10 Fourier Series

10.1 Periodic Functions

- Definition : A function $f(x)$ is called periodic, if it is defined for all real x and if there is some positive number p such that

$$f(x+p) = f(x) \qquad \text{for all } x$$

- The number p is called a period of $f(x)$
Example: $\sin(x+2\pi) = \sin x$, $\cos(x+2\pi) = \cos x$
(note) $f = c$=constant for every x
- Examples of non-periodic functions : x, x^2, x^3, e^x, $\cosh x, \cdots$
- $f(x+2p) = f[(x+p)+p] = f(x+p) = f(x)$
- For any integer n

$$f(x+np) = f(x) \qquad \text{for all } x$$

- If $f(x)$ and $g(x)$ have period p, then the function $h(x) = af(x) + bg(x)$, a, b: constant

$$h(x+p) = af(x+p) + bg(x+p) = af(x) + bg(x) = h(x)$$
$$\therefore\ h(x+p) = h(x) \quad : \text{period } p$$

- Fundamental period: a smallest period $p(>0)$ of $f(x)$

$$\cos x \quad \text{and} \quad \sin x \quad \rightarrow \quad 2\pi$$
$$\cos 2x \quad \text{and} \quad \sin 2x \quad \rightarrow \quad \pi$$
$$f = \text{constant} \rightarrow \ \text{no fundamental period}$$

10.2 Trigonometric Series

- Trigonometric functions with a period of 2π

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cdots \quad \cos nx, \quad \sin nx, \quad \cdots$$

- Trigonometric Series

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

where a_0, a_1, b_1, a_2, a_3, b_2, b_3 $\cdots$ are real constants.

- Trigonometric system

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

If the series (1) converges, its sum will be a function of period 2π.

10.3 Fourier Series

Euler Formulas for the Fourier Coefficients

$f(x)$ is a periodic function of period 2π and is integrable over a period. Assume that eqn (1) converges and (1)=$f(x)$.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2}$$

Given $f(x)$, how to compute the coefficients?

- Determination of the constant term a_0

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi}[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)]dx \\
\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty}(\int_{-\pi}^{\pi} a_n \cos nxdx + \int_{-\pi}^{\pi} b_n \sin nxdx) \\
&\because \ a_n \int_{-\pi}^{\pi} \cos nxdx = b_n \int_{-\pi}^{\pi} \sin nxdx = 0 \\
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx
\end{aligned}$$

- Determination of the coefficients a_n
  - Multiplying (2) by $\cos mx$ where m is any fixed positive integer,

$$\int_{-\pi}^{\pi} f(x)\cos mxdx = \int_{-\pi}^{\pi}[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)]\cos mxdx$$

  - R.H.S of the above equation

$$a_0\int_{-\pi}^{\pi}\cos mxdx + \sum_{n=1}[a_n\int_{-\pi}^{\pi}\cos nx\cdot\cos mxdx + b_n\int_{-\pi}^{\pi}\sin nx\cdot\cos mxdx]$$

  - 1st integral:

$$a_0\int_{-\pi}^{\pi}\cos mxdx = 0$$

  - 2nd and 3rd integrals:

$$\begin{aligned}
\int_{-\pi}^{\pi}\cos nx\cdot\cos mxdx &= \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)xdx + \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx \\
\int_{-\pi}^{\pi}\sin nx\cdot\cos mxdx &= \frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)xdx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(n-m)xdx
\end{aligned}$$

- When $n \neq m$ and the integration is performed over the common period of 2π,

$$\int_{-\pi}^{\pi} \cos(n+m)x dx = \int_{-\pi}^{\pi} \cos(n-m)x dx = 0$$

$$\int_{-\pi}^{\pi} \sin(n+m)x dx = \int_{-\pi}^{\pi} \sin(n-m)x dx = 0$$

- When $n = m$, all the integrals are zero except

$$\int_{-\pi}^{\pi} \cos(n-m)x dx = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\therefore\ a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \qquad m = 1, 2, 3, \cdots$$

- Determination of the coefficients b_n
  - Multiplying (2) by $\sin mx$ where m is any fixed positive integer,

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)] \sin mx dx$$

  - R.H.S of the above equation

$$a_0 \int_{-\pi}^{\pi} \sin mx dx + \sum_{n=1}^{\infty} [a_n \int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx]$$

  - 1st integral:

$$a_0 \int_{-\pi}^{\pi} \sin mx dx = 0$$

  - 2nd and 3rd integrals:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(m+n)x dx - \frac{1}{2} \int_{-\pi}^{\pi} \sin(n-m)x dx \\ \int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x dx \end{aligned}$$

