9 Performance Specifications

9.1 Design Tradeoffs

Consider again the feedback system shown in Fig. 1[1].

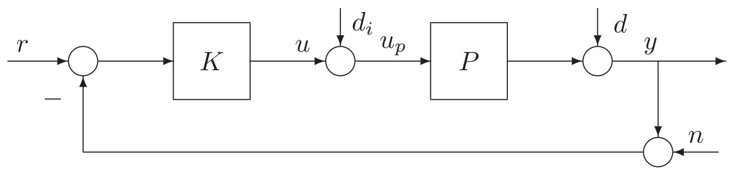

Figure 1: Standard feedback control configuration

For further discussion, it is convenient to define the *input loop transfer matrix* L_i and *output loop transfer matrix,* L_0, as

$$L_i = KP, \quad L_0 = PK$$

respectively, where L_i is obtained from breaking the loop at the input u of the plant while L_o is obtained from breaking the loop at the output y of the plant. The *input sensitivity* matrix is defined as the transfer matrix from d_i to u_p:

$$S_i = (I + L_i)^{-1}, \quad u_p = S_i d_i$$

The *output sensitivity* matrix is defined as the transfer matrix from d to y:

$$S_o = (I + L_O)^{-1}, \quad y = S_o d.$$

The *input* and *output complementary sensitivity* matrices are defined as

$$T_i = I - S_i = L_i(I + L_i)^{-1}$$

$$T_o = I - S_o = L_o(I + L_o)^{-1},$$

respectively. The word complementary is used to signify the fact that T is the complement of $S, T = I - S$. The matrix $I + L_i$ is called the *input return difference matrix* and $I + L_o$ is called the *output return difference matrix.*

[1] Taken from Chap. 6, Essentials of Robust Control, by Zhou

It is easy to see that the closed-loop system, if internally stable, satisfies the following equations:

$$\begin{aligned} y &= T_o(r-n) + S_o P d_i + S_o d && (1)\\ r-y &= S_o(r-d) + T_o n - S_o P d_i && (2)\\ u &= K S_o(r-n) - K S_o d - T_i d_i && (3)\\ u_p &= K S_o(r-n) - K S_o d + S_i d_i && (4) \end{aligned}$$

These four equations show the fundamental benefits and design objectives inherent in feedback loops. For example, equation (1) shows that the effects of d on the plant output y can be made "small" by making the output sensitivity function S_o small. Similarly, equation (4) shows that the effects of d_i on the plant input u_p can be made small by making the input sensitivity function S_i small.

The notion of smallness for a transfer matrix in a certain range of frequencies can be made explicit using frequency-dependent singular values. For example, $\bar{\sigma}(S_o) < 1$ over a frequency range would mean that the effects of disturbance d at the plant output are effectively desensitized over that frequency range.

Hence, good disturbance rejection at the plant output (y) would require that

$$\begin{aligned} \overline{\sigma}(S_o) &= \overline{\sigma}\left((I+PK)\right)^{-1} = \frac{1}{\underline{\sigma}(I+PK)} && \text{(for disturbance at plant output, } d\text{)}\\ \overline{\sigma}(S_o P) &= \overline{\sigma}((I+PK)^{-1}P) = \overline{\sigma}(PS_i) && \text{(for disturbance at plant input, } d_i\text{)} \end{aligned}$$

be made small. And good disturbance rejection at the plant input (u_p) would require that

$$\begin{aligned} \overline{\sigma}(S_i) &= \overline{\sigma}((I+KP)^{-1}) = \frac{1}{\underline{\sigma}(I+KP)} && \text{(for disturbance at plant input, } d_i\text{),}\\ \overline{\sigma}(S_i K) &= \overline{\sigma}(K(I+PK)^{-1}) = \overline{\sigma}(KS_o) && \text{(for disturbance at plant output, } d\text{)} \end{aligned}$$

be made small, particularly in the low-frequency range where d and d_i are usually significant.

