14 Review of Fourier Stuff, Partial Differential Equations

14.1 Basic Concepts of Partial Differential Equations

- order, linear, homogeneous vs nonhomogeneous
- boundary conditions (BC) : given on the boundary of the region
- initial conditions (IC) : u, $\frac{\partial u}{\partial t}, \cdots$ at time $t = 0$
- The second order differential equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F$$

- Discriminant $B^2 - AC$

$$\begin{aligned} B^2 - AC &= 0 : \quad \text{Parabolic} \\ B^2 - AC &> 0 : \quad \text{Hyperbolic} \\ B^2 - AC &< 0 : \quad \text{Elliptic} \end{aligned}$$

Example 1. Important linear partial differential equations of the second order

- 1-dimensional wave equation

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

  Discriminant: $B^2 - AC = 0 - 1 \cdot (-c^2) = c^2 > 0$: hyperbolic

- 1-dimensional heat equation

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2}$$

  Discriminant: $B^2 - AC = 0 - 0 \cdot c^2 = 0$: parabolic

- 2-dimensional Laplace equation (steady state heat equation with no heat generation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

  Discriminant: $B^2 - AC = 0 - 1 \cdot 1 = -1$: elliptic

- 2-dimensional Poisson equation (steady state heat equation with heat generation)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

- 2-dimensional wave equation

$$\frac{\partial^2 u}{\partial^2 t} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

- 3- dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Theorem 1 Fundamental Theorem (Superposition or linearity principle)
If u_1 and u_2 are any solutions of a linear homogeneous partial differential equation in some region R, then

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 and c_2 is also a solution of that equation in R.

Example 2. Find a solution $u(x, y)$ of the partial differential equation

$$u_{xx} - u = 0$$

Solution. $u(x, y) = A(y) \cdot e^x + B(y) \cdot e^{-x}$

Example 3. Solve the partial differential equation

$$u_{xy} = -u_x$$

Solution.

$$u_x = P, \quad \Rightarrow \quad P_y = -P$$

$$\frac{P_y}{P} = -1 \quad \Longrightarrow \quad \ln P = -y + \tilde{c}(x)$$

$$\therefore P = c(x) \cdot e^{-y}$$

$$u(x, y) = f(x)e^{-y} + g(y) \quad \text{where} \quad f(x) = \int c(x)dx$$

14.2 Modeling; Vibrating String; Wave Equation

Physical Assumptions

1. Homogeneous, perfectly elastic string.
2. Gravitational force is negligible, compared to the tension.
3. String performs small transverse motions in a vertical plane.

Derivation of the Differential Equation

Force balance

$$x : \quad T_1 \cos\alpha = T_2 \cos\beta = T = \text{const}$$

$$y: \quad T_2 \cdot \cos\beta - T_1 \cdot \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\bigg|_x \qquad \text{and} \qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x}$$

- By Taylor expansion around x,

$$\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} = \left(\frac{\partial u}{\partial x}\right)\bigg|_x + \frac{\partial^2 u}{\partial x^2}\bigg|_x \Delta x + \cdots$$

$$\left[\left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\bigg|_x\right] = \left[\frac{\partial u}{\partial x}\bigg|_x + \frac{\partial^2 u}{\partial x^2}\bigg|_x \Delta x - \frac{\partial u}{\partial x}\bigg|_x\right] = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \cdot \frac{\partial^2 u}{\partial x^2}$$

- One-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \left(c^2 = \frac{T}{\rho}\right)$$

- Unit of c

$$[c^2] = \left[\frac{T}{\rho}\right] = \left[\frac{\text{kg} \cdot \text{m/s}^2}{\text{kg/m}}\right] = [\text{m/s}]^2$$

$$\therefore [c] = [\text{m/s}] \qquad \text{(propagation velocity)}$$