  - When $n \neq m$ and the integration is performed over the common period of 2π, the above integrals are zero.
  - When $n = m$, the above integrals are zero for the same reason except

$$\frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x dx = \frac{1}{2} \int_{-\pi}^{\pi} dx = \pi$$

$$\therefore b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx \qquad m = 1, 2, 3, \cdots$$

Summary of These Calculations:
Fourier Coefficients, Fourier Series, Euler formula ——————————————————-

$$\begin{aligned} &(a) \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ &(b) \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx dx \qquad n = 1, 2, \cdots \\ &(c) \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx dx \qquad n = 1, 2, \cdots \end{aligned} \tag{3}$$

Example 1. Rectangular wave

$$\begin{aligned} f(x) &= -k \qquad \text{if } -\pi < x < 0 \\ f(x) &= k \qquad \text{if } 0 < x < \pi \\ f(x + 2\pi) &= f(x) \end{aligned}$$

Sol)

$$\therefore a_0 = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx dx = \frac{1}{\pi}[\int_{-\pi}^{0}(-k)\cos nx dx + \int_{0}^{\pi} k\cos nx dx] \\ &= \frac{1}{\pi}[-k\frac{\sin nx}{n}|_{-\pi}^{0} + k\frac{\sin nx}{n}|_{0}^{\pi}] = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx dx = \frac{1}{\pi}[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi} k\sin nx dx] \\ &= \frac{1}{\pi}[k\frac{\cos nx}{n}|_{-\pi}^{0} - k\frac{\cos nx}{n}|_{0}^{\pi}] = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] \\ &= \frac{2k}{n\pi}(1 - \cos n\pi) \end{aligned}$$

- Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc :

$$\cos n\pi = \begin{pmatrix} -1 & \text{for odd } n \\ 1 & \text{for even } n \end{pmatrix}$$

$$1 - \cos n\pi = \begin{pmatrix} 2 & \text{for odd } n \\ 0 & \text{for even } n \end{pmatrix}$$

- Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_5 = \frac{4k}{5\pi}, \cdots$$

$$f(x) = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$$

- The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \quad S_2 = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x), \quad etc.$$

- At $x = \pi/2$, $f(\pi/2) = k$

$$f(\frac{\pi}{2}) = k = \frac{4k}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - + \cdots)$$

$$\therefore 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4} \quad \sharp.$$

Orthogonality of the Trigonometric System

- The trigonometric system, 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ... $\cos nx$, $\sin nx$, $\cdots$ is orthogonal on the interval $-\pi \le x \le \pi$.

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cdot \cos nx dx &= 0 \qquad (m \neq n) \\ \int_{-\pi}^{\pi} \sin mx \cdot \sin nx dx &= 0 \qquad (m \neq n) \end{aligned}$$

and for any integers m and n (including $m = n$)

$$\int_{-\pi}^{\pi} \cos mx \cdot \sin nx dx = 0$$

$$\because \int_{-\pi}^{\pi} \cos mx \cdot \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x dx = 0$$

10.4 Convergence and Sum of Fourier Series

Theorem 1 (Representation by a Fourier series)
If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series (2) of $f(x)$ [with coefficients (3)] is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left-hand and right-hand limit of $f(x)$ having continuous first and second derivatives.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{f(x)\cdot \sin nx}{n\pi}\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cdot \sin nx dx \\ &= \frac{f'(x)\cdot \cos nx}{n^2\pi}\Big|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cdot \cos nx dx \end{aligned}$$

- In the above equation,

$$\frac{f'(x)\cdot \cos nx}{n^2\pi}\Big|_{-\pi}^{\pi} = [f'(\pi)\frac{\cos n\pi}{n^2\pi} - f'(-\pi)\frac{\cos(-n\pi)}{n^2\pi}] = 0$$

$$\cos(-n\pi) = \cos n\pi, \qquad f'(\pi) = f(-\pi)$$

$\because$ continuity and periodicity of the function $f(x)$.

- Since f'' is continuous in the interval of integration, we have $|f''(x)| < M$ for an appropriate constant M. Furthermore, $|\cos nx| \le 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M dx = \frac{2M}{n^2}$$

- Similarly, $|b_n| < 2M/n^2$ for all n,

$$\therefore |f(x)| < |a_0| + 2M(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots$$

- It converges !

Example 2. Convergence at a jump as indicated in Theorem 1

The square wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k. Hence the average of these limit is 0. The Fourier series (5) of the square wave does indeed converge to the value when $x = 0$ because then all its terms are 0. Similarly for other jumps. This is in agreement with Theorem 1.