Note that

$$\begin{aligned} \underline{\sigma}(PK) - 1 &\le \underline{\sigma}(I+PK) \le \underline{\sigma}(PK) + 1\\ \underline{\sigma}(KP) - 1 &\le \underline{\sigma}(I+KP) \le \underline{\sigma}(KP) + 1 \end{aligned}$$

then

$$\begin{aligned} \frac{1}{\underline{\sigma}(PK)+1} \le \overline{\sigma}(S_o) \le \frac{1}{\underline{\sigma}(PK)-1}, &\text{ if } \quad \underline{\sigma}(PK) > 1\\ \frac{1}{\underline{\sigma}(KP)+1} \le \overline{\sigma}(S_i) \le \frac{1}{\underline{\sigma}(KP)-1}, &\text{ if } \quad \underline{\sigma}(KP) > 1 \end{aligned}$$

These equations imply that

$$\overline{\sigma}(S_o) \ll 1 \quad \Longleftrightarrow \quad \underline{\sigma}(PK) \gg 1 \tag{5}$$
$$\overline{\sigma}(S_i) \ll 1 \quad \Longleftrightarrow \quad \underline{\sigma}(KP) \gg 1 \tag{6}$$

Now suppose P and K are invertible:then

$$\underline{\sigma}(PK) \gg 1 \; or \; \underline{\sigma}(KP) \gg 1 \quad \Longleftrightarrow \quad \overline{\sigma}(S_oP) = \overline{\sigma}((I+PK)^{-1}P) \approx \overline{\sigma}(K^{-1}) = \frac{1}{\underline{\sigma}(K)} \tag{7}$$
$$\underline{\sigma}(PK) \gg 1 \; or \; \underline{\sigma}(KP) \gg 1 \quad \Longleftrightarrow \quad \overline{\sigma}(KS_o) = \overline{\sigma}(K(I+PK)^{-1}) \approx \overline{\sigma}(P^{-1}) = \frac{1}{\underline{\sigma}(P)} \tag{8}$$

Hence good performance at plant output (y) requires, in general,
- large output loop gain $\underline{\sigma}(L_o) = \underline{\sigma}(PK) \gg 1$ in the frequency range where d is significant for desensitizing d (from (5)), and
- large enough controller gain $\underline{\sigma}(K) \gg 1$ in the frequency range where d_i is significant for desensitizing d_i (from (7)).
Similarly, good performance at plant input (u_p) requires, in general,
- large input loop gain $\underline{\sigma}(L_i) = \underline{\sigma}(KP) \gg 1$ in the frequency range where d_i is significant for desensitizing d_i (from (6)), and
- large enough plant gain $\underline{\sigma}(P) \gg 1$ in the frequency range where d is significant, which cannot be changed by controller design, for desensitizing d (from (8)).

Remark. In general, $S_o \neq S_i$ (whereas $=$ holds for a scalar P). Hence, small $\overline{\sigma}(S_o)$ does not necessarily imply small $\overline{\sigma}(S_i)$; in other words, good disturbance rejection at the output does not necessarily mean good disturbance rejection at the plant input.

Hence, *good multivariable feedback loop design boils down to achieving high loop (and possibly controller) gain in the necessary frequency range.*

Despite the simplicity of this statement, feedback design is by no means trivial. This is true because loop gains cannot be made arbitrarily high over arbitrarily large frequency ranges. Rather, they must satisfy certain performance tradeoff and design limitations.

- commands and disturbance error reduction vs. stability under the model uncertainty
  Assume that the plant model is perturbed to $(I+\triangle)P$ with $\triangle$ stable, and assume that the system is nominally stable(i.e., the closed-loop system with $\triangle = 0$ is stable). Now the perturbed closed-loop system is stable if

  $$\begin{aligned} det(I+(I+\triangle)PK) &= det(I+PK)det(I+(I+PK)^{-1}\triangle PK) \\ &= det(I+PK)det(I+\triangle PK(I+PK)^{-1}) \\ &= det(I+PK)det(I+\triangle T_o) \end{aligned}$$

  has no right-half plane zero. Here we used the property $det(I+MN) = det(I+NM)$. This would, in general, amount to requiring that $\|\triangle T_o\|$ be small or that $\overline{\sigma}(T_o)$ be small at those frequencies where $\triangle$ is significant, typically at high-frequency range, which, in turn, implies that the loop gain, $\overline{\sigma}(L_o)$, should be small at those frequencies.

- disturbance rejection vs. the sensor noise reduction
Large $\underline{\sigma}(L_o(jw))$ values over a large frequency range make errors due to d small. However, in equation (1), they also make errors due to n large because this noise "passed through" over the same frequency range, that is,
$$y = T_o(r-n) + S_oPd_i + S_od \approx (r-n)$$
Note that n is typically significant in the high-frequency range.

