13 Fourier Transform

13.1 Fourier Cosine and Sine Transforms

- integral transform: a transformation that produces from given functions new functions that depend on a different variable and appear in the form of an integral.
ex) Laplace transform, Fourier transform

Fourier Cosine Transforms
For an even function $f(x)$,

$$\text{(a)} \quad f(x) = \int_0^\infty A(\omega) \cos \omega x \, d\omega \quad \text{where} \quad \text{(b)} \quad A(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \cos \omega v \, dv \qquad (1)$$

Set $A(\omega) = \sqrt{2/\pi} \cdot \hat{f}_c(\omega)$.
Fourier cosine transform of $f(x)$:

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \qquad (2)$$

Inverse Fourier cosine transform of $\hat{f}_c(\omega)$:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega \qquad (3)$$

Fourier Sine Transforms
For an odd function $f(x)$,

$$\text{(a)} \quad f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega \quad \text{where} \quad \text{(b)} \quad B(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \sin \omega v \, dv \qquad (4)$$

Set $B(\omega) = \sqrt{2/\pi} \cdot \hat{f}_s(\omega)$.
Fourier sine transform of $f(x)$:

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \qquad (5)$$

Inverse Fourier sine transform of $\hat{f}_s(\omega)$:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega \qquad (6)$$

Notations: $\mathcal{F}_c(f) = \hat{f}_c, \quad \mathcal{F}_s(f) = \hat{f}_s, \quad \mathcal{F}_c^{-1}, \quad \mathcal{F}_s^{-1}$

Example 1. Fourier cosine and Fourier sine transforms

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a. \end{cases}$$

Solution.

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \cos \omega x dx = \sqrt{\frac{2}{\pi}} k \left(\frac{\sin a\omega}{\omega} \right)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin \omega x dx = \sqrt{\frac{2}{\pi}} \cdot \left(-\frac{k}{\omega} \right) \cos \omega x |_0^a = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos a\omega}{\omega} \right)$$

Example 2. $\hat{f}_c(e^{-x})$.
Solution.

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos \omega x dx = \sqrt{\frac{2}{\pi}} \cdot \frac{e^{-x}}{1 + w^2} (-\cos \omega x + \omega \sin \omega x) |_0^\infty$$

$$= \frac{\sqrt{2/\pi}}{1 + \omega^2}$$

13.2   Properties of Fourier Cosine/Sine Transforms

- Linear operations

$$\text{(a)} \quad \hat{f}_c(af + bg) = a\hat{f}_c(f) + b\hat{f}_c(g)$$
$$\text{(b)} \quad \hat{f}_s(af + bg) = a\hat{f}_s(f) + b\hat{f}_s(g)$$

$$\because \hat{f}_c(af + bg) = \sqrt{\frac{2}{\pi}} \int_0^\infty [af(x) + bg(x)] \cos \omega x dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \omega x dx$$

$$= a\hat{f}_c(f) + b\hat{f}_c(g)$$

Theorem   (Cosine and sine transforms of derivatives)
Let $f(x)$ be continuous and absolutely integrable on the x-axis,
let $f'(x)$ be piecewise continuous on each finite interval, and
let $f(x) \to 0$ as $x \to \infty$. Then

$$\begin{array}{ll} \text{(a)} & \hat{f}_c[f'(x)] = \omega \hat{f}_s[f(x)] - \sqrt{\frac{2}{\pi}} f(0) \\ \text{(b)} & \hat{f}_s[f'(x)] = -\omega \hat{f}_c[f(x)] \end{array} \quad . \tag{7}$$

Proof.

$$\hat{f}_c[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \cos \omega x \big|_0^\infty + \omega \int_0^\infty f(x) \cdot \sin \omega x \, dx \right]$$

$$= \omega \hat{f}_s[f(x)] - \sqrt{\frac{2}{\pi}} f(0)$$

$$\hat{f}_s[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \sin \omega x \big|_0^\infty - \omega \int_0^\infty f(x) \cos \omega x \, dx \right]$$

$$= -\omega \hat{f}_c[f(x)] . \qquad \sharp$$

Similarly,

$$
\begin{array}{ll}
\text{(a)} & \hat{f}_c[f''(x)] = \omega \hat{f}_s[f'(x)] - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \hat{f}_c[f(x)] - \sqrt{\frac{2}{\pi}} f'(0) \\
\text{(b)} & \hat{f}_s[f''(x)] = -\omega \hat{f}_s[f'(x)] = -\omega^2 \hat{f}_s[f(x)] + \sqrt{\frac{2}{\pi}} \omega f(0)
\end{array}
\tag{8}
$$

Example 3. (An application of (??)) Find the Fourier cosine transform of $f(x) = e^{-ax}$, where $a > 0$.