10.5 Functions of Any Period $p = 2L$

- If a function $f(x)$ of period $p = 2L$ has a Fourier series, the series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

$$\begin{aligned}
&\text{(a)} \quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&\text{(b)} \quad a_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \qquad n = 1, 2, \cdots \\
&\text{(c)} \quad b_n = \frac{1}{2L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \qquad n = 1, 2, \cdots
\end{aligned} \tag{4}$$

Proof.

$$\begin{aligned}
v &= \frac{\pi x}{L} \quad \rightarrow \quad x = \frac{Lv}{\pi} \\
x &= \pm L \quad \rightarrow \quad v = \pm\pi
\end{aligned}$$

$$\Longrightarrow f(x) = g(v)$$

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cdot \cos nv + b_n \cdot \sin nv)$$

Coefficients:

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)dv \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv dv \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv dv
\end{aligned} \tag{5}
$$

$$v = \frac{\pi x}{L}, \quad dv = \frac{\pi}{L}dx$$

(5) gives (4) !

Interval of integration : In (4), we may replace the interval of integration by any interval of length $p = 2L$, for example, by the interval $0 \le x \le 2L \quad \sharp$.

Example 3. Periodic square wave

$$
\begin{aligned}
f(x) &= 0 \qquad \text{if } -2 < x < -1 \\
f(x) &= k \qquad \text{if } -1 < x < 1 \\
f(x) &= 0 \qquad \text{if } 1 < x < 2
\end{aligned}
$$

$$p = 2L = 4, \quad \therefore L = 2$$

Solution.

$$
\begin{aligned}
a_0 &= \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-1}^{1} kdx = \frac{k}{2} \\
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx \\
&= \frac{2k}{2n\pi}\sin\frac{n\pi x}{2}\,|_{-1}^{1} = \frac{2k}{n\pi}\sin\frac{n\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= 0 \qquad \text{if } n \text{ is even} \\
a_n &= \frac{2k}{n\pi} \qquad \text{if } n = 1, 5, 9, \cdots \\
a_n &= \frac{-2k}{n\pi} \qquad \text{if } n = 3, 7, 11, \cdots
\end{aligned}
$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cdot\sin\frac{n\pi x}{2}dx = \frac{-k}{n\pi}\cos\frac{n\pi x}{2}|_{-1}^{1} = 0$$

$$\therefore f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right) \qquad \sharp$$

Example 4. Half - wave rectifier

$$
\begin{aligned}
u(t) &= 0 \qquad \text{if } -L < t < 0 \\
u(t) &= E\cdot\sin\omega t \qquad \text{if } 0 < t < L
\end{aligned}
$$

$$p = 2L = \frac{2\pi}{\omega}$$

Solution.

$$\begin{aligned} a_0 &= \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\cdot\sin\omega t dt = -\frac{E}{2\pi}\cos\omega t|_0^{\pi/\omega} \\ &= -\frac{E}{2\pi}(\cos\pi - \cos 0) = -\frac{E}{2\pi}(-1-1) = \frac{E}{\pi} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{\omega}{\pi}\int_0^{\pi/\omega} E\cdot\sin\omega t\cos n\omega t dt \\ &= \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin(1+n)\omega t + \sin(1-n)\omega t]dt \end{aligned}$$

- $a_1 = 0$ if $n = 1$.
- If $n = 2,\ 3,\ \cdots,$

$$\begin{aligned} a_n &= \frac{\omega E}{2\pi}[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}]_0^{\pi/\omega} \\ &= \frac{E}{2\pi}[-\frac{\cos(1+n)\pi + 1}{1+n} - \frac{\cos(1-n)\pi + 1}{1-n}]_0^{\pi/\omega} \end{aligned}$$

- If n is odd $\cos(n+1)\pi = \cos(1-n)\pi = 1$,

$$\therefore a_n = 0$$

$$\begin{aligned} a_n &= \frac{E}{2\pi}(\frac{2}{1+n} + \frac{2}{1-n}) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n = 2, 4, 6, \cdots) \\ b_n &= \frac{\omega}{\pi}\int_0^{\pi/\omega} E\cdot\sin\omega t\cdot\sin n\omega t dt \\ &= \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[-\cos(1+n)\omega t + \cos(1-n)\omega t]dt \end{aligned}$$

- If $n = 1$,

$$b_1 = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}(-\cos 2\omega t + 1)dt = \frac{E}{2}$$

- If $n = 2, 3, \cdots,$

$$b_n = \frac{\omega E}{2\pi}[-\frac{\sin(1+n)\omega t}{(1+n)\omega} + \frac{\sin(1-n)\omega t}{(1-n)\omega}]_0^{\pi/\omega} = 0$$

$$\therefore u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos\omega t + \cdots) \qquad \sharp$$

10.6 Fourier cosine/sine series

Even and Odd Functions

- A function $y = g(x)$ is even if
$$g(-x) = g(x) \quad \text{for all } x$$
- A function $h(x)$ is odd if
$$h(-x) = -h(x) \quad \text{for all } x$$
- $\cos nx$: even, $\sin nx$: odd

