3 Controllability and Observability

We begin this section with the following familiar result:

Lemma 3.1 *$A \in \mathbb{R}^{n\times n}$ has all its evals in the open LHP iff there exists $P \in \mathbb{R}^{n\times n}$, $P = P^T > 0$ such that*

$$A^T P + PA < 0\ .$$

3.1 Controllability

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(t_0) = x_0 & (1) \\ y &= Cx + Du & (2) \end{aligned}$$

Definition 3.2 (Controllability) *The dynamical system described by* (1) *or the pair (A, B) is said to be controllable if, for any initial state $x(t_0) = x_0, t_1 > 0$ and final state x_1, there exists a (piecewise continuous) input $u(\cdot)$ such that the solution of equation* (1) *satisfies $x(t_1) = x_1$. Otherwise, (A, B) is said to be uncontrollable.*

Theorem 3.3 (Controllability) *The following are equivalent:*

(i) (A,B) is controllable.

(ii) The controllability Gramian

$$W_c(t) \triangleq \int_o^t e^{A\tau} BB^* e^{A^*\tau} d\tau \tag{3}$$

is positive definite for any $t > 0$.

(iii)

$$\mathcal{C} \triangleq \begin{bmatrix} B\ AB\ A^2B\ \cdots A^{n-1}B \end{bmatrix} \tag{4}$$

has full-rank.

(iv) The matrix $[A - \lambda I, B]$ has full-row rank for all $\lambda \in \mathbb{C}$.

(v) Let λ and x be any eigenvalue and any corresponding left eigenvector of A (i.e., $x^* A = x^* \lambda$); then $x^* B \neq 0$.

(vi) The eigenvalues of $A + BF$ can be freely assigned (with the restriction that complex eigenvalues are in conjugate pairs) by a suitable choice of F.

Remark. The conditions *(iv)* and *(v)* are called Popov-Belevitch-Hautus (PBH) tests.

Definition 3.4 (Stabilizability) *The dynamical system described by* (1) *or the pair* (A, B) *is said to be (state-feedback) stabilizable if there exists a state feedback* $u = Kx$ *such that* $A + BK$ *is stable.*

Theorem 3.5 (A, B) *is stabilizable iff there exist* $W \in \mathbb{R}^{n\times n}$ *and* $R \in \mathbb{R}^{n_u\times n}$ *such that* $W = W^T > 0$ *and*

$$AW + WA^T + BR + R^T B^T < 0\ .$$

Proof.

3.2 Observability

Definition 3.6 (Observability) *The dynamical system described by* (2) *or the pair* (C, A) *is said to be observable if, for any* $t_1 > 0$*, the initial state state* $x(0) = x_0$ *can be determined from the time history of the input* $u(t)$ *and the output* $y(t)$ *over the interval* $[0, t_1]$*. Otherwise,* (C, A) *is said to be unobservable.*

Theorem 3.7 (Observability) *The following are equivalent:*

(i) *(C,A) is observable.*

(ii) *The observability Gramian*

$$W_o(t) \triangleq \int_o^t e^{A^*\tau} C^* C e^{A\tau} d\tau \tag{5}$$

is positive definite for any $t > 0$.

(iii)

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{6}$$

has full-rank.

(iv) *The matrix* $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ *has full-row rank for all* $\lambda \in \mathbb{C}$.

(v) *Let* λ *and* x *be any eigenvalue and any corresponding right eigenvector of* A *(i.e.,* $Ay = \lambda y$*); then* $Cy \neq 0$.

(vi) *The eigenvalues of* $A + LC$ *can be freely assigned (with the restriction that complex eigenvalues are in conjugate pairs) by a suitable choice of* L.

(vii) (A^*, C^*) *is controllable.*

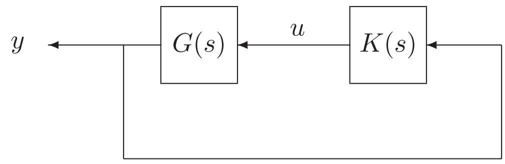

Figure 1: observer-based controller

Remark. The conditions *(iv)* and *(v)* are called Popov-Belevitch-Hautus (PBH) tests.

Definition 3.8 (Detectability) *The pair (C, A) is said to be detectable if there exists a matrix L such that $A + LC$ is stable.*

Theorem 3.9 *(C, A) is detectable iff there exist $P \in \mathbb{R}^{n\times n}$, $P = P^T > 0$ and $H \in \mathbb{R}^{n\times n_y}$ such that*

$$A^T P + PA + HC + C^T H^T < 0 \, .$$

Proof. similar to Thm 3.5.

3.3 Observer-based Controllers

If a system is controllable and the states are available for feedback, then clearly, the c.l. poles can be assigned arbitrarily through a constant feedback. Often, the designer knows y and u only.

An observer is a dynamical system with input u, y and output $\hat{x}$, which asymptotically estimated the state, i.e., $\hat{x} \to x$ for (all) initial states and for every input. An observer for (2) exists iff (C, A) is detectable, in which case, a full-order Luenberger observer is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} + Du - y)$$

where L is a matrix that makes $A + LC$ stable.

Then with $u = K\hat{x}$, the total system state equations are

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

and with $e := x - \hat{x}$, these equations become

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + LC & 0 \\ -LC & A + BK \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix} \, .$$

Now if (A, B) is controllable and (C, A) is observable, the closed-loop poles (eigenvalues of $A + LC$ and $A + BK$) can be arbitrarily assigned.

The closed-loop system is shown in Fig. 1, with the observer-based controller denoted as

$$u = K(s)y$$

and

$$\begin{aligned} K(s) &= \left[\begin{array}{c|c} A+BK+LC+LDK & -L \\ \hline K & 0 \end{array}\right] \\ G(s) &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] . \end{aligned}$$

From this construction, we can see that a system is output feedback stabilizable iff (A, B) is stabilizable and (C, A) is detectable.

3.4 Lyapunov Equations

The equation

$$AX + XA^* = -P \, , \tag{7}$$

where $A \in \mathbb{F}^{n\times n}$ and $P \in \mathbb{F}^{n\times n}$ are given matrices, is called the Lyapunov equation.

Lemma 3.10 *There exists a unique solution X for (7), iff*

$$\lambda_i(A) + \bar{\lambda}_j(A) \neq 0, \ \forall i, j \, . \tag{8}$$

Theorem 3.11 *Let $A \in \mathbb{F}^{n\times n}$ be a given stable matrix. Then for any $P \in \mathbb{F}^{n\times n}$, the unique solution solving (7) is given by*

$$X = \int_0^\infty e^{A^*\tau} P e^{A\tau} d\tau \, . \tag{9}$$

Proof.

It follows that the observability Gramian W_o of (C, A) can be obtained from

$$A^* W_o + W_o A + C^* C = 0 \, .$$

(Similarly, the controllability Gramian W_c of (A, B) can be obtained from $AW_c + W_c A^* + BB^* = 0$.)

Theorem 3.12 *Suppose $A, Q \in \mathbb{F}^{n\times n}$ are given, and A is stable and $Q = Q^* \geq 0$. Then Then $(Q^{1/2}, A)$ is observable iff*

$$X := \int_0^\infty e^{A^*\tau} Q e^{A\tau} d\tau > 0 \, . \tag{10}$$

Proof.

Theorem 3.13 *Suppose that (C, A) is detectable and that X is any solution to $A^*X + XA = -C^*C$. (i.e. there is no apriori assumption on the uniqueness of solution) Then $X \geq 0$ iff A is stable.*