4 System Norms

4.1 Optimal Feedback Control

Q. Why do people use 'feedback' control (vs. open-loop control) ?

Consider a tracking problem, with disturbance rejection, measurement noise, and control input signal limitations, as shown in Fig. 1. We would want to design a controller to keep tracking errors and control input signal small for all reasonable reference commands, sensor noise, and external force disturbances. Thus, a reasonable performance objective is the closed-loop "gain" from exogenous influences (reference commands, sensor noise, and external force disturbances) to regulated variables (tracking errors and control input signal). The magnitude of certain closed-loop TM norms are use as performance objectives in popular optimal control methods. Often, there exists a tradeoff between tracking/disturbance error reduction and minimizing sensitivity to measurement noise or uncertainty (we delay formal discussion to later).

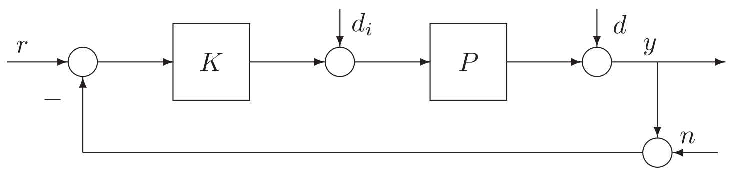

Figure 1: Standard feedback control configuration

The main subject of discussion in this lecture is the standard LTI feedback optimization setup. There are three basic concepts behind the standard feedback optimization setup:

- the notion of a multi-input, multi-output (MIMO) linear time-invariant (LTI) finite order system,
- the notion of an internally stable feedback interconnection of two MIMO LTI systems, and
- the notion of a system norm.

Let T denote the closed-loop mapping from the outside influences to the regulated variables,

$$\underbrace{\begin{bmatrix} \text{tracking error} \\ \text{control input} \end{bmatrix}}_{\text{regulated variables}} = T \underbrace{\begin{bmatrix} \text{reference} \\ \text{external force} \\ \text{noise} \end{bmatrix}}_{\text{outside influences}} \tag{1}$$

We can associate good performance with the gain of T being small. To quantify the term *gain* mathematically, we need to define some additional things.

First, we review a well-known property of an analytic function. Let $S \in \mathbb{C}$ be an open set, and let $f(s)$ be a cpx-valued ftn defined on S. Then $f(s)$ is said to be analytic at a point $z_0 \in S$ if it is differentiable at z_0 and also at each point in some nbhd of z_0.

If $f(s)$ is analytic at z_0 then f has conti derivatives of all orders at z_0, and it has a power series representation at z_0. $f(s)$ is said to be analytic in S if it is analytic at each point of S. A matrix-valued ftn is analytic in S if every element of the matrix is analytic in S. For eample, all real rational stable transfer matrices are analytic in RHP.

Theorem 4.1 (max modulus thm) *If $f(s)$ is defined and conti on a closed-bounded set S and analytic on the interior of S, then*

$$\max_{s \in S} |f(s)| = \max_{s \in \partial S} |f(s)|$$

where ∂S denotes the boundary of S.

4.2 Norms of Signals and Systems

For a scalar signal $e(t)$ in the time domain, we often use the 2-norm (or $\mathcal{L}_2$ norm), which is defined as

$$||e||_2 := \left(\int_{-\infty}^{\infty} e^2(t) dt \right)^{\frac{1}{2}} .$$

If this integral is finite, then we say that the signal e is *square integrable* and $e \in \mathcal{L}_2$ (Lebesgue). Similarly, for vector-valued signals, the 2-norm is defined as

$$||e||_2 := \left(\int_{-\infty}^{\infty} ||e(t)||_2^2 dt \right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} ||e^T(t) e(t)|| dt \right)^{\frac{1}{2}} .$$