Three Key Facts for the Present Discussion

- If $g(x)$ is an even function, then
$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx.$$
- If $h(x)$ is an odd function, then
$$\int_{-L}^{L} h(x)dx = 0.$$
- The product of an even and an odd function is odd.
Proof)
$$q = gh \quad g\text{: even, } h\text{: odd}$$
$$q(-x) = g(-x)\cdot h(-x) = g(x)(-h(x)) = -g(x)h(x) = -q(x)$$

Theorem (Fourier cosine series, Fourier sine series)

- The Fourier series of an even function of period $2L$ is a "Fourier cosine series".
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{n\pi x}{L} \qquad (f:\ \text{even})$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}dx, \qquad n = 1, 2, \cdots$$

- The Fourier series of an odd function of period $2L$ is a "Fourier sine series".

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi x}{L} \qquad (f: \text{ odd})$$

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} dx$$

Proof.

- Derivation of the coefficients of the even series

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx = \frac{1}{2L} \left[\int_{-L}^{0} f(x)dx + \int_0^L f(x)dx \right]$$

In the even function, $f(-x) = f(x)$.

$$-x = x' \quad \Longrightarrow \quad -dx = dx'$$

$$\int_{-L}^{0} f(x)dx = -\int_{L}^{0} f(-x')dx' = \int_0^L f(x')dx'$$

$$\therefore a_0 = \frac{1}{L} \int_0^L f(x)dx$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \\ &= \frac{1}{L} \left[\int_{-L}^{0} f(x) \cdot \cos \frac{n\pi x}{L} dx + \int_0^L f(x) \cdot \cos \frac{n\pi x}{L} dx \right] \end{aligned}$$

$$x = -x' \quad \Longrightarrow \quad dx = -dx'$$

$$\begin{aligned} \int_{-L}^{0} f(x) \cdot \cos \frac{n\pi x}{L} dx &= \int_{L}^{0} f(-x') \cdot \cos(-\frac{n\pi x'}{L})(-dx') \\ &= \int_0^L f(x') \cos \frac{n\pi x'}{L} dx' \end{aligned}$$

$$\therefore a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos \frac{n\pi}{L} x dx$$

- Derivation of the coefficients of an odd function

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{1}{L} [\int_{-L}^{0} f(x) \cdot \sin \frac{n\pi x}{L} dx + \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} dx] \end{aligned}$$

Since f is an odd function, $f(-x) = -f(x)$ and $\sin(-x) = -\sin x$.

$$x = -x' \quad \Longrightarrow \quad dx = -dx'$$

$$\begin{aligned} \int_{-L}^{0} f(x) \cdot \sin \frac{n\pi x}{L} dx &= \int_{L}^{0} f(-x') \cdot \sin(-\frac{n\pi x'}{L})(-dx') \\ &= \int_{0}^{L} f(x') \sin \frac{n\pi x'}{L} dx' \end{aligned}$$

$$\therefore b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} dx \qquad \sharp$$

- For an even function of a period of 2π,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos nx \qquad (f: \text{ even})$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{\pi} f(x) dx \\ a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx dx, \qquad n = 1, 2, \cdots \end{aligned}$$

- Similarly, for an odd 2π-periodic function

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin nx \qquad (f: \text{ odd})$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx, \qquad n = 1, 2, \cdots \qquad \sharp$$

Theorem (Sum of functions)
The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2. The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

Example . Rectangular wave with $p = 2\pi$

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f_1(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} k \sin nx dx \\ &= \frac{2k}{n\pi}(1 - \cos n\pi) \\ &= \begin{cases} \frac{4k}{n\pi} & \text{odd } n \\ 0 & \text{even } n \end{cases} \end{aligned}$$

$$f(x) = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots) \qquad \sharp$$

Example . Rectangular pulse with $p = 4$ $(L = 2)$

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

We can use the formula. Also, we can use the result of the above example to obtain

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x + \cdots) \qquad \sharp$$

Example . Sawtooth wave

$$f(x) = x + \pi \qquad \text{if } -\pi < x < \pi$$
$$f(x + 2\pi) = f(x)$$

$$f = f_1 + f_2 \qquad \text{where } f_1 = x \quad \text{and } f_2 = \pi$$
$$\therefore f_2 = \pi$$

Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x \cdot \sin nx dx \\ &= \frac{2}{\pi}[-\frac{x \cdot \cos nx}{n}|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx dx] = -\frac{2}{n}\cos n\pi \end{aligned}$$

Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4$, $\cdots$

$$f(x) = \pi + 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots \qquad \sharp$$