16 Heat Equation

16.1 Derivation

- Energy conservation in the control volume

$$\dot{E}_{\text{st}} = \dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_{\text{gen}}$$

where
$\dot{E}_{\text{st}}$: Energy stored in the control volume
$\dot{E}_{\text{in}}$: Energy entering in the control volume through the surfaces
$\dot{E}_{\text{out}}$: Energy exiting out of the control volume through the surfaces
$\dot{E}_{\text{gen}}$: Energy generated inside the control volume

$$\dot{E}_{\text{st}} = \frac{\partial}{\partial t}(dm \cdot c_p T) = \rho c_p dV \frac{\partial T}{\partial t} = \rho c_p \frac{\partial T}{\partial t} dxdydz$$

ρ: density (kg/m^3), c_p: heat capacity (J/kg·K)

$$\dot{E}_{\text{in}} = \dot{q}_x dydz + \dot{q}_y dzdx + \dot{q}_z dxdy$$

$$\dot{E}_{\text{out}} = \dot{q}_{x+dx} dydz + \dot{q}_{y+dy} dzdx + \dot{q}_{z+dz} dxdy$$

Fourier's law of heat conduction

$$\dot{q}_x = -k\frac{\partial T}{\partial x}, \qquad \dot{q}_y = -k\frac{\partial T}{\partial y}, \qquad \dot{q}_z = -k\frac{\partial T}{\partial z}$$

k: thermal conductivity (W/m·K)
By Taylor series expansion, ignoring higher order terms

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{\partial \dot{q}_x}{\partial x}dx \quad \dot{q}_{y+dy} = \dot{q}_y + \frac{\partial \dot{q}_y}{\partial y}dy \quad \dot{q}_{z+dz} = \dot{q}_z + \frac{\partial \dot{q}_z}{\partial z}dz$$

$$\begin{aligned}
\dot{E}_{\text{in}} - \dot{E}_{\text{out}} &= (\dot{q}_x dydz + \dot{q}_y dzdx + \dot{q}_z dxdy) \\
&- \left(\dot{q}_x dydz + \frac{\partial \dot{q}_x}{\partial x}dxdydz + \dot{q}_y dzdx + \frac{\partial \dot{q}_y}{\partial y}dydzdx + \dot{q}_z dxdy + \frac{\partial \dot{q}_z}{\partial z}dzdxdy\right) \\
&= -\left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right)dxdydz \\
&= \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]dxdydz
\end{aligned}$$

$$\dot{E}_{\text{gen}} = \dot{g}\,dxdydz \qquad \dot{g}: \text{ volumetric heat generation (W/m}^3\text{)}$$

$$\rho c_p \frac{\partial T}{\partial t} dxdydz = \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dxdydz + \dot{g}dxdydz$$

For $k = k(x, y, z)$,

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g}$$

For k =constant,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{g} = \rho c_p \frac{\partial T}{\partial t}$$

For steady-state,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{g} = 0$$

No heat generation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

$$\therefore \nabla^2 T = 0$$

- Transient heat conduction equation with no heat generation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T, \qquad \alpha = \frac{k}{c\rho} : \text{thermal diffusivity } (\mathrm{m^2/s})$$

16.2 One-dimensional heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

- Boundary conditions

$$T(0, t) = 0, \qquad T(L, t) = 0 \qquad \text{for all } t \tag{2}$$

- Initial condition

$$T(x, 0) = f(x) \tag{3}$$

- **Step I:** Two ODEs

$$\begin{aligned} T(x,t) &= F(x) \cdot G(t) \\ \Rightarrow &\quad F\dot{G} = \alpha F'' G \\ \Rightarrow &\quad \frac{\dot{G}}{\alpha G} = \frac{F''}{F} = -p^2 \; = \text{constant} \\ &\quad F'' + p^2 F = 0 \qquad (4) \\ &\quad \dot{G} + \alpha p^2 G = 0 \qquad (5) \end{aligned}$$

- **Step II:** Satisfying BCs

- A general solution of (**??**): $F(x) = A\cos px + B\sin px$
  From the BC (**??**), $T(0,t) = F(0)G(t) = 0$, and $T(L,t) = F(L)G(t) = 0$.
  Since $G \equiv 0$ would give a trivial solution $u \equiv 0$, we get $F(0) = F(L) = 0$.

