Chapter 2. Stresses, Strains & Effective Stresses

Key words : Stress, Strain, Stress-strain relationship, Subsurface stress distribution, Effective stress

1. Stresses ?

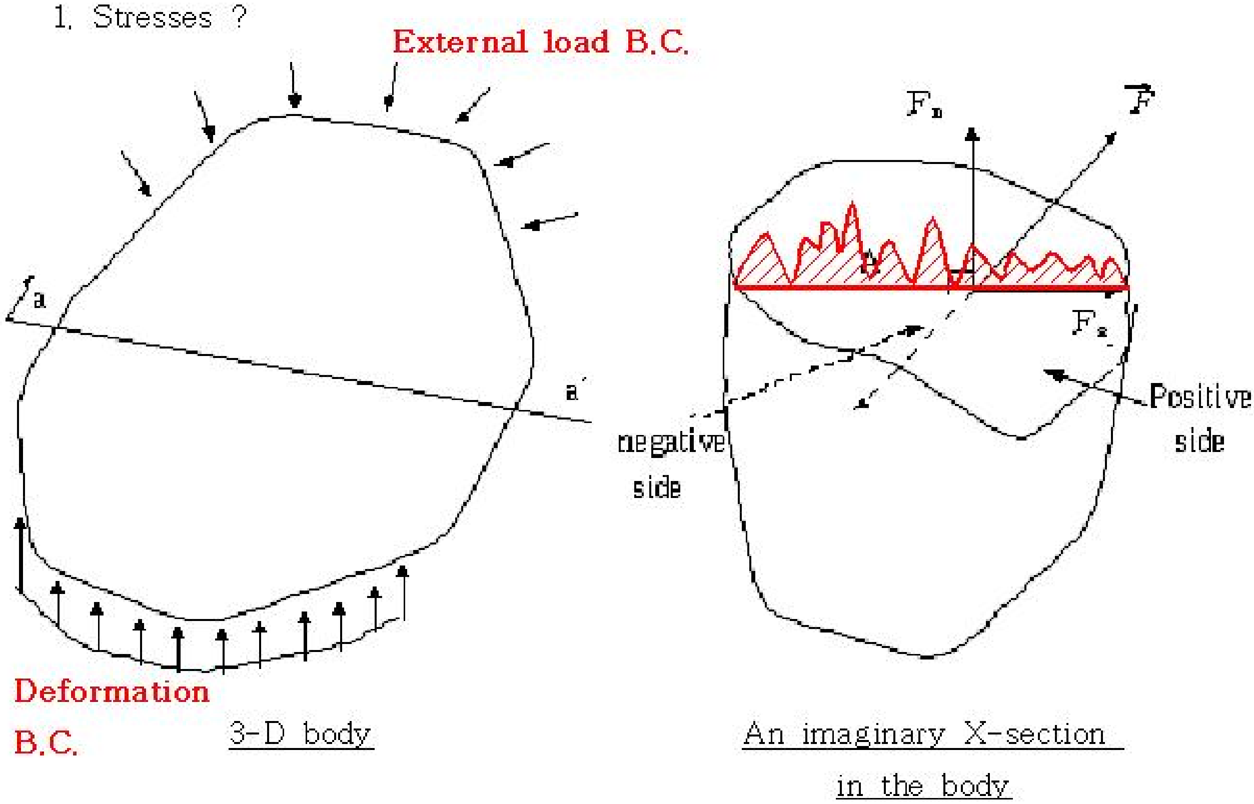


Fig.1

○ F : the force transmitted thru the area A ← **Internal force**

∵ **reaction exerted within the body**

○ The magnitude of the average forces per unit area are :

F_n / A : **the average normal stress**

F_s / A : **the average shear stress**

○ Stress at a point,

$$\lim_{A \to 0} \frac{F_n}{A} = \sigma \quad , \quad \lim_{A \to 0} \frac{F_s}{A} = \tau$$

○ Pressure (in fluid) = **Positive normal stress, or Compressive stress**

↓

no **tensile**

no **shear**

→ stresses accepted (i.e., no **such strength**)

2. Body forces vs. Stresses

○ Stresses : act on surface elements inside or on the boundary of the body

ex) contact force between solid bodies
hydrostatic pressure between a solid body and a fluid

