# 21 Analyticity

## 21.1 Limit, Continuity, Analyticity

**Limit** : $\lim_{z \to z_0} f(z) = l$
if $f$ is defined in a nbhd of $z_0$ and if for every $\varepsilon > 0$ we can fund $\delta > 0$ s.t. for all $z \neq z_0$, $|z - z_0| < \delta$, we have $|f(z) - l| < \varepsilon$.

**Continuity** : $\lim_{z \to z_0} f(z) = f(z_0)$
a function $f(z)$ is said to be continuous at $z = z_0$.

**Derivative.**

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\Delta z = z - z_0, z = z_0 + \Delta z$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

***along whatever path*** $z$ approaches $z_0$ the value approaches the same value.

**Ex** Differentiability, Derivative.

$$f(z) = z^2$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Differentiation rules:

$$(cf)' = cf', \quad (f + g)' = f' + g', \quad (fg)' = f'g + fg'$$

$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$

$$(z^n)' = nz^{n-1} \text{ (n : integer)}$$

**Ex.** $\bar{z}$ not differentiable.

$$f(z) = \bar{z} = x - iy.$$

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{(z + \Delta z)} - z}{\Delta z} = \frac{\Delta \bar{z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \qquad \begin{array}{l} \Delta x \to 0 : -1. \\ \Delta y \to 0 : +1. \end{array}$$

**Definition (Analyticity).**
A function $f(z)$ is said to be **analytic** in a domain D if $f(z)$ is defined and **differentiable**

**at all points** of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of $z_0$. Also, by an analytic function we mean a function that is analytic in some domain.

**Ex** Polynomials, rational functions.
polynomials $\quad 1, z, z^2, \cdots$
quotient of two polynomials $g(z)$ , $h(z)$

$$f(z) = \frac{g(z)}{h(z)} : \text{rational function.}$$

## 21.2 Cauchy-Riemann Equations

$$w = f(z) = u(x, y) + iv(x, y)$$

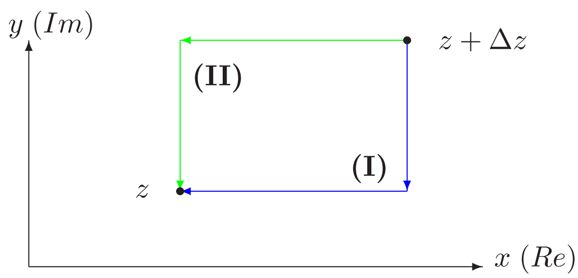
Cauchy-Riemann equations:

$$u_x = v_y, \quad u_y = -v_y \tag{2}$$

$f$ is analytic in a domain D if and only if the first partial derivatives of $u, v$ satisfy the two so-called Cauchy-Riemann equations everywhere in D.

**Theorem 1** (Cauchy-Riemann equations).
Let $f(z) = u(x, y) + iu(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at $z$ itself. Then at that point, the first-order partial derivatives of $u$ and $v$ exist and satisfy the Cauchy-Riemann equations (2). Hence if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (2) at all points of D.
**Proof.**

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad \Delta z = \Delta x + i\Delta y. & (3) \\ &= \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y} & (4) \end{aligned}$$



i) path I. After $\Delta y = 0$ then $\Delta z = \Delta x$.

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \\ &= u_x + i v_x \end{aligned} \tag{5}$$

ii) Path II. After $\Delta x = 0$ then $\Delta z = i\Delta y$.

$$\begin{aligned} f'(z) &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+y+\Delta y) - v(x,y)}{i\Delta y} \\ &= -i u_y + v_y \end{aligned} \tag{6}$$

(5)=(6) $\qquad \therefore u_x = v_y$ and $v_x = -u_y$.

**Theorem 2** (Cauchy-Riemann equations: not only necessary, but also sufficient)
If two real-valued continuous functions $u(x,y)$ and $v(x,y)$ of two real variables $x$ and $y$ have continuous derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in D.

**Example .** An analytic function of constant absolute value is constant. Show that if $f(z)$ is analytic in a domain D and $|f(z)| = k$ =constant in D, then $f(z)$ =const in D.
**Solution.** $u^2 + v^2 = k^2$.

$$\Rightarrow \quad uu_x + vv_x = 0, \; uu_y + vv_y = 0$$

Cauchy-Riemann equations: $v_x = -u_y$, & $v_y = u_x$

$$u \cdot u_x - v \cdot u_y = 0 \qquad \text{(A)} \tag{7}$$
$$u \cdot u_y - v \cdot u_x = 0 \qquad \text{(B)} \tag{8}$$