$$\begin{aligned} F(0) &= A = 0 \\ F(L) &= B\sin pL = 0 \quad \Rightarrow \quad \sin pL = 0 \quad \Rightarrow \quad p = \frac{n\pi}{L}, \qquad n = 1,\ 2,\ 3,\ \cdots \end{aligned}$$

  Setting $B = 1$,

$$F_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1,\ 2,\ 3,\ \cdots$$

- From (**??**)

$$\dot{G}_n + \lambda_n^2 G_n = 0 \quad \text{where} \quad \lambda_n = \alpha p_n = \sqrt{\alpha}\frac{n\pi}{L}.$$

  General solution

$$G_n(t) = B_n e^{-\alpha p_n^2 t} \qquad n = 1,\ 2,\ \cdots$$

Therefore,

$$T_n(x,t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\alpha p_n^2 t} \qquad (n = 1,\ 2,\ \cdots) \tag{6}$$

Eigenfunctions: $B_n \sin\frac{n\pi x}{L} e^{-\alpha p_n^2 t}$
Eigenvalue: $\lambda_n = \sqrt{\alpha}\cdot p_n = \sqrt{\alpha}\frac{n\pi}{L}$

- **Step III:**Solution of the Entire Problem

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\alpha p_n^2 t}, \qquad \left(p_n = \frac{n\pi}{L}\right) \tag{7}$$

- From (**??**)

$$T(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} dx = f(x)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L} dx \qquad (n = 1,\ 2,\ \cdots) \quad \sharp \tag{8}$$

Example 1. Sinusoidal initial temperature
ρ=8.92 g/cm^3, c=0.092 cal/(g °C), k=0.95 cal/(cm sec °C), L=80 cm.

$$T(x,0) = f(x) = 100\cdot\sin\frac{\pi x}{80} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

By orthogonality of the trigonometric functions,

$$B_1 = 100, \quad B_2 = B_3 = \cdots = 0$$

$$p_1 = \frac{\pi}{L}, \qquad \alpha = \frac{k}{\rho c} = \frac{0.95}{8.92 \times 0.092} = 1.158 \text{ cm}^2/\text{sec}$$

$$T(x,t) = 100 \cdot \sin\frac{\pi x}{80} \cdot e^{-1.158\frac{\pi^2}{L^2}t}$$

At $x = L/2$=40 cm,

$$100 \cdot e^{-1.158\frac{\pi^2}{80^2}t} = 50$$

$$t = \frac{\ln 0.5}{-1.158\frac{\pi^2}{80^2}t} = 388 \text{ seconds}$$

Example 2. Speed of decay

$$T(x,0) = f(x) = 100 \cdot \sin\frac{3\pi x}{80}$$

$$\therefore T(x,t) = 100 \cdot \sin\frac{3\pi x}{80} e^{-1.158\cdot\frac{3^2\pi^2}{80^2}t}$$

At $x = 40/3$ cm,

$$50 = 100 \cdot \sin\frac{3\pi x}{80} e^{-10.422\frac{\pi^2}{80^2}}$$

$$t = \frac{\ln 0.5}{-10.422\frac{\pi^2}{80^2}} \approx 43 \text{ seconds}$$

Example 3. Triangular initial temperature in a bar

$$T(x,0) = f(x) = \begin{cases} x & \text{if } \ 0 < x < L/2 \\ L-x & \text{if } \ L/2 < x < L \end{cases}$$

$$\begin{aligned}
B_n &= \frac{2}{L}\left[\int_0^{L/2} x \sin\frac{n\pi x}{L} dx + \int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L} dx\right] \\
&= \frac{2}{L}\left[-\frac{L}{n\pi} x \cos n\pi x L\Big|_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2} \cos\frac{n\pi x}{L}\right] \\
&+ \frac{2}{L}\left[-\frac{L}{n\pi}(L-x)\cos\frac{n\pi x}{L}\Big|_{L/2}^{L} - \frac{L}{n\pi}\int_{L/2}^{L}\cos\frac{n\pi x}{L}dx\right] \\
&= \frac{2}{L}\left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\Big|_0^{L/2} + \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L}\right] \\
&= \frac{2L}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \sin n\pi + \sin\frac{n\pi}{2}\right) = \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}$$