- large loop gains vs. the magnitude of control
Large loop gains outside of the bandwidth of P - that is, $\underline{\sigma}(L_o(jw)) \gg 1$ or $\underline{\sigma}(L_i(jw)) \gg 1$ while $\overline{\sigma}(P(jw)) \ll 1$ - can make the control activity (u) quite unacceptable, which may cause the saturation of actuators. This follows from
$$u = KS_o(r-n-d) - T_id_i = S_iK(r-n-d) - T_id_i \approx P^{-1}(r-n-d) - d_i$$
Here, we have assumed P to be square and invertible for convenience. The resulting equation shows that disturbances and sensor noise are actually amplified at u whenever the frequency range significantly exceeds the bandwidth of P, since for w such that $\overline{\sigma}(P(jw)) \ll 1$ we have
$$\underline{\sigma}[P^{-1}(jw)] = \frac{1}{\overline{\sigma}[P(jw)]} \gg 1$$

- Similarly, the controller gain, $\overline{\sigma}(K)$, should also be kept not too large in the frequency range where the loop gain is small in order not to saturate the actuators. This is because for small loop gain $\overline{\sigma}(L_o(jw)) \ll 1 \text{ or } \overline{\sigma}(L_i(jw)) \ll 1$
$$u = KS_o(r-n-d) - T_id_i \approx K(r-n-d)$$
Therefore, it is desirable to keep $\overline{\sigma}(K)$ not too large when the loop gain is small.

To summarize the above discussion, we note that good performance requires in some frequency range, typically some low-frequency range $(0, w_l)$,
$$\underline{\sigma}(PK) \gg 1, \quad \underline{\sigma}(KP) \gg 1 \quad \underline{\sigma}(K) \gg 1$$
and good robustness and good sensor noise rejection require in some frequency range, typically some high-frequency range (w_h, ∞),
$$\overline{\sigma}(PK) \ll 1, \quad \overline{\sigma}(KP) \ll 1, \quad \overline{\sigma}(K) \leq M$$
where M is not too large. These design requirements are shown graphically in Fig. 2. The specific frequencies w_l and w_h depend on the specific applications and the knowledge one has of the disturbance characteristics, the modeling uncertainties, and the sensor noise levels.

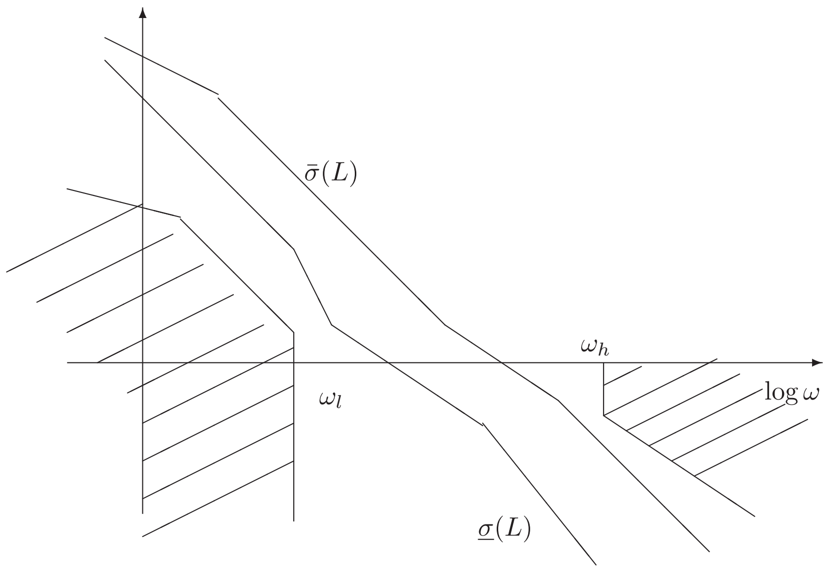

Figure 2: Desired loop gain

9.2 Weighted Performance

We saw that the performance objectives of a feedback sys can be specified by conditions on S or T, etc. Suppose that we have the following condition on a scalar system:

$$|S(j\omega)| \quad \leq \quad \begin{cases} \epsilon & \forall \omega \leq \omega_0 \\ M & \forall \omega > \omega_0 \end{cases}$$

where $S(j\omega) = \frac{1}{1+P(j\omega)K(j\omega)}$.
Equivalently,

$$|W_e(j\omega)S(j\omega)| \leq 1 \quad \forall \omega$$

where the weighting function (usually a rational transfer ftn) satisfies

$$|W_e(j\omega)| \quad = \quad \begin{cases} 1/\epsilon & \forall \omega \leq \omega_0 \\ 1/M & \forall \omega > \omega_0 \end{cases}$$

This idea is generalized to weighted performance specifications in MIMO control design. Advantage of using weighted performance includes:

- Some components of a vector signal might be more important than others.
- Each component might be in different units or orders of magnitude.
- We might be interested in error rejection in low freq range only, and interested in other objectives in other freq range, etc.