○ Body forces : act on the elements of volume (or mass) of the body

ex) **gravitational, magnetic, inertia forces**

3. Stresses after Transformed

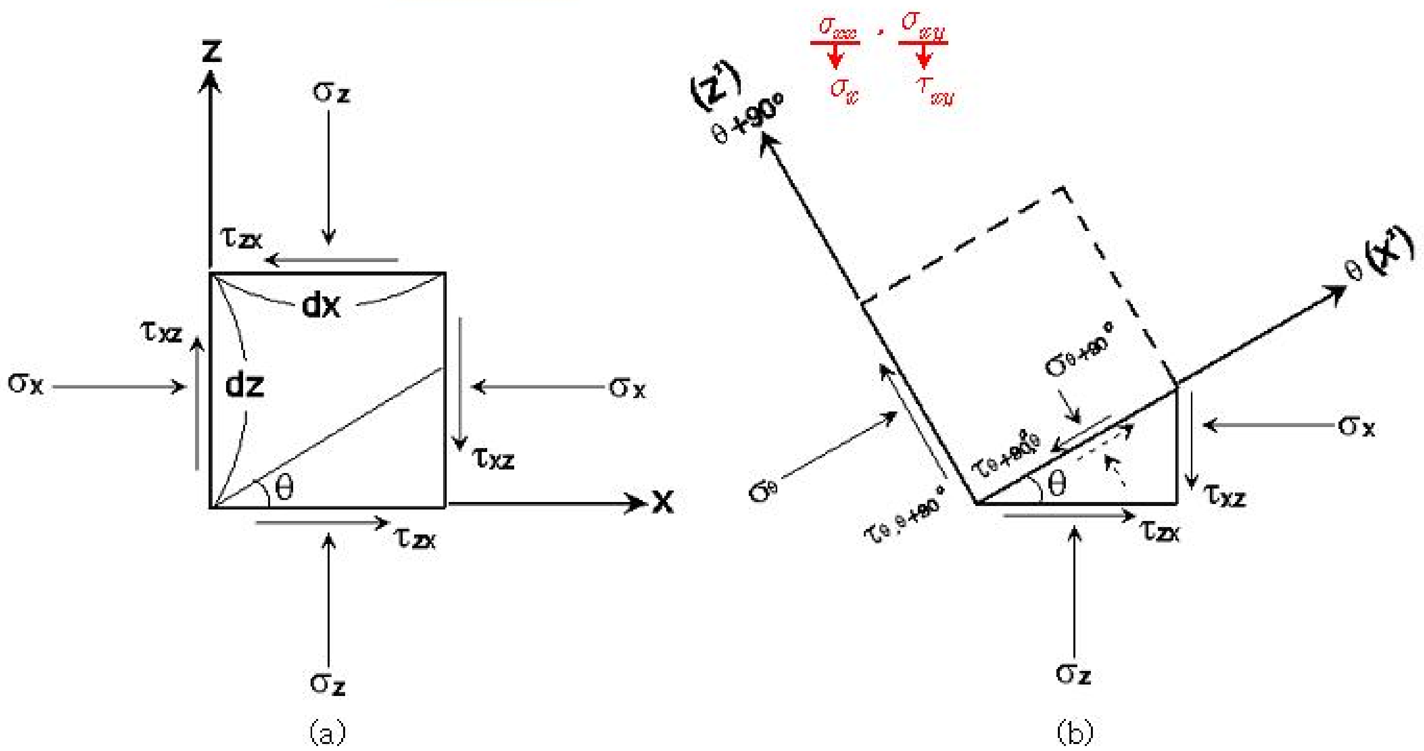


Fig. 2

○ $\Sigma F_{\theta+90^\circ}=0,$

$$\sigma_{\theta+90^\circ}(dx/\cos\theta)-\sigma_x(dx\ \tan\theta)\sin\theta + \tau_{xz}(dx\ \tan\theta)\cos\theta-\sigma_z(dx)\cos\theta + \tau_{zx}(dx)\sin\theta=0$$

$$\rightarrow \quad \sigma_{\theta+90^\circ}=\sigma_x\sin^2\theta+\sigma_z\cos^2\theta-\tau_{xz}\sin\theta\cos\theta$$

-- ①

f(2θ) →

$$=\frac{\sigma_x+\sigma_z}{2}+\frac{\sigma_z-\sigma_x}{2}\cos 2\theta-\tau_{xz}\sin 2\theta$$

○ $\Sigma F_{\theta}=0,$

$$\tau_{\theta+90^\circ,\theta}=(\sigma_z-\sigma_x)\sin\theta\cos\theta-\tau_{xz}(\sin^2\theta-\cos^2\theta)$$