For a signal $e(t) \in \mathcal{L}_\infty$, the $\mathcal{L}_\infty$ norm is

$$||e||_\infty := \text{ess.} \sup_t |e(t)| = \inf\{B : |e| \leq B (a.e.)\} .$$

For $f(t), g(t) \in \mathcal{L}_2$, let

$$F(j\omega) = \mathcal{F}\{f(t)\} \triangleq \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt ,$$

and

$$f(t) = \mathcal{F}^{-1}\{F(j\omega)\} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega \; .$$

Then

$$\mathcal{F}\{f(t) * g(t)\} = F(j\omega)G(j\omega) \; ,$$

where

$$f(t) * g(t) \triangleq \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \; : \text{ convolution.}$$

Theorem 4.2 (Parseval) *For* $f(t), g(t) \in \mathcal{L}_2$*,*

$$\int_{-\infty}^{\infty} f(t)g(t)dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)G^*(j\omega)d\omega \; ,$$

in particular,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega \; .$$

A state-space model for a finite order CT LTI system H with input $u(t)$, output $y(t)$, and state $x(t)$ has the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where A, B, C, D are constant matrices with real entries. Given an input $u(t)$ and the initial state vector $x(0)$, the output $y(t)$ is defined according to the formula

$$y(t) = Ce^{At}x(0) + Du(t) + \int_0^{\infty} Ce^{A\tau}Bu(t-\tau)d\tau \; .$$

The transfer matrix (transfer function in the case when both u and y are scalar) of the system is defined for all complex s such that $sI - A$ is invertible by

$$G(s) = D + C(sI - A)^{-1}B \; .$$

Two mathematically convenient measures of the TM $G(s)$ in the frequency domain are the matrix $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms (Hardy):

- **The $\mathcal{H}_\infty$ norm** $||G||_\infty$ of G is defined as the supremum (minimal upper bound) of the largest singular number of its TM over the imaginary axis:

  $$||G||_\infty \triangleq \sup_{Re(s)>0} \sigma_{\max}(G(j\omega)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(j\omega)),$$

  where, for an k-by-m complex matrix M, $\sigma_{\max}(M) = \max_{u\in C^m, |u|=1} |Mu|$ and $|v|$ denotes the standard Hermitian norm (length) of vector v.

- **The $\mathcal{H}_2$ norm** $||G||_2$ of a finite order stable CT LTI system with $D = 0$ is defined by the integral

$$\begin{aligned}||G||_2 &= \int_0^{\infty} \mathrm{tr}(g(t)g(t)')dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G^*(j\omega)G(j\omega))d\omega \\ &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} ||G(j\omega)||_F^2 \mathrm{d}\omega\right]^{\frac{1}{2}}\end{aligned}$$

where $g(t) = Ce^{At}B$ is the impulse response matrix of G, and $||\cdot||_F$ is the Frobenius norm.

4.3 Interpretations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

$\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms are frequently used as the cost measure in feedback optimization. Both these TM norms have time-domain interpretations. Suppose that the initial condition is $x(0) = 0$, then

- for a unit intensity white noise input u, the steady-state variance of output y is $||G||_2$.
- The $\mathcal{L}_2$ (RMS) gain from u to y is $||G||_\infty$:

$$\max_{d\neq 0} \frac{||y||_2}{||u||_2} = ||G||_\infty$$

This section describes interpretations of the norms as performance measures.

4.3.1 $||H||_\infty$ as $\mathcal{L}_2$ gain

The $\mathcal{L}_2$ gain of a continuous time linear system (strictly speaking, $\mathcal{L}_2$-to-$\mathcal{L}_2$ gain of a continuous time system with input u and output y) is defined as the minimal $\gamma \geq 0$ such that

$$\inf_{T\geq 0} \int_0^T \gamma^2|u(t)|^2 - |y(t)|^2 dt > -\infty$$

for all input/output pairs u and y where input u is square integrable over arbitrary finite intervals.