If n is even, $B_n = 0$.
Otherwise n is odd,

$$B_n = \frac{4L}{n^2\pi^2} \quad (n = 1,\ 5,\ 9,\ \cdots) \quad \text{and} \quad B_n = -\frac{4L}{n^2\pi^2} \quad (n = 3,\ 7,\ 11,\ \cdots)$$

$$T(x,t) = \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}\exp\left\{-\alpha\left(\frac{\pi}{L}\right)^2 t\right\} - \frac{1}{9}\sin\frac{3\pi x}{L}\exp\left\{-\alpha\left(\frac{3\pi}{L}\right)^2 t\right\} + - \cdots\right]$$

Example 4. Bar with insulated ends. Eigenvalue 0

- Governing equation

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} = 0$$

- Boundary conditions

$$T_x(0,t) = 0, \qquad T_x(L,t) = 0$$

- Initial condition

$$T(x,0) = f(x)$$

- Separation of variables

$$T(x,t) = F(x)\cdot G(t)$$
$$\dot{G}F = \alpha G F''$$
$$\tfrac{\dot{G}}{\alpha G} = \tfrac{F''}{F} = -p^2$$

$$F'' + p^2 F = 0$$
$$\dot{G} + \alpha p^2 G$$

$$F(x) = A\cos px + B\sin px$$
$$T_x(0,t) = F'(0)G(t) = 0, \qquad T_x(L,t) = F'(L)G(t) = 0$$
$$\therefore F'(0) = F'(L) = 0$$
$$F'(x) = -Ap\sin px + Bp\cos px$$

$$F'(0) = Bp = 0, \quad \Rightarrow \quad B = 0$$
$$F'(L) = -Ap\sin pL = 0, \quad \Rightarrow \quad \sin pL = 0$$
$$\therefore p = p_n = \frac{n\pi}{L} \qquad (n = 0,\ 1,\ 2,\ \cdots)$$
$$F_n(x) = \cos\frac{n\pi x}{L} \qquad (n = 0,\ 1,\ 2,\ \cdots)$$
$$G_n(t) = A_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

- Eigenfunctions

$$T_n(x,t) = F_n(x)G_n(t) = A_n\cos\frac{n\pi x}{L}e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \qquad (n = 0,\ 1,\ 2,\ \cdots)$$

- Eigenvalues

$$\lambda_n = \sqrt{\alpha}\frac{n\pi}{L}$$

$$T(x,t)=\sum_{n=0}^{\infty}T_n(x,t)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}$$

- With the initial condition

$$T(x,0)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}=f(x)$$

$$A_0=\frac{1}{L}\int_0^L f(x)dx, \qquad A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad (n=0,\ 1,\ 2,\ \cdots)$$

Example 5. Triangular initial temperature in a bar with insulated ends

$$T(x,0)=f(x)=\begin{cases} x & \text{if } \ 0<x<L/2 \\ L-x & \text{if } \ L/2<x<L \end{cases}$$

$$\begin{aligned}
A_0 &= \frac{1}{L}\int_0^L f(x)dx=\frac{1}{L}\cdot\frac{1}{2}L\cdot\frac{L}{2}=\frac{L}{4} \\
A_n &= \frac{2}{L}\left[\int_0^{L/2}x\cos\frac{n\pi x}{L}dx+\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right] \\
&= \frac{2}{L}\left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\Big|_0^{L/2}-\frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi x}{L}dx\right] \\
&\quad +\frac{2}{L}\left[\frac{L}{n\pi}(L-x)\sin\frac{n\pi x}{L}\Big|_{L/2}^{L}+\frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi x}{L}dx\right] \\
&= \frac{2}{L}\left[\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}+\left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi x}{L}\Big|_0^{L/2}-\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}-\left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}\right] \\
&= \frac{2L}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1+\cos\frac{n\pi}{2}-\cos n\pi\right)
\end{aligned}$$

$$\therefore A_n=\frac{2L}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}-\cos n\pi-1\right)$$

