2 Review of Linear Algebra

2.1 Notation and Definitions

- $\mathbb{R}$: the set of real numbers
- $\mathbb{C}$: the set of complex numbers
- $\mathbb{F}$: either $\mathbb{R}$ or $\mathbb{C}$
- $\mathbb{F}^n$: the vector space over $\mathbb{F}$
- $\mathbb{F}^{m\times n}$: the set of all $m \times n$ matrices with entries in $\mathbb{F}$. The elements in the ith row and jth column of a matrix A is denoted by A_{ij}.
- For $A \in \mathbb{F}^{m\times n}$, A^T is the transpose of A, and A^* denotes the complex-conjugate transpose.
- $A \in \mathbb{F}^{m\times n}$ is a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. Then the kernel (or null space) of the linear transformation A is

  $$KerA = N(A) = \{x \in \mathbb{F}^m : Ax = 0, \mathbb{F}^n\}\ .$$

  Let $a_1, \cdots, a_n$ denote the columns of A, then the image or range of A is

  $$ImA = \text{span}\{a_1, \cdots, a_n\} := \{\alpha_1 a_1 + \cdots + \alpha_n a_n : \alpha_i \in \mathbb{F}\} := \{y \in \mathbb{F}^m : y = Ax,\ x \in \mathbb{F}^n\}\ .$$

2.2 Eigenvalues and Eigenvectors

The eigenvalues of $M \in \mathbb{C}^{n\times n}$ are the n roots of its characteristic polynomial $p(\lambda) := det(\lambda I - A)$.

- The maximal modulus $\rho(A) := \max_{1\le i\le n} |\lambda_i|$
- The maximal real modulus $\rho_R(A) := \max_{\lambda_i\in\mathbb{R}} |\lambda_i|$

Eigenvalues and (right) eigenvectors satisfy

$$Av_i = \lambda_i v_i \tag{1}$$

The eigenvector v defines a one-dimensional subspace that is invariant with respect to premultiplication by A since $A^k v = \lambda^k v,\ \forall k$. A subspace $S \subset \mathbb{C}^n$ is $A - invariant$, if $Ax \in S$ for every $x \in S$. In other words, the image of S under A is contained in S, i.e. $As \subset S$. For example, $\{0\}, \mathbb{C}^n, KerA, ImA$ are all $A - invariant$ subspaces.

Facts

- The eigenvalues of a matrix are continuous functions of the entries of the matrix.
- Given a matrix $A \in \mathbb{R}^{n\times n}$, there exists a matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^*Q = I, \quad \text{and} \quad Q^*AQ = \Lambda \text{ upper triangular}$$

(Recall Gram-Schmidt orthonormalization process) Since Q^*AQ is upper triangular, the eigenvalues of Q^*AQ are the diagonal entries. Since $Q^{-1} = Q^*$, the eigenvalues of Q^*AQ are the same as the eigenvalues of A.

2.3 Vector & Matrix Norms

- the vector p-norm of $x \in \mathbb{F}^n$

$$||x||_p := (\sum_1^n |x_i|^p)^{1/p}$$

In particular,

$$||x||_1 := \sum_1^n |x_i|$$

$$||x||_2 = \sqrt{\sum_1^n |x_i|^2}$$

$$||x||_\infty := \max_{1\leq i\leq n} |x_i|$$

Each of these satisfies the following properties: For any $x, y \in \mathbb{F}^n$ and $\alpha \in \mathbb{F}$

(i) $||x|| \geq 0$
(ii) $||x|| = 0$ iff $x = 0$
(iii) $||\alpha x|| = |\alpha|||x||$
(iv) $||x + y|| \leq ||x|| + ||y||$

Facts For $x \in \mathbb{F}$,

- $||x||^2 = x^*x$
- if $Q \in \mathbb{F}^{n\times n}, Q^*Q = I$, then $||Qx|| = ||x||$

- the induced p-norm of a matrix $A \in \mathbb{C}^{m\times n}$

$$||A||_p := \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{||Ax||_p}{||x||_p}$$

Recall the viewpoint of A as a mapping from a vector space $\mathbb{C}^n$ equipped with a vector norm $||\cdot||_p$ to another vector space $\mathbb{C}^m$ equipped with a vector norm $||\cdot||_p$. From this point of view, the induced norms can be interpreted as the maximum amplification obtainable from input to output.