Solution.

$$(e^{-ax})'' = a^2 \cdot e^{-ax} \quad \Longrightarrow \quad a^2 f(x) = f''(x)$$

$$a^2 \hat{f}_c(f) = \hat{f}_c(f'') = -\omega^2 \hat{f}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \hat{f}_c(f) + a\sqrt{\frac{2}{\pi}}$$

$$\Rightarrow (a^2 + \omega^2) \hat{f}_c(f) = a\sqrt{\frac{2}{\pi}}$$

$$\therefore \hat{f}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + \omega^2} \right) \qquad (a > 0)$$

13.3   Fourier Transform: Fourier Integral in the Complex Form

Recall: real Fourier integral

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

$$\text{where} \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos\omega v\, dv,$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin\omega v\, dv.$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v)[\cos\omega v\cos\omega x + \sin\omega v\sin\omega x]dvd\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(v)\cos(\omega x - \omega v)dv\right]d\omega$$

Let

$$F(\omega) = \int_{-\infty}^\infty f(v)\cos(\omega x - \omega v)dv \ : \ \text{even function},$$

then

$$f(x) = \frac{1}{\pi}\int_0^\infty F(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)\cos(\omega x - \omega v)dv\right]d\omega \tag{9}$$

Let

$$G(\omega) = \int_{-\infty}^\infty f(v)\sin(\omega x - \omega v)dv: \text{odd function},$$

so

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)\sin(\omega x - \omega v)dv\right]d\omega = 0 \tag{10}$$

(??) + i(??) with $e^{ix} = \cos x + i\sin x$ (Euler formula)

$$f(v)\cos(\omega x - \omega v) + if(v)\sin(\omega x - \omega v) = f(v)e^{i(\omega x - \omega v)}$$

Complex Fourier Integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(v)e^{i\omega(x-v)}dvd\omega \tag{11}$$

13.4   Fourier Transform and Its Inverse

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(v)e^{-i\omega v}dv\right]e^{i\omega x}d\omega$$

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<u>Fourier transform</u> of f:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx \tag{12}$$

<u>Inverse Fourier transform</u> of $\hat{f}(\omega)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x} d\omega \tag{13}$$

Existence condition for the Fourier Transform:
1. $f(x)$ is piecewise continuous on every finite interval.
2. $f(x)$ is absolutely integrable on the x-axis.

Example 1. Find the Fourier transform of

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^a ke^{-i\omega x} dx = \frac{k}{\sqrt{2\pi}} \left(\frac{e^{-i\omega a} - 1}{-iw} \right) = \frac{k(1 - e^{-ia\omega})}{i\omega\sqrt{2\pi}}$$

Example 2. Find the Fourier transform of e^{-ax^2}, where $a > 0$.

$$\begin{aligned} F(e^{-ax^2}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[-ax^2 - ix\omega] dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{a}x + \frac{i\omega}{2\sqrt{a}} \right)^2 + \left(\frac{i\omega}{2\sqrt{a}} \right)^2 \right] dx \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{4a} \right) \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{a}x + \frac{i\omega}{2\sqrt{a}} \right)^2 \right] dx \end{aligned}$$

Set

$$v = \sqrt{a}x + \frac{i\omega}{2\sqrt{a}} \qquad \Rightarrow \qquad dx = \frac{dv}{\sqrt{a}}$$

$$\therefore I = \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{a}x + \frac{i\omega}{2\sqrt{a}} \right)^2 \right] dx = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2} dv$$

i) Error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{a}} \int_0^x e^{-v^2} dv$$

$$erf(0) = 0, \qquad erf(\infty) = 1$$

$$\int_0^{\infty} e^{-v^2} dv = \frac{\sqrt{\pi}}{2} \quad \Rightarrow \quad \therefore I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\frac{\pi}{a}}$$

ii) $(u, v) \Rightarrow (r, \theta)$

$$I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$I^2 = \frac{1}{a} \int_{-\infty}^{\infty} e^{-u^2} du \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{1}{a} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)} dudv$$

$$r^2 = u^2 + v^2, \qquad dudv = rdrd\theta$$

$$\begin{aligned}
I^2 &= \frac{1}{a} \int_0^{2\pi} \int_0^{\infty} e^{-r^2} rdrd\theta = \frac{1}{a} \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} rdr \\
&= \frac{2\pi}{a} \left(-\frac{1}{2} e^{-r^2} \right) |_0^{\infty} = \frac{\pi}{a}
\end{aligned}$$

$$\therefore \hat{f}(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{4a} \right) \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$$

13.5 Physical Interpretation: Spectrum

- Spectral representation: the spectral density $\hat{f}(\omega)$ measures the intensity of $f(x)$ int the frequency interval between ω and $\omega + \Delta\omega$.