-- ②

f(2θ) →

$$=\frac{\sigma_z-\sigma_x}{2}\sin 2\theta+\tau_{xz}\cos 2\theta$$

$\sigma_{\theta} \leftarrow \sigma_{\theta+90^\circ}\,(\theta \rightarrow \theta - 90^\circ)$

$\tau_{\theta,\theta+90^\circ} = \tau_{\theta+90^\circ,\theta}$ always ---------------------- ③

○ If $\tan 2\theta = \dfrac{2\tau_{xz}}{\sigma_x - \sigma_z}$, $\tau_{\theta+90^\circ,\theta} = 0$

($\theta = \theta_{cr}$, $\sigma_{\theta_{cr}} = \sigma_1$ or σ_3 : principal stresses
if $\sigma_{\theta_{cr}} = \sigma_1$, $\sigma_3 = \sigma_{\theta_{cr}+90^\circ}$)

4. Stress transformation matrix

$$\begin{bmatrix} \sigma_{x'} (= \sigma_{\theta}) \\ \sigma_{z'} (= \sigma_{\theta+90^\circ}) \\ \tau_{x'z'} (= \tau_{\theta,\theta+90^\circ}) \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & -\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix}$$

$\quad \| \qquad\quad \|$
$\tau_{z'x'} \quad \tau_{\theta+90^\circ,\theta}$

5. Mohr circle

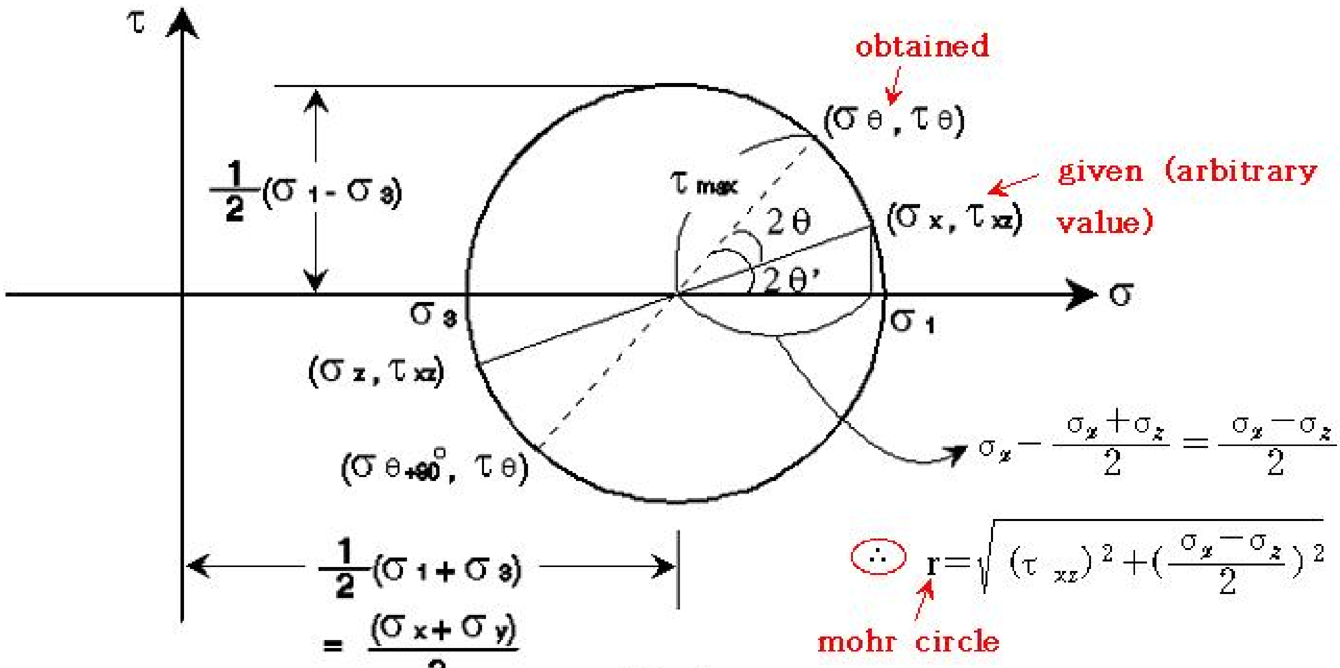


Fig.3

○ σ_θ, τ_θ in terms of σ_1 & σ_3

(From Eqs. ① & ② ; $\sigma_x = \sigma_1$, $\sigma_z = \sigma_3$, $\tau_{xz} = 0$)

$$\sigma_\theta = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\cos 2\theta'$$

$$\tau_\theta = \frac{\sigma_1 - \sigma_3}{2}\sin 2\theta' \qquad \text{------------------ ④}$$