In particular,

$$\begin{aligned} ||A||_1 &:= \max_{1\leq j\leq m} \sum_{i=1}^{n} |A_{ij}| \quad \text{max. column sum} \\ ||A||_2 &:= \sqrt{\lambda_{\max}(A^*A)} \\ ||A||_\infty &:= \max_{1\leq i\leq n} \sum_{j=1}^{m} |A_{ij}| \quad \text{max. row sum} \end{aligned}$$

- Frobenius norm is another often used matrix norm:

$$||A||_F := \sqrt{\text{trace}(A^*A)}$$

Unless specified otherwise, we will use $||x|| := ||x||_2$ to denote the Euclidean 2-norm of x, and $||A|| := ||A||_2$ to denote the induced 2-norm of A.

2.4 Symmetric & Hermitian Matrices

Definition

- real, symmetric matrices $\mathcal{S}^{n\times n} := \{M \in \mathbb{R}^{n\times n} : M^T = M\}$
- complex, Hermitian matrices $\mathcal{H}^{n\times n} := \{M \in \mathbb{C}^{n\times n} : M^* = M\}$
- complex, normal matrices $\mathcal{N}^{n\times n} := \{M \in \mathbb{C}^{n\times n} : M^*M = MM^*\}$

Note that $\mathcal{S}^{n\times n} \subset \mathcal{H}^{n\times n} \subset \mathcal{N}^{n\times n}$.

Facts

- Hermitian matrices have real eigenvalues.
- A normal matrix M has an orthogonal set of eigenvectors, i.e. there exist matrices $Q, \Lambda \in \mathbb{C}^{n\times n}$ with

$$Q^*Q = I, \quad \Lambda \text{ diagonal}, \quad \text{and} \quad M = Q\Lambda Q^*$$

- For $M \in \mathcal{H}^{n\times n}$,

$$\{x^*Mx : ||x||_2 = 1\} = [\lambda_{\min}(M),\ \lambda_{\max}(M)]$$

This can be used to show that

$$\begin{aligned} ||M||_2^2 &:= \max_{||x|| \le 1} ||Mx||^2 \\ &= \max_{||x||=1} ||Mx||^2 \\ &= \max_{||x||=1} x^* M^* M x \\ &= \lambda_{\max}(M^* M) \end{aligned}$$

$||M||_2$ is called the maximum singular value of M.

2.5 Singular Values

Definition 2.1 (Maximum and Minimum Singular Values)

$$\bar{\sigma}(A) = \max_{||x||=1} ||Ax||_2 \tag{2}$$

$$\underline{\sigma}(A) = \min_{||x||=1} ||Ax||_2 \tag{3}$$

Note that

$$\bar{\sigma}^2(A) = \max_i \lambda_i(A^* A) \tag{4}$$

$$\underline{\sigma}^2(A) = \min_i \lambda_i(A^* A) \tag{5}$$

Theorem 2.2 (Singular Value Decomposition Theorem)
Let A be an $m \times n$ complex-valued matrix of rank k. Then, there is an $m \times m$ unitary matrix U (i.e. $U^* = U^{-1}$), an $n \times n$ unitary matrix V (i.e. $V^* = V^{-1}$), and an $m \times n$ diagonal matrix Σ such that

$$A = U \Sigma V^*$$

Here the diagonal entries of Σ can be arranged to be nonincreasing: all these entries are nonnegative, and exactly k of them are strictly positive.