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \qquad \text{: total energy}$$

- Harmonic oscillator

$$\begin{aligned}
my'' + ky &= 0 \\
my'y'' + ky'y &= 0
\end{aligned}$$

$$\text{By integration,} \qquad \frac{1}{2}m(y')^2 + \frac{1}{2}ky^2 = E_0 = \text{constant}$$

- General solution is

$$y = a_1 \cos\omega_0 x + b_1 \sin\omega_0 x = c_1 e^{i\omega_0 x} + c_{-1} e^{-i\omega_0 x}, \qquad \omega_0^2 = \frac{k}{m}$$

where $c_1 = (a_1 - ib_1)/2$, $c_{-1} = \bar{c}_1 = (a_1 + ib_1)/2$.
- Set $A = c_1 e^{i\omega_0 x}$, $B = c_{-1} e^{-i\omega_0 x}$.

$$y = A + B, \quad \rightarrow \quad v = y' = A' + B' = i\omega_0(A - B)$$

$$\begin{aligned}
E_0 &= \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \frac{1}{2}(i\omega_0)^2(A - B)^2 + \frac{1}{2}k(A + B)^2 \\
&= \frac{1}{2}k[-(A - B)^2 + (A + B)^2] = 2kAB \\
&= 2kc_1 e^{i\omega_0 x} c_{-1} e^{-i\omega_0 x} = 2kc_1 c_{-1} = 2kc_1\bar{c}_1 = 2k|c_1|^2
\end{aligned}$$

$\therefore$ The energy is proportional to the square of the amplitude $|c_1|$.

13.6 Linearity. Fourier Transformation of Derivatives

Theorem 1 (Linearity of the Fourier transform)
The Fourier transform is a linear operation, that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b,

$$\hat{f}(af + bg) = a\hat{f}(f) + b\hat{f}(g)$$

Proof.

$$
\begin{aligned}
\hat{f}[af(x) + bg(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]e^{-i\omega x}dx \\
&= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx \\
&= a\hat{f}[f(x)] + b\hat{f}[g(x)]
\end{aligned}
$$

Theorem 2 (Fourier transform of the derivative of $f(x)$)
Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\hat{f}[f'(x)] = i\omega\hat{f}[f(x)] \tag{14}$$

Proof.

$$
\begin{aligned}
\hat{f}[f'(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}} \left[f(x)e^{-i\omega x}|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \right] \\
&\quad \text{Since } f(x) \to 0 \text{ as } |x| \to \infty, \\
&= i\omega\hat{f}[f(x)]
\end{aligned}
$$

$$\text{Similarly,} \quad \hat{f}(f'') = i\omega\hat{f}(f') = (i\omega)^2\hat{f}(f) = -\omega^2\hat{f}[f(x)]$$

Example 3. Find the Fourier transform of xe^{-x^2} from Table III, Sec. 10.11

$$
\begin{aligned}
\hat{f}(xe^{-x^2}) &= \hat{f}\left(-\frac{1}{2}(e^{-x^2})'\right) = -\frac{1}{2}\hat{f}[(e^{-x^2})'] \\
&= -\frac{1}{2}i\omega\hat{f}(e^{-x^2}) = -\frac{i\omega}{2} \cdot \frac{1}{\sqrt{2}}e^{-\omega^2/4} = -\frac{i\omega}{2\sqrt{2}}e^{-\omega^2/4}
\end{aligned}
$$

13.7 Convolution

- The convolution $f * g$ of functions f and g is defined by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp \tag{15}$$

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- The *convolution* of functions correspond to the *multiplication of their Fourier transforms.*

Theorem 3 (Convolution theorem)

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then,

$$\hat{f}(f * g) = \sqrt{2\pi}\hat{f}(f)\hat{f}(g) \tag{16}$$

Proof.

$$
\begin{aligned}
F(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x} dp dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x} dx dp
\end{aligned}
$$

- We now take $x - p = q$. $x = p + q$ and $dx = dq$.

$$
\begin{aligned}
\hat{f}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(q)e^{-i\omega(p+q)} dq dp \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-i\omega p} dp \int_{-\infty}^{\infty} g(q)e^{-i\omega q} dq \\
&= \sqrt{2\pi}\hat{f}(f)\hat{f}(g)
\end{aligned}
$$

- By taking the inverse Fourier transform of the both sides,

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{i\omega x} d\omega \tag{17}$$

Example 4. Find the inverse of the Fourier transform of

$$F(\omega) = \frac{1}{2\pi(6 + 5i\omega - \omega^2)}.$$

- Since $-\omega^2 = (i\omega)^2$,

$$F(\omega) = \frac{1}{2\pi\{6 + 5i\omega + (i\omega)^2\}} = \frac{1}{2\pi} \cdot \frac{1}{2 + i\omega} \cdot \frac{1}{3 + i\omega}.$$

- From Table III. Sec. 10.11,

$$g(x) = \begin{cases} e^{-2x} & x > 0 \\ 0 & x < 0 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} e^{-3x} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$f(x) = \int_0^x e^{-2p}e^{-3(x-p)} dp = e^{-3x} \int_0^x e^p dp = \begin{cases} e^{-2x} - e^{-3x} & x > 0 \\ 0 & x < 0 \end{cases} \quad \sharp$$