6 Inverse of a Matrix

6.1 Cramer's Rule & Inverse   for solving $A\mathrm{x} = \mathrm{b}$

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \quad A\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} := B_1$$

$\rightarrow$ Take determinants : $(detA)x_1 = detB_1$

$$\therefore \; x_1 = \frac{detB_1}{detA}$$

Similarly,

$$A\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} := B_2$$

$\rightarrow$ Take determinants : $(detA)x_2 = detB_2$

$$\therefore \; x_2 = \frac{detB_2}{detA}$$

<**Cramer's Rule**>   If $detA \neq 0$, $A\mathrm{x} = \mathrm{b}$ has the unique soln. $$x_j = \frac{detB_j}{detA}$$ where B_j has the column j of A replaced by the vector b.

Example .

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 \qquad &= 0 \\ -4x_1 + \qquad + x_3 &= 0 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & \\ -4 & & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & \\ 0 & & 1 \end{vmatrix} = 1$$

$$\Longrightarrow \qquad x_j = \frac{|B_j|}{|A|} \qquad \therefore \left(\begin{array}{l} x_1 = \frac{1}{7} \\ x_2 = \frac{2}{7} \\ x_3 = \frac{4}{7} \end{array} \right.$$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$|B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & & 0 \end{vmatrix} = 4$$

6.2 Using Cramer's rule to compute the inverse

- $AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

  $\Downarrow$

  b in the above example

  $$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \text{the first column of } A^{-1}$$

- In the above example, note that

$$|B_1| = C_{11}, \quad |B_2| = C_{12}, \quad |B_3| = C_{13}$$

$$\therefore \quad \text{the first col. of } A^{-1} = \begin{bmatrix} \frac{C_{11}}{|A|} \\ \\ \frac{C_{12}}{|A|} \\ \\ \frac{C_{13}}{|A|} \end{bmatrix}$$

- use b $= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to get the second column of A^{-1} :

$$|B_1| = \begin{vmatrix} 0 & a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = C_{21}$$

$$|B_2| = \begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} = C_{22} \quad \Rightarrow \quad \text{the 2nd col. of } A^{-1} \; \| \; \begin{bmatrix} \frac{C_{21}}{|A|} \\ \frac{C_{22}}{|A|} \\ \frac{C_{23}}{|A|} \end{bmatrix}$$

$$|B_3| = \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{31} & 0 \end{vmatrix} = C_{23}$$

- use b $= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to get the third column of A^{-1} :

$$\Rightarrow \begin{bmatrix} \frac{C_{31}}{|A|} \\ \frac{C_{32}}{|A|} \\ \frac{C_{33}}{|A|} \end{bmatrix}$$

$$\therefore \quad \boxed{(A^{-1})_{ij} = \frac{C_{ji}}{detA}}$$

If we define the cofactor matrix $C = [C_{ij}]$ then,

$$A^{-1} = \frac{C^T}{detA}$$

Example .

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} \quad \Rightarrow detA = 1$$

$$\Rightarrow \quad M_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow C_{11} = 1$$

$$M_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow C_{12} = -1$$

$$M_{13} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow C_{13} = 0$$

$$\vdots$$

$$A^{-1} = \frac{C^T}{detA} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(The inverse of a triangular matrix is triangular)

6.3 Inverse & Rank

Theorem (Existence of the inverse) (Theorem 1, Sec. 6.7)
The inverse A^{-1} of an $n \times$ matrix A exists if and only if rank $A = n$, hence (by Theorem 3, Sec. 6.6) if and only if $detA \neq 0$. Hence A is nonsingular if rank $A = n$, and is singular if rank $A < n$.

Proof. Consider the linear system

$$(2) \qquad \mathrm{Ax} = \mathrm{b}$$

with the given matrix A as coefficient matrix. If the inverse exists, then multiplication from the left on both sides gives

$$\mathrm{A^{-1}Ax = x = A^{-1}b}.$$

This shows that (2) has a unique solution x, so that A must have rank n by the Fundamental Theorem in Sec. 6.5

Conversely, let rank A = n. Then by the same theorem, the system (2) has a unique solution x for any b, and the back substitution following the Gauss elimination (in Sec 6.3) shows that its components x_j are linear combinations of those of b, so that we can write

$$(3) \qquad \mathrm{x = Bb}.$$

Substitution into (2) gives

$$\mathrm{Ax = A(Bb) = (AB)b = Cb = b} \qquad \mathrm{(C = AB)}$$

for any b. Hence C = AB = I, the unit matrix. Similarly, if we substitue (2) into (3) we get

$$\mathrm{x = Bb = B(Ax) = (BA)x}$$

for any x (and b = Ax). Hence BA = I. Together, B = $\mathrm{A^{-1}}$ exists.