- If n is odd, $A_n=0$.
- If $n=2,\ 6,\ 10,\cdots$,

$$A_n=-\frac{8L}{n^2\pi^2}$$

- If $n=4,\ 8,\ 12,\cdots$, $A_n=0$.

$$T(x,t)=\frac{L}{4}-\frac{8L}{\pi^2}\left\{\frac{1}{2^2}\cos\frac{2\pi x}{L}\exp\left[-\alpha\left(\frac{2\pi}{L}\right)^2 t\right]+\frac{1}{6^2}\cos\frac{6\pi x}{L}\exp\left[-\alpha\left(\frac{6\pi}{L}\right)^2 t\right]+\cdots\right\}$$

16.3 Steady-State Two-Dimensional Heat Flow

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

- If the heat flow is steady, then $\partial T / \partial t = 0$.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{9}$$

- Boundary value problem:
Dirichlet problem if T is prescribed on C,
Neumann problem if the normal derivative $T_n = \partial T / \partial n$ is prescribed on C,
Mixed problem if T is prescribed on a portion of C and T_n on the rest of C.

16.3.1 Dirichlet Problem in a Rectangle R

- Governing equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} = 0$$

- Boundary conditions

$$\begin{aligned} &T(0, y) = 0, \qquad && T(a, y) = 0 \\ &T(x, 0) = 0, \qquad && T(x, b) = f(x) \end{aligned}$$

- Separation of variables

$$T(x, y) = F(x) \cdot G(y)$$

$$\frac{1}{F} \cdot \frac{d^2 F}{dx^2} = -\frac{1}{G} \cdot \frac{d^2 G}{dy^2} = -k = -p^2$$

$$\frac{d^2 F}{dx^2} + p^2 F = 0$$

- Boundary conditions

$$\begin{aligned} T(0, y) = F(0) G(y) = 0, \quad &\Rightarrow \quad F(0) = 0 \\ T(a, y) = F(a) G(y) = 0, \quad &\Rightarrow \quad F(a) = 0 \end{aligned}$$

$$F(x) = A \cos px + B \sin px$$

$$F(0) = A = 0 \quad \text{and} \quad F(a) = B \sin pa = 0$$

$$\therefore\ p_n = \frac{n\pi}{a}, \quad F_n(x) = \sin \frac{n\pi x}{a} \qquad (n = 1,\ 2,\ \cdots)$$

$$\frac{d^2 G_n}{dy^2} - \left(\frac{n\pi}{a} \right)^2 G_n = 0$$

$$\therefore\ G_n(y) = A_n \sinh\frac{n\pi y}{a} + B_n \cosh\frac{n\pi y}{a}$$
$$T_n(x,0) = 0 = F_n(x)G_n(0) = 0 \quad \Rightarrow \quad G_n(0) = 0$$
$$G_n(0) = B_n = 0$$
$$\therefore\ G_n(y) = A_n \sinh\frac{n\pi y}{a}$$

- Eigenfunctions

$$T_n(x,y) = F_n(x)\cdot G_n(y) = A_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

- Last boundary condition to obtain A_n

$$T(x,y) = \sum_{n=1}^{\infty} T_n(x,y)$$

$$T(x,b) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

- By using the orthogonality of the trigonometric functions,

$$A_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

$$T(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

where

$$A_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

16.4 Solution by Fourier Integrals and Transforms

- Heat equation in a bar that extends to infinity on both sides (and is laterally insulated, as before):

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \tag{10}$$

  - No Boundary Conditions, only the initial condition:

$$T(x,0) = f(x) \qquad (-\infty < x < \infty) \tag{11}$$

  - Substituting $T(x,t) = F(x)G(t)$ into (1),

$$\begin{aligned} F'' + p^2 F &= 0 \\ \dot{G} + \alpha p^2 G &= 0 \end{aligned}$$
$$F(x) = A\cos px + B\sin px \qquad \text{and} \qquad G(t) = e^{-\alpha p^2 t}$$