The informal rationale behind the definition of the $\mathcal{L}_2$ norm is as follows: for zero initial conditions, we expect the energy of the output to be bounded by the energy of the input times the $\mathcal{L}_2$ gain squared. Since non-zero initial conditions can produce nonzero output even for zero input, the actual definition says that the difference between the energies must be bounded on one side. $\mathcal{L}_2$ gain is a key concept in robustness analysis. The importance of the $\mathcal{H}_\infty$ norm is largely due to the fact that, for a stable finite order LTI system, $\mathcal{H}_\infty$ norm equals $\mathcal{L}_2$ gain.

Theorem 4.3 *$\mathcal{L}_2$ gain of a stable finite order LTI system equals its $\mathcal{H}_\infty$ norm.*

Proof.

Consider the CT case (the DT case is similar). Let $G(s)$ be the transfer function of the system. To show that $\mathcal{L}_2$ gain cannot be larger than $\mathcal{H}_\infty$ norm, use the Parseval theorem. Consider the case of zero initial conditions first. For an arbitrary input signal u and for $T > 0$ let u_T denote the signal defined by

$$u_T(t) = \begin{cases} u(t) & t < T \\ 0 & t \geq T \end{cases}$$

Let y and y_T denote the response of the system to u and u_T respectively, both assuming zero initial conditions. Then u_T, y_T are square integrable over $t \in (0, \infty)$ (for y_T this is true since A is a Hurwitz matrix), and hence both have Fourier transforms u_T and y_T respectively. In addition, by causality, $y(t) = y_T(t)$ for $t < T$. Hence

$$\begin{aligned}
\int_0^T |y(t)|^2 dt &\leq \int_0^\infty |y(t)|^2 dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |y_T(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega) u_T(j\omega)|^2 d\omega \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} ||G||_\infty^2 |u_T(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi} ||G||_\infty^2 \int_0^\infty |u_T(j\omega)|^2 dt \\
&= ||G||_\infty^2 \int_0^T |u(t)|^2 dt
\end{aligned}$$

To show that $\mathcal{H}_\infty$ norm cannot be larger than $\mathcal{L}_2$ gain, consider zero initial conditions and sinusoidal inputs with unit 2-norm, whose frequency ω_0 is a freq where

$$\sigma_{\max}(G(j\omega_0)) = ||G||_\infty$$

See the text pg. 51 for details.

4.3.2 $\mathcal{H}_2$ norm and $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ gain

$\mathcal{L}_2$-to-$\mathcal{L}_\infty$ gain of a stable state space model is defined as the supremum of the amplitude of its time domain response to an input signal of unit energy.

Theorem 4.4 *$\mathcal{L}_2$-to-$\mathcal{L}_\infty$ gain of a stable LTI system with a scalar output equals its $\mathcal{H}_2$ norm.*

Proof. Consider the continuous-time case (the DT case is similar). To show that $\mathcal{H}_2$ norm is not smaller than the $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ gain, use the standard Cauchy-Schwartz inequality:

$$|y(T)|^2 = \left| \int_0^T h(t)u(T-t)dt \right|^2 \leq \int_0^T |h(t)|^2 dt \int_0^T |u(t)|^2 dt \; .$$

Since the inequality becomes equality for $u(t) = h(T-t)$, the $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ gain actually equals the $\mathcal{H}_2$ norm.

4.3.3 $\mathcal{H}_2$ norm and variance of white noise response

$\mathcal{H}_2$ norm of a system is also an measure of system sensitivity to white noise input. The continuous time white noise (with zero mean, unit variance) is a slightly complicated concept: it is a generalized random process $f = f(t)$ (something akin to the Dirac delta in the world of deterministic functions), which can be characterized by its effect in integration: if

$$\xi = \int_{t_1}^{t_2} h(t)f(t)dt \; ,$$

where $h(t)$ is a row vector of appropriate length, then

$$\mathrm{E}\,\xi = 0, \quad |\xi|^2 = \int_{t_1}^{t_2} |h(t)|^2 dt \; .$$

Combining this with the definition of $\mathcal{H}_2$ norm, we conclude that, for a stable LTI system, the asymptotic (as $t \to \infty$) variance of white noise response equals square of $\mathcal{H}_2$ norm.

