss-sectional area

$$\overline{u} = U + u'' \qquad \overline{u''} = 0$$

$$\overline{c} = C + c'' \qquad \overline{c''} = 0$$

U, C = cross-sectional average of the velocity and concentration

$$\overline{\overline{\frac{\partial(C+c'')}{\partial t}}} + \overline{\overline{(U+u'')\frac{\partial(C+c'')}{\partial x}}} = \overline{\overline{(D_m+\varepsilon_x)\frac{\partial^2(C+c'')}{\partial x^2}}} + \overline{\overline{(D_m+\varepsilon_y)\frac{\partial^2(C+c'')}{\partial y^2}}}$$

By Reynolds rule

$$\frac{\partial C}{\partial t}+U\frac{\partial C}{\partial x}=(D_m+\varepsilon_x)\frac{\partial^2 C}{\partial x^2}+(D_m+\varepsilon_y)\frac{\partial^2 C}{\partial y^2}-\frac{\partial\left(\overline{\overline{u"c"}}\right)}{\partial x} \tag{3.6}$$

Neglect $\frac{\partial^2 C}{\partial y^2}$ because after lateral mixing is completed

$$\frac{\partial C}{\partial y}\approx 0;\ C=\bar{C}\neq f\left(y\right)$$

Then

$$\frac{\partial C}{\partial t}+U\frac{\partial C}{\partial x}=(D_m+\varepsilon_x)\frac{\partial^2 C}{\partial x^2}+\frac{\partial\left(-\overline{\overline{u"c"}}\right)}{\partial x}$$

Taylor (1953, 1954) show that advective transport associated with $u^{"}$ is proportional to the longitudinal gradient of C

→ longitudinal dispersion

$$-\overline{\overline{u"c"}}\propto-\frac{\partial C}{\partial x}$$

$$-\overline{\overline{u"c"}}=K\frac{\partial C}{\partial x}$$

$$\frac{\partial}{\partial x}\left(-\overline{\overline{u"c"}}\right)=\frac{\partial}{\partial x}\left(K\frac{\partial C}{\partial x}\right)$$

K = longitudinal dispersion coefficient

In turbulent uniform flow

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \left(D_m + \varepsilon_x + K\right)\frac{\partial^2 C}{\partial x^2}$$

$$(D_m + \varepsilon_x)\frac{\partial C}{\partial x} << \overline{-u"c"}$$

1%   99%

Because the velocity distribution influences $u^{"}$

→ Lateral diffusion plays a large role in determining the distribution of $c^{"}$, both velocity distribution and lateral diffusion contribute to longitudinal dispersion.

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = K\frac{\partial^2 C}{\partial x^2} \tag{3.7}$$

→ 1-D Dispersion Equation

- Limitation proposed by Chatwin (1970)

$$t > \frac{0.4h^2}{D(\varepsilon_t)}$$

$$x > \frac{0.4uh^2}{\varepsilon_t}$$

3.2.4 Relative Importance of Dispersion

To investigate the relative importance of dispersion, use dimensionless term as

$$H = \frac{dispersion\ rate}{advective\ rate} = \frac{K\dfrac{\partial C}{\partial x}}{UC} = \frac{K}{U}\cdot\frac{1}{C}\frac{\partial C}{\partial x} = \frac{K}{U}\cdot\frac{\partial(\ln C)}{\partial x}$$

$$\left.\begin{array}{l} H << 1 \\ H < H_c \approx 0.01 \end{array}\right] \rightarrow \text{ dispersive transport may be neglected}$$

3.2.5 Conclusion

diffusion

= transport associated with fluctuating component of molecular action and with turbulent action

= transport in a given direction at a point in the flow due to the difference between the true advection in that direction and the time average of the advection in that direction

dispersion = transport associated with the variation of the velocity across the flow section

= transport in a given direction due to the difference between the true advection in that direction and the spatial average (=cross-sectional average) of the advection in that direction

THE DIFFUSION AND DISPERSION SPECTRUM

$D_{ij}, cm^2/\sec$	
10^{-6}	
10^{-5}	Molecular Diffusion in Water
10^{-4}	
10^{-3}	
10^{-2}	
10^{-1}	Molecular Diffusion Gases
1	
	Eddy Diffusion - Pipes
10	
100	Eddy Diffusion - Streams
10000	
10^{4}	Longitudinal Dispersion - Streams
10^{5}	
10^{6}	Longitudinal Tidal Dispersion - Estuaries
10^{7}	
10^{8}	
10^{9}	
10^{10}	Eddy Diffusion - Atmosphere