  - A solution of (**??**) is

$$T(x,t;p) = FG = (A\cos px + B\sin px)e^{-\alpha p^2 t}. \tag{12}$$

- Since $f(x)$ in (**??**) is not assumed to be periodic, it is natural to use **Fourier integrals** instead of Fourier series.

$$T(x,t) = \int_0^\infty T(x,t;p)dp = \int_0^\infty [A(p)\cos px + B(p)\sin px]e^{-\alpha p^2 t}dp \tag{13}$$

Determination of $A(p)$ and $B(p)$ from the Initial Condition:

$$T(x,0) = \int_0^\infty [A(p)\cos px + B(p)\sin px]dp = f(x) \tag{14}$$

$$A(p) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos pv dv, \qquad B(p) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin pv dv.$$

Fourier integral (**??**) with $A(p)$ and $B(p)$

$$\begin{aligned} T(x,0) &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty (f(v)\cos pv\cos px + f(v)\sin pv \sin px)dv\right] dp \\ &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(v)\cos(px - pv)dv\right] dp \end{aligned}$$

- Similarly, (**??**) becomes

$$\begin{aligned} T(x,t) &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(v)\cos(px - pv)e^{-\alpha p^2 t}dv\right] dp \\ &= \frac{1}{\pi}\int_{-\infty}^\infty f(v)\left[\int_0^\infty e^{-\alpha p^2 t}\cos(px - pv)dp\right] dv \end{aligned} \tag{15}$$

Evaluating the inner integral by the formula

$$\int_0^\infty e^{-s^2}\cos 2bs\, ds = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

Choose $p = s/\sqrt{\alpha t}$ as a new variable of integration and set

$$b = \frac{(x-v)}{2\sqrt{\alpha t}} \quad \Rightarrow \quad 2bs = (x-v)p \quad \text{and} \quad ds = \sqrt{\alpha t}dp$$

$$\begin{aligned} \int_0^\infty e^{-\alpha p^2 t}\cos(px - pv)dp &= \frac{1}{\sqrt{\alpha t}}\int_0^\infty e^{-s^2}\cos 2bs\, ds = \frac{\sqrt{\pi}}{2\sqrt{\alpha t}}e^{-b^2} \\ &= \frac{1}{2}\left(\frac{\pi}{\alpha t}\right)^{1/2}\exp\left\{-\frac{(x-v)^2}{4\alpha t}\right\} \end{aligned}$$

By inserting this result into (**??**),

$$T(x,t) = \frac{1}{2\sqrt{\pi\alpha t}}\int_{-\infty}^\infty f(v)\exp\left\{-\frac{(x-v)^2}{4\alpha t}\right\} dv$$

Taking $z = (v-x)/2\sqrt{\alpha t}$,

$$v = x + 2z\sqrt{\alpha t} \quad \Longrightarrow \quad dv = 2\sqrt{\alpha t}dz$$

$$T(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2z\sqrt{\alpha t}) e^{-z^2} dz. \tag{16}$$

Example 1. Temperature an infinite bar

$$f(x) = \begin{cases} T_0 = \text{const} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

- From (**??**)

$$T(x,t) = \frac{T_0}{2\sqrt{\pi \alpha t}} \int_{-1}^{1} \exp\left\{-\frac{(x-v)^2}{4\alpha t}\right\} dv$$

- Introducing $z = (v-x)/2\sqrt{\alpha t}$,

$$T(x,t) = \frac{T_0}{\sqrt{\pi}} \int_{(-1-x)/2\sqrt{\alpha t}}^{(1-x)/2\sqrt{\alpha t}} e^{-z^2} dz \tag{17}$$

16.5 Use of Fourier Transforms

Example 2. Temperature in the infinite bar in Example 1.

$$f(x) = \begin{cases} T_0 = \text{const} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

- Let $\hat{T} = \mathcal{F}(T)$ denote the Fourier transform of T, regarded as a function of x.

$$\begin{aligned} \mathcal{F}(T_t) &= \alpha \mathcal{F}(T_{xx}) = \alpha(-\omega^2)\mathcal{F}(T) = -\alpha\omega^2 \hat{T} \\ \mathcal{F}(T_t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T_t e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} T e^{-i\omega x} dx = \frac{\partial \hat{T}}{\partial t} \\ \Rightarrow \qquad & \frac{\partial \hat{T}}{\partial t} = -\alpha\omega^2 \hat{T} \\ \therefore \qquad & \hat{T}(\omega, t) = C(\omega) e^{-\alpha\omega^2 t} \end{aligned}$$