Let $\frac{1}{2}(\sigma_1 + \sigma_3) = \sigma_{av}$, $\frac{1}{2}(\sigma_1 - \sigma_3) = \tau_{max}$ ---------------- ⑤

Sub. into Eq.④

$$\sigma_\theta = \sigma_{av} + \tau_{max}\cos 2\theta' ,$$

$$\tau_\theta = \tau_{max}\sin 2\theta' \qquad \text{---------------- ⑥}$$

To eliminate θ, manipulate the following.

$$\cos 2\theta' = \sqrt{1 - \sin^2 2\theta'} = \sqrt{1 - \frac{\tau_\theta^2}{\tau_{max}^2}} \qquad \text{---------------- ⑦}$$

Sub. into Eq.⑥,

$$\sigma_\theta - \sigma_{av} = \tau_{max}\sqrt{1 - \frac{\tau_\theta^2}{\tau_{max}^2}}$$

or $(\sigma_\theta - \sigma_{av})^2 + \tau_\theta^2 = \tau_{max}^2$ ---------------- ⑧

,which is the equation of a circle. (center at σ_{av} & $r = \tau_{max}$)

↑ $\frac{1}{2}(\sigma_1 + \sigma_3)$ ↑ $\frac{1}{2}(\sigma_1 - \sigma_3)$

○ σ_1 & σ_3 in terms of σ_x, σ_z & τ_{xz}

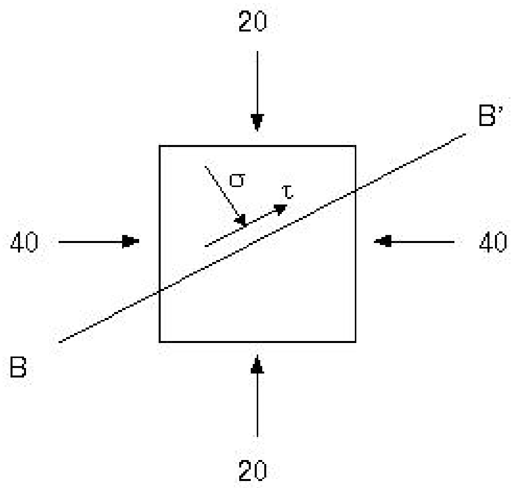
$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_z) + \sqrt{\tau_{xz}^2 + (\frac{\sigma_x - \sigma_z}{2})^2}$$

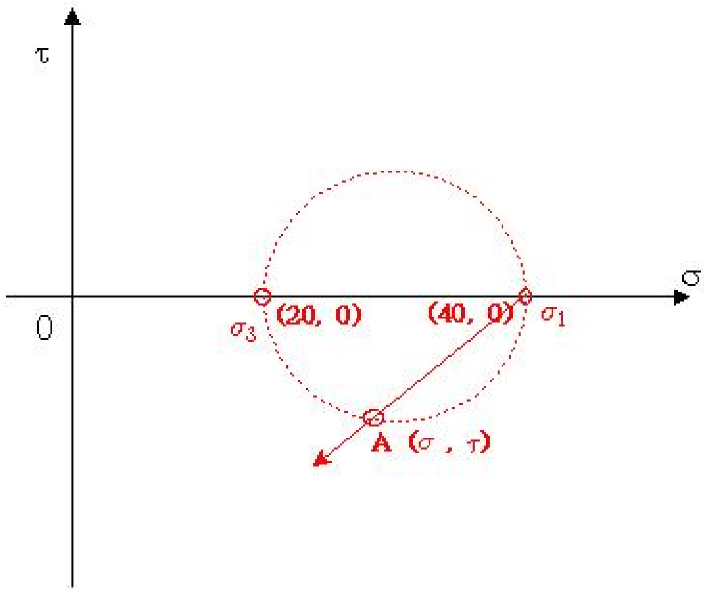
$$\sigma_3 = \frac{1}{2}(\sigma_x + \sigma_z) - \sqrt{\tau_{xz}^2 + (\frac{\sigma_x - \sigma_z}{2})^2}$$

○ In fact,

$\sigma_1 + \sigma_3 = \sigma_x + \sigma_z = \sigma_\theta + \sigma_{\theta + 90°} =$ constant **(1st Invariant)**