For example, look at the figure 3. Weighting ftns can be chosen to reflect the design objectives and knowledge of the disturbances and sensor noise, expected system inputs, and the relative importance of the outputs. Then, we choose a contoller K such that certain weighted signals are made small in some sense, such as H_2 or H_∞.

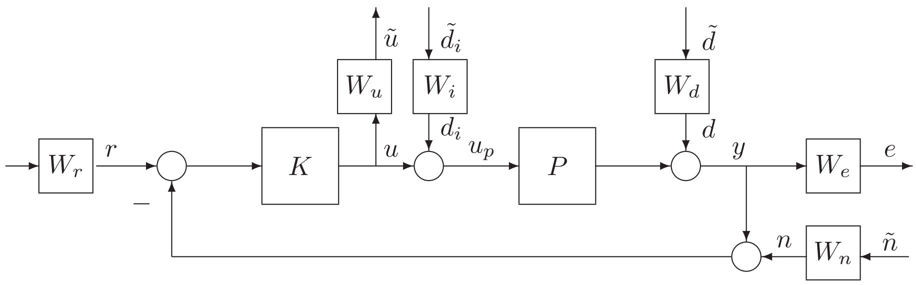

Figure 3: Closed-loop system with weighting blocks

$\mathcal{H}_2$ **Performance.**

For simplicity, assume $d_i = 0n = 0$, and $\tilde{d}(t) = \eta\delta(t)$ with $E\eta\eta^* = I$, i.e. impulse noise with random input direction.

- Want to minimize the expected energy of the error due to $\tilde{d}$:

$$E||e||_2^2 = E\int_0^\infty ||e||^2 dt = ||W_e S_o W_d||_2^2$$

- Don't want to saturate actuators:

$$E||\tilde{u}||_2^2 = ||W_u K S_o W_d||_2^2$$

Thus, our perf criterion would look something like

$$E[||e||_2^2 + \rho||\tilde{u}||_2^2] = \left\| \begin{bmatrix} W_e S_o W_d \\ \sqrt{\rho} W_u K S_o W_d \end{bmatrix} \right\|_2^2$$

$\rho :$ tradeoff btween disturbance rejection at the output and control effort.

But remember that LQG formulation allows only additive noise, and the plant uncertainty cannot be formally addressed.

$\mathcal{H}_\infty$ **Performance.**

- Want the tolerance to uncertainties by limiting high loop gains. e.g. minimize

$$\sup_{||\tilde{d}||_2 \leq 1} ||e||_2 = ||W_e S_o W_d||_\infty$$

subject to some restrictions on

$$\sup_{||\tilde{u}||_2 \leq 1} ||\tilde{u}||_2 = ||W_u K S_o W_d||_\infty .$$

Thus,

$$\sup_{||\tilde{d}||_2 \leq 1} \{||e||_2^2 + \rho||\tilde{u}||_2^2\} = \left|\left| \begin{bmatrix} W_e S_o W_d \\ \sqrt{\rho} W_u K S_o W_d \end{bmatrix} \right|\right|_\infty^2$$

- Or, want to limit the weighted complementary sensitivity function:

$$\left|\left| \begin{bmatrix} W_e S_o W_d \\ \sqrt{\rho} W_1 T_o W_2 \end{bmatrix} \right|\right|_\infty .$$

9.3 Selection of Weighting Functions

Selection of weighting functions is very important and it is not trivial. Let's consider a 2nd order SISO example.

$$\begin{aligned} L &= PK = \frac{\omega_n^2}{s(s + 2\xi\omega_n)} \\ S &= \frac{1}{1+L} = \frac{s(s+2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2} \end{aligned}$$

and recall that

- speed of the response $\leftrightsquigarrow \omega_n$
- overshoot of the response $\leftrightsquigarrow \xi$

How to choose our weighting functions?

- Recall the tracking error Eq. (2) :

$$r - y = S(r-d) + Tn - SPd_i .$$

  We want to keep $|S|$ small over a frequency range where r and d are significant. How to choose W_e?

- Recall Eq. (3) :

$$u = KS(r - n - d) - Td_i \ .$$

  We want $|KS|$ to roll off as fast as possible beyond the desired control bandwidth so that the high-freq noises are attenuated as much as possible. How to choose W_u?