Ex. $[U, \Sigma, V]$ =`svd` (A)

Lemma 2.3 (Properties of Singular Values)

1.
$$\sigma_i(cA) = |c| \sigma_i(A) \quad \textit{for all singular values of } A \tag{6}$$

2.
$$\bar{\sigma}(A + B) \le \bar{\sigma}(A) + \bar{\sigma}(B) \qquad \textit{(triangle inequality)} \tag{7}$$

3.
$$\bar{\sigma}(A^{-1}) = [\, \underline{\sigma}(A) \,]^{-1} \quad \textit{for any nonsigular matrix } A \tag{8}$$

4.
$$\underline{\sigma}(A+B) \leqq \underline{\sigma}(A) + \bar{\sigma}(B) \tag{9}$$
5.
$$\underline{\sigma}(A+B) \geqq \underline{\sigma}(A) - \bar{\sigma}(B) \tag{10}$$
6.
$$\bar{\sigma}(AB) \leqq \bar{\sigma}(A)\,\bar{\sigma}(B) \text{ for any square matrices } A \text{ and } B \tag{11}$$
7.
$$\underline{\sigma}(AB) \geqq \underline{\sigma}(A)\,\underline{\sigma}(B) \text{ for any square matrices } A \text{ and } B \tag{12}$$

2.6 Positive Definite Matrices

Definition: $M \in \mathcal{H}^{n\times n}$ is

- positive definite ($M > 0$) if $x^*Mx > 0,\ \forall x \in \mathbb{C}^n, x \neq 0$
- positive semi-definite ($M \geq 0$) if $x^*Mx \geq 0,\ \forall x \in \mathbb{C}^n, x \neq 0$

Negative definite matrices can be defined similarly.

Facts

- $M > 0$ if and only if $\lambda_{\min} > 0$.
- For any $M \in \mathbb{C}^{m\times n}$,
$$\bar{\sigma}(M) < \beta \quad \Longleftrightarrow \quad M^*M - \beta^2 I_n < 0 \quad \Longleftrightarrow \quad MM^* - \beta^2 I_m < 0\ .$$
- For $M = M^* \geq 0$, there is a unique matrix S such that
$$S = S^*, \quad S \geq 0, \quad \text{and } S^2 = M\ .$$
S is called the Hermitian square-root of M and denoted $M^{\frac{1}{2}}$. Also, $(M^{-1})^{\frac{1}{2}} = (M^{\frac{1}{2}})^{-1}$, so we can write $M^{-\frac{1}{2}}$.
- For $M \in \mathcal{H}^{n\times n}$ and $L \in \mathbb{C}^{n\times n}$, with L invertible,
$$M > 0 \quad \Longleftrightarrow \quad L^*ML > 0$$
- For $A \in \mathcal{H}^{n\times n},\ B \in \mathcal{H}^{m\times m}$,
$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} > 0 \quad \Longleftrightarrow \quad A > 0 \text{ and } B > 0$$
- For $A \in \mathcal{H}^{n\times n}, B \in \mathcal{H}^{m\times m},\ C \in \mathbb{C}^{n\times m}$,
$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} > 0 \quad \Longleftrightarrow \quad B > 0 \text{ and } A - CB^{-1}C > 0$$
- The space of positive definite matrix is convex. Later we'll use it in LMI.

2.7 Matrix Inversion Formulas

Let A be a square matrix partitioned as

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are also square matrices.

Now suppose that A_{11} is nonsingular. Then A is nonsingular iff $\Delta_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$ is nonsingular.

Dually, if A_{22} is nonsingular, then A is nonsingular iff $\Delta_{22} := A_{11} - A_{12}A_{22}^{-1}A_{21}$ is nonsingular.

Δ_{11} (Δ_{22}) is called the Schur complement of A_{11} (A_{22}) in A.

If A is nonsingular,

$$\begin{aligned} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} &= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}\Delta_{11}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}\Delta_{11}^{-1} \\ -\Delta_{11}^{-1}A_{21}A_{11}^{-1} & \Delta_{11}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \Delta_{22}^{-1} & -\Delta_{22}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}\Delta_{22}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}\Delta_{22}^{-1}A_{12}^{-1}A_{22}^{-1} \end{bmatrix} \end{aligned}$$

Suppose A_{11} and A_{22} are both nonsingular. Then

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}$$