2 Gauss Elimination

2.1 Inverse matrices

A is **invertible** if there exists a matrix A^{-1} such that

$$A^{-1}A = I \qquad \text{and} \qquad AA^{-1} = I$$

1. inverse exists $\Leftrightarrow$ elimination produces n pivots (row exchanges allowed)
2. A cannot have two different inverses.
   If $BA = I,\ AC = I$, then $B(AC) = (BA)C \ \Rightarrow B = C$.
   i.e. A left-inverse and right-inverse are the same.
3. If A is invertible, the one and only solution to $A\mathrm{x} = \mathrm{b}$ is $\mathrm{x} = A^{-1}\mathrm{b}$.
4. If there is a nonzero vector x such that $A\mathrm{x} = 0$, then A cannot have an inverse. If A is invertible, $x = 0$ is the only solution to $A\mathrm{x} = 0$
5. A is invertible $\Leftrightarrow$ *determinant* is not zero.

$$\text{ex)} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

6. diagonal matrix

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_n \end{bmatrix}$$

7. If A and B are invertible, then so is AB.
   $(AB)^{-1} = B^{-1}\ A^{-1}$

2.2 Calculating A^{-1} by Gauss - Jordan Elimination

$$\begin{array}{ccc} AA^{-1} & = & I \\ \shortparallel & & \\ A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} & = & \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \end{array}$$

Gauss - Jordan method solves three systems of equations($Ax_i = e_i$) together.

Example . Compute the inverse of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Apply elimination to the augmented matrix

$$\begin{bmatrix} A & e_1 & e_2 & e_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\frac{1}{2}$ row 1 + row 2
$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\frac{2}{3}$ row 2 + row 3
$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

upper triangular. continue
$\frac{3}{4}$ row 3 + row 2
$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$\frac{2}{3}$ row 2 + row 1
$\Longrightarrow$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

divide each row by its pivots
$\Longrightarrow$

$$
\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right] = \left[\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \end{array}\right]
$$

i.e

$$
\boxed{\begin{array}{c} \text{G-J process multiplies} \\ A^{-1}\ [\text{A} \quad \text{I}] \\ \text{to get } [\text{I} \quad A^{-1}] \end{array}}
$$

- Let A be a square matirx.
  A^{-1} exists (and Gauss-Jordan finds it) exactly when A has n pivots.
  If $AC = I$ then $CA = I$ and $C = A^{-1}$.

Example . If L is lower triangular with 1's on the diagonal, so is L^{-1}.

$$
\begin{aligned}
\left[\begin{array}{cc} L & I \end{array}\right] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 1 & 0 \\ * & * & 1 & 0 & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & * & 1 & 0 \\ 0 & * & 1 & * & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & * & 1 & 0 \\ 0 & 0 & 1 & * & * & 1 \end{array}\right] = \left[\begin{array}{c|c} I & L^{-1} \end{array}\right]
\end{aligned}
$$

2.3 Elimination = LU Factorization

(Stang Pg.83)

Example . $A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}$

$$
E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U
$$

$$
\underbrace{E_{21}^{-1}U}_{\text{LU=A}} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A
$$

Example . A is 3 × 3, <u>no row exchange needed</u>.

$$\text{then,} \quad E_{21}A$$

$$E_{31}(E_{21}A)$$

$$E_{32}(E_{31}E_{21}A) = U$$

$$\Longrightarrow \quad A = \underbrace{(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})}_{\text{lower triangular 1's on the diagonal}} \quad U = LU$$

lower multiplier l_{ij} goes directly into (i,j) position

Example .

$$A = \begin{bmatrix} 2 & 1 & \overline{0} \\ 1 & 2 & 1 \\ \underline{0} & 1 & 2 \end{bmatrix}$$

$A_{31} = 0 \rightarrow$ don't need the step E_{31}

$$= \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ \underline{0} & 2/3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 & \overline{0} \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix}}$$

observation :

- If a row of A starts with 0,
  then so does that row of L

- If a column of A starts with 0,
  then so does that column of U

In the above example,

Row 3 of U = Row 3 of A - l_{31} (Row1 of U) $- l_{32}$(Row2 of U)

$\Longrightarrow$ Row3 of A

= l_{31}(Row1 of U) + l_{32}(Row2 of U) + 1 ·(Row3 of U)

= Row3 of LU

$\Longrightarrow$ This shows why A = LU.

2.4 LDU Factorization

$\Longrightarrow$ Divide U by diagonal elements so that U has 1's on the diagonal.

In the previous example,

$$A = \begin{bmatrix} . & . & . \\ . & L & . \\ . & . & . \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{U_{12}}{d_1} & \frac{U_{13}}{d_1} \\ 0 & 1 & \frac{U_{23}}{d_2} \\ 0 & 0 & 1 \end{bmatrix}$$

2.5 Transposes

A^T = <u>transpose</u> of A

column of A^T = row of A

$(A^T)_{ij} = A_{ji}$

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$
- For any vectors x and y,

$(Ax)^T y = x^T A^T y = x^T (A^T y)$

- <u>symmetric</u> matrix : $A^T = A$ $(a_{ji} = a_{ij})$
- inverse of an invertible, symmetric matrix is also symmetric.

  $\because (A^{-1})^T = (A^T)^{-1} = A^{-1}$

- Choose any matrix R, then $R^T R$ is square, and

  $(R^T R)^T = R^T (R^T)^T = R^T R$

$\therefore R^T R$ is a symmetric matrix.

(RR^T is also a symmetric matrix.)

In the previous example,

$$\underbrace{A}_{sysmetric} = LDU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = LDL^T$$

$\Longrightarrow$ If $A = A^T$, $A = LDU$ (with no row exchange),

then $U = L^T$, and $A = LDL^T$.