(A)$\cdot u$+(B)$\cdot v$ : $(u^2 + v^2) \cdot u_x = 0$.
(A)$\cdot(-v)$+(B)$\cdot u$ : $(u^2 + v^2) \cdot u_y = 0$.
i) if $k^2 = u^2 + v^2 = 0$ then $u = v = 0 \to f = 0$.
ii) $u_x = u_y = 0$ by C-R eq. $v_x = v_y = 0 \quad \Rightarrow u = v = \text{const} \to f = \text{const}$. ♯

**Polar form.** $z = r(\cos\theta + i\sin\theta)$

$$f(z) = u(r,\theta) + iv(r,\theta)$$
$$x = r\cos\theta, y = r\sin\theta. \to r = \sqrt{x^2+y^2}, \theta = \arctan\frac{y}{x}$$
$$r_x = \frac{x}{(x^2+y^2)^1/2} = \frac{x}{r}, \qquad r_y = \frac{y}{(x^2+y^2)^1/2} = \frac{y}{r}$$
$$\theta_x = \frac{1}{1+(y/x)^2}(-\frac{y}{x^2}) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2}$$

$$\theta_y = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$u_x = u_r \cdot r_x + u_\theta \cdot \theta_x = \frac{x}{r} \cdot u_r - \frac{y}{r^2} u_\theta.$$

$$u_y = v_r \cdot r_y + v_\theta \cdot \theta_y = \frac{y}{r} \cdot v_r + \frac{x}{r^2} v_\theta.$$

Since $u_x = v_y$ $\qquad rx \cdot u_r - yu_\theta = ryv_r + xv_\theta \cdots$ (A)

$$u_y = u_r \cdot r_y + u_\theta \cdot \theta_y = \frac{y}{r} u_r + \frac{x}{r^2} u_\theta.$$

$$-v_x = -v_r \cdot r_x - v_\theta \cdot \theta_x = -\frac{x}{r} \cdot v_r + \frac{y}{r^2} v_\theta.$$

Since $u_y = -v_x$ $\qquad r_y u_r + xu_\theta = -rxv_r + yv_\theta \cdots$ (B)
(A)$\times x+$ (B)$\times y$ : $u_r \cdot r \cdot (x^2+y^2) = v_\theta(x^2+y^2) \quad \Rightarrow u_r = \frac{1}{r} v_\theta$
(A)$\times y-$(B)$\times x$ : $-(y^2+x^2)u_\theta = r(y^2+x^2)v_r \quad \Rightarrow v_r = -\frac{1}{r} u_\theta.$

$$\therefore\ u_r = \frac{1}{r} v_\theta, v_r = -\frac{1}{r} u_\theta \qquad (r > 0)$$

## 21.3 Laplace's Equation. Harmonic Functions. $\Rightarrow$ (Solution of Laplace's Equation)

**Theorem 3** (Laplace's equation)
If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then $u$ and $v$ satisfy Laplace's equation.

$$\begin{aligned} \nabla^2 u &= u_{xx} + u_{yy} = 0. \\ \text{and} \quad \nabla^2 v &= v_{xx} + v_{yy} = 0. \end{aligned}$$

respectively, in D and have continuous second partial derivatives in D.
**Proof.**

i) $\left.\begin{array}{ll} u_x = v_y & \Rightarrow u_{xx} = v_{yx} \\ u_y = -v_x & \Rightarrow u_{yy} = -v_{yx} \end{array}\right) \Rightarrow u_{xx} + u_{yy} = 0$

ii) $\left.\begin{array}{ll} u_x = v_y & \Rightarrow u_{xy} = v_{yy} \\ u_y = -v_x & \Rightarrow u_{yx} = -v_{xx} \end{array}\right) \Rightarrow v_{xx} + v_{yy} = 0$ ♯

**Remark.**

- Solns of Laplace eqn with conti 2nd order partial derivatives are called **harmonic** ftns, and their theory is called potential theory.
- Real and imaginary parts of an analytic ftn are harmonic ftns.

- If two harmonic functions $u$ and $v$ satisfy the Cauchy-Riemann equations in a domain D, they are the real and imaginary parts of an analytic function $f$ in D. Then $v$ is said to be a conjugate harmonic function of $u$ in D.

**Example .** How to find a conjugate harmonic function by the Cauchy-Riemann equations.

$$u = x^2 - y^2 - y$$

sol) $u_x = 2x, u_{xx} = 2, u_y = -2y - 1, u_{yy} = -2.$

$$u_{xx} + u_{yy} = 2 - 2 = 0 \qquad \therefore \text{harmonic function.}$$

By C-R. $v_y = u_x = 2x, v_x = -u_y = 2y + 1.$

$$v = 2xy + h(x) \Rightarrow v_x = 2y + dh/dx.$$

$$\therefore dh/dx = 1 \Rightarrow h(x) = x + c$$

$$v = 2xy + x + c.$$