11 μ Analysis

11.1 Problem setting

We now know that any interconnection can be rearranged to fit the general framework shown in Fig. 1. The system labeled P is the open-loop interconnection and contains all of the known elements including the nominal plant model and performance/unvertainty weighting ftns. The Δ block is the uncertain element from the set $\mathbf{\Delta}$, and K is the controller.

The set of systems to be controlled is described by the LFT:

$$\{F_U(P,\Delta):\ \Delta \in \mathbf{\Delta}, \max_{\omega} \bar{\sigma}(\Delta(j\omega)) \leq 1\}\ .$$

And the design objective is to find a stabilizing controller K such that for all such perturbations Δ, the CL sys is stable and satisfies

$$||F_L(\underbrace{F_U(P,\Delta)}_{\text{perturbed plant}}, K)||_\infty \leq 1\ .$$

Since $F_L(F_U(P,\Delta),K) = F_U(F_L(P,K),\Delta)$, the design objective is to find a nominally stabilizing controller K such that for all $\Delta \in \mathbf{\Delta}$, $\max_\omega(\Delta) \leq 1$, the CL sys is stable and satisfies

$$||F_U(F_L(P,K),\Delta)||_\infty \leq 1\ .$$

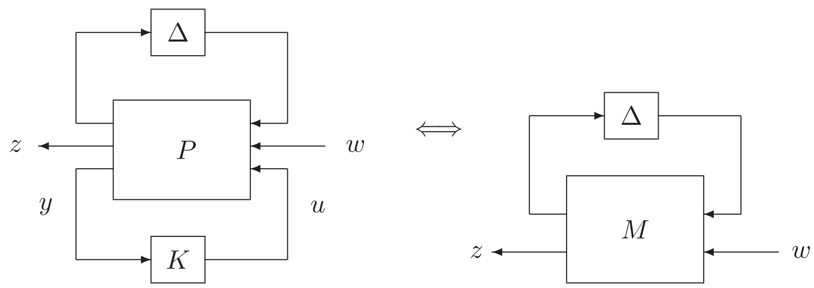

Figure 1: General Framework

Denote

$$M(s) = F_L(P,K) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

then the general framework reduces to the figure on the right in Fig. 1, where

$$z = F_u(M,\Delta)w = [M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}]w.$$

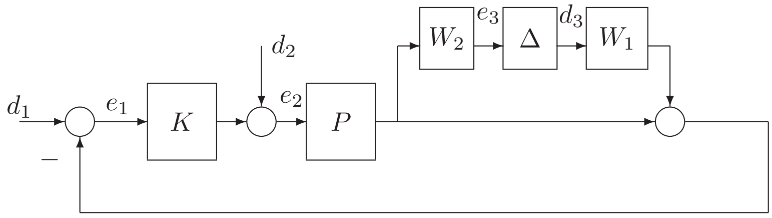

Figure 2: Feedback control for multiplicative perturbed systems

Now, let

$$w := \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad z := \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

Then the system is robustly stable for all $\Delta(s) \in \mathcal{RH}_\infty$ with $||\Delta||_\infty < 1$ iff $F_u(M, \Delta) \in \mathcal{RH}_\infty$ for all admissible Δ with $M_{11} = -W_2PK(I + PK)^{-1}W_1$, which is guaranteed by $||M_{11}||_\infty \le 1$.

If a system is built from uncertain components, then, in general, the uncertainty in the system level is structured, involving a large number of real parameters. The exact stability analysis for real parameters is a bit more difficult, so we will begin with norm-bounded dynamical uncertainty. Since the interconnection model M can always be chosen so that $\Delta(s)$ is block diagonal and $||\Delta||_\infty < 1$ by absorbing any weights, we shall assume that $\Delta(s)$ takes the form

$$\Delta(s) = \{ \operatorname{diag}[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_F] : \delta_i(s) \in \mathcal{RH}_\infty, \ \Delta_j \in \mathcal{RH}_\infty\}$$

with $||\delta_i||_\infty < 1, \ ||\Delta_j||_\infty < 1$.

Note that

1. $||M_{11}||_\infty \le 1$ implies stability, but not conversely. This test ignores the known block diagonal structure of the uncertainties, and is equivalent to regarding Δ as unstructured. This can be arbitrary conservative since stable systems can have arbitrarily large $||M_{11}||_\infty$.
2. Testing each $\delta_i(\Delta_j)$ individually (assuming no uncertainty in other channels) can be arbitrarily optimistic, because it ignores interaction between $\delta_i(\Delta_j)$'s.

And the difference between the stability margins obtained by the above two approaches can be arbitrarily far apart. To analyze the exact robust stability/performance for structured uncertainty, we define a matrix function called "structured singular value".

11.2 Structured Singular Value

Let $u = Mv$, $v = \Delta u$, $\Delta \in \boldsymbol{\Delta}$. Then $(I - M\Delta)u = 0$. Under what condition there exist nonzero u, v that satisfy this relationship? When $\det(I - M\Delta) = 0$, we can find a $\Delta \in \mathbf{\Delta}$ such that nontrivial solutions exist, and in fact lots of such (u, v) exist, which can be arbitrarily large.

Given a matrix $M \in \mathbb{C}^{p\times q}$, what is the smallest perturbation matrix $\Delta \in \mathbb{C}^{q\times p}$ such that

$$\det(I - M\Delta) = 0 \ ?$$

i.e. what is

$$\inf\{\bar{\sigma}(\Delta) : \ \det(I - M\Delta)\} = 0 \ ?$$

Definition 11.1 *For $M \in \mathbb{C}^{n\times n}$, $\mu_\Delta(M)$ is defined as*

$$\mu_\Delta(M) \triangleq \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $(I - M\Delta)$ singular, in which case $\mu_\Delta(M) \triangleq 0$.

Note that, if $\mathbf{\Delta_1} \subseteq \mathbf{\Delta_2}$, then

$$\mu_{\Delta_1}(M) \le \mu_{\Delta_2}(M) \ .$$

Lemma 11.2 *For $\mathbf{\Delta} = \mathbb{C}^{n\times n}$, $\mu_\Delta(M) = \bar{\sigma}(M)$.*

Proof.
Suppose $M = O_{n\times n}$, then no Δ works, so $\mu_\Delta(M) = 0 = \bar{\sigma}(O_{n\times n})$.

Suppose $M \neq O_{n\times n}$, $\bar{\sigma}(M) > 0$. Pick any $\Delta \in \mathbb{C}^{n\times n}$ s.t. $\bar{\sigma}(\Delta) < \frac{1}{\bar{\sigma}(M)}$. Then

$$\bar{\sigma}(M\Delta) \le \bar{\sigma}(M)\bar{\sigma}(