- Initial condition : $\hat{T}(\omega, 0) = C(\omega) = \hat{f}(\omega) = \mathcal{F}(f)$

$$\hat{T}(\omega, t) = \hat{f}(\omega) e^{-\alpha\omega^2 t}$$

$$T(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-\alpha\omega^2 t} e^{i\omega x} d\omega \tag{18}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-iv\omega} dv$$

$$T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{-\alpha\omega^2 t} e^{i\omega(x-v)} d\omega\right] dv$$

$$e^{-\alpha\omega^2 t} e^{i\omega(x-v)} = e^{-\alpha\omega^2 t}\cos(\omega x - \omega v) + i e^{-\alpha\omega^2 t}\sin(\omega x - \omega v)$$

- Sine is an odd function:

$$i\int_{-\infty}^{\infty} e^{-\alpha\omega^2 t}\sin\omega(x-v) d\omega = 0$$

$$\therefore\ T(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{0}^{\infty} e^{-\alpha\omega^2 t}\cos\omega(x-v) d\omega\right] dv$$

Example 3. Solution in Example 1 by the method of convolution
The beginning is as in Example 2 and leads to (**??**):

$$\begin{aligned} T(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{-\alpha\omega^2 t} e^{i\omega x} d\omega \\ &= \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) e^{i\omega x} d\omega \ = \ \mathcal{F}^{-1}(\hat{f}(\omega)\hat{g}(\omega)) \\ \text{where} \qquad & \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\alpha\omega^2 t} \end{aligned}$$

Revoking the convolution integral in Fourier transform

$$T(x,t) = (f*g)(x) \ = \int_{-\infty}^{\infty} f(p) g(x-p) dp \tag{19}$$

Use

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$$

with $\alpha t = 1/4a$ or $a = 1/4\alpha t$, then

$$F(e^{-x^2/4\alpha t}) = \sqrt{2\alpha t} e^{-\alpha\omega^2 t} = \sqrt{2\alpha t}\sqrt{2\pi}\hat{g}(\omega)$$

Hence $\hat{g}$ has the inverse

$$g(x) = \frac{1}{\sqrt{2\alpha t}\sqrt{2\pi}} e^{-x^2/4\alpha t}$$

Replacing x with $x - p$ and substituting this into (**??**),

$$T(x,t) = (f*g)(x) = \frac{1}{2\sqrt{\pi\alpha t}}\int_{-\infty}^{\infty} f(p)\exp\left\{-\frac{(x-p)^2}{4\alpha t}\right\} dp.$$

Example 4. (Fourier sine transform applied to the heat equation) A laterally insulated bar extends from $x = 0$ to infinity.
- Initial condition: $T(x,0) = f(x)$.
- Boundary condition: $T(0,t) = 0 \quad \Rightarrow \quad f(0) = 0$.

$$\begin{aligned} \mathcal{F}_s(T_t) &= \frac{\partial \hat{T}_s}{\partial t} = \alpha\mathcal{F}_s(T_{xx}) = -\alpha\omega^2\mathcal{F}_s(T) = -\alpha\omega^2\hat{T}_s(\omega,t) \\ \therefore \qquad & \hat{T}_s(\omega,t) = C(\omega) e^{-\alpha\omega^2 t} \end{aligned}$$

$$
\begin{aligned}
T(x,0) &= f(x) \quad \Rightarrow \quad \hat{T}_s(\omega,0) = \hat{f}_s(\omega) = C(\omega) \\
\therefore & \qquad \hat{T}_s(\omega,t) = \hat{f}_s(\omega) e^{-\alpha\omega^2 t}
\end{aligned}
$$

- Taking the inverse Fourier sine transform and substituting

$$
\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p) \sin \omega p \; dp \ ,
$$

$$
T(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p) e^{-\alpha\omega^2 t} \sin \omega p \ \sin \omega x \; dp \; d\omega
$$