4.4 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Spaces

- $\mathcal{H}_2$ Space
  consists of matrix functions $F(s)$ that are analytic and bounded in the open right-half plane, with the norm is defined as

  $$||F||_2 := \sup_{\sigma>0} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}[F^*(\sigma + j\omega)F(\sigma + j\omega)]d\omega \right\}$$

  It can be shown that

  $$||F||_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}[F^*(j\omega)F(j\omega)]d\omega \; .$$

- $\mathcal{H}_\infty$ Space
  consists of matrix functions that are analytic and bounded in the open right-half plane, with the norm is defined as

  $$||F||_\infty := \sup_{\mathrm{Re}(s)>0} \bar{\sigma}[F(s)]$$

It can be shown that (by generalization of maximum modulus theorem for matrix functions)

$$||F||_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}[F(j\omega)] \; .$$

The rational subspace of $\mathcal{H}_\infty$, denoted by $\mathcal{RH}_\infty$, consists of all proper and real rational stable transfer matrices.

4.5 Computing $\mathcal{H}_2$ norms

Lemma 4.5 *Consider a TM*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

with A stable. Then we have

$$||G||_2^2 = \mathrm{tr}(B^* W_o B) = \mathrm{tr}(C W_c C^*) \tag{2}$$

where W_c, W_o are controllability and observability Gramians that can be obtained from Lyapunov Equations:

$$A W_c + W_c A^* + B B^* = 0 \quad A^* W_o + W_o A + C^* C = 0 \; . \tag{3}$$

Proof. Since G is stable, we have

$$g(t) = \begin{cases} C e^{At} B & t \geq 0 \\ 0 & t < 0 \end{cases}$$

and

$$\begin{aligned} ||G||_2^2 &= \int_0^\infty \mathrm{tr}\{g^*(t) g(t)\} dt \;=\; \int_0^\infty \mathrm{tr}\{g(t) g^*(t)\} dt \\ &= \int_0^\infty \mathrm{tr}\{B^* e^{A^* t} C^* C e^{At} B\} dt \;=\; \int_0^\infty \mathrm{tr}\{C e^{At} B B^* e^{A^* t} C^*\} dt \\ &= \mathrm{tr}(B^* W_o B) \;=\; \mathrm{tr}(C W_c C^*). \end{aligned}$$

And Eqn. (3) follow from the result shown in the controllability/observability chapter.

4.6 Computing $\mathcal{H}_\infty$ norms

The infinity norm of a scalar transfer function G can be interpreted as the distance in the complex plane from the origin to the farthest point on the Nyquist plot of G, or as the peak value on the Bode magnitude plot of $|G(j\omega)|$.

Lemma 4.6 *Let $\gamma > 0$ and G is a proper real rational TM with no poles in $j\omega$ axis.*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Then $||G||_\infty < \gamma$ iff $\bar{\sigma}(D) < \gamma$ and the Hamiltonian matrix H has no eigenvalues on the $j\omega$ axis where

$$H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix} \tag{4}$$

and $R := \gamma^2 I - D^*D$.

Lemma 4.6 lets us use the following iterative algorithm to compute $||\cdot||_\infty$.

Bisection Algorithm

1. Set γ_u and γ_l such that $\gamma_l \leq ||G||_\infty \leq \gamma_u$;
2. If $(\gamma_u - \gamma_l)/\gamma_l \leq \varepsilon$, stop; $||G|| \approx (\gamma_u + \gamma_l)/2$. Otherwise go to the next step.
3. Set $\gamma = (\gamma_u + \gamma_l)/2$;
4. Test if $||G||_\infty < \gamma$ by calculating the eigenvalues of H for the given γ;
5. If H has eigenvalues on $j\omega$ axis, set $\gamma_l = \gamma$; otherwise set $\gamma_u = \gamma$; go back to step 2.

Remark. Since $||G||_\infty < \gamma$ iff $||\gamma^{-1}G||_\infty < 1$, we will assume that $\gamma = 1$ without loss of generality.

Example. `hinfnorm(sys); h2norm(sys);`