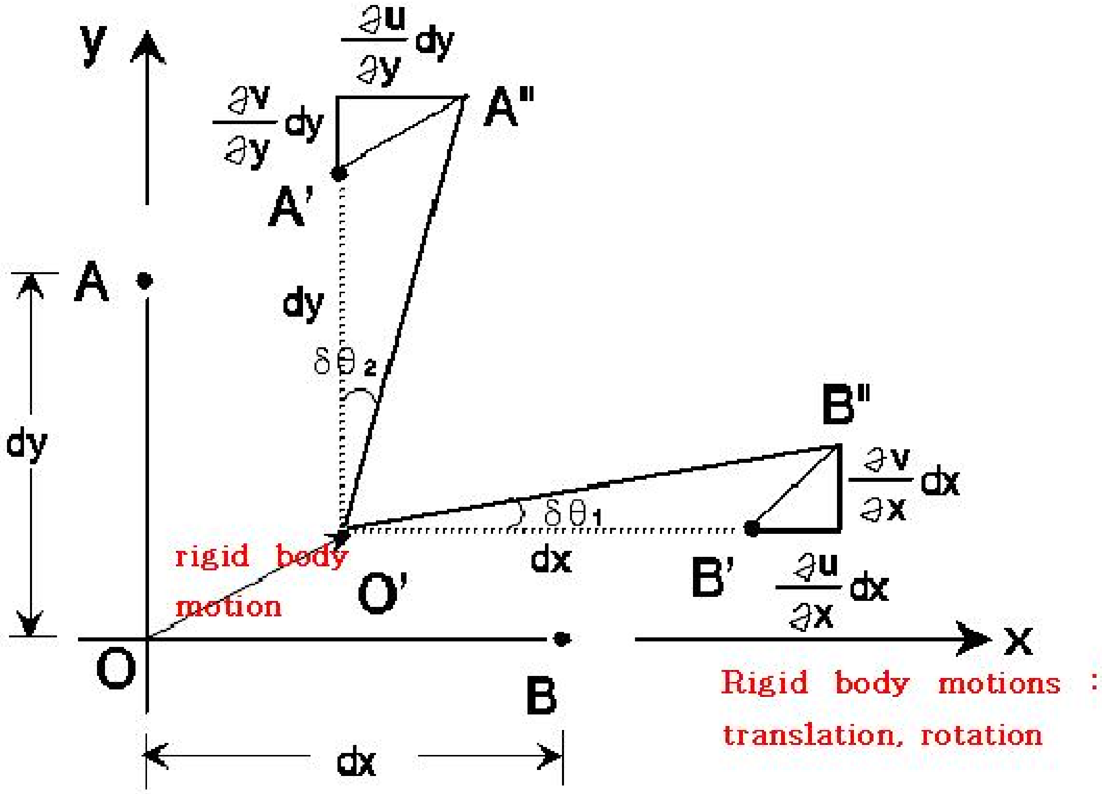
6. Origin of Planes(Pole, O_P)





A line through O_P and any point A of the Mohr circle will be parallel to the plane on which the stresses given by point A act.

7. Strains (Infinitesimal → **2nd order terms neglected → e.g.,** $(\frac{\partial u}{\partial x})^2 = 0$)



where OB = O'B', OA = O'A'

u: displacement function in x direction

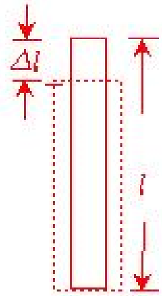
v: displacement function in y direction

$$O'B'' = [(dx + \frac{\partial u}{\partial x}dx)^2 + (\frac{\partial v}{\partial x}dx)^2]^{\frac{1}{2}}$$

$$\varepsilon_x = \frac{\Delta L}{L} = \frac{[\ (\quad)^2 + (\quad)^2]^{\frac{1}{2}} - dx}{dx}$$

$$= [(1 + \frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2]^{\frac{1}{2}} - 1$$

$$\fallingdotseq 1 + \frac{\partial u}{\partial x} - 1 = \frac{\partial u}{\partial x}$$



$$\epsilon = \frac{\Delta l}{l}$$

Similarly,

$$\varepsilon_y = \frac{\partial v}{\partial y}, \quad \left(\varepsilon_z = \frac{\partial w}{\partial z}\right)$$

○ Change in angle (radian)

$$\gamma_{xy} = \frac{\frac{\partial v}{\partial x} dx}{dx} + \frac{\frac{\partial u}{\partial y} dy}{dy}$$

$$= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \left(= \quad \delta\theta_1 + \delta\theta_2 \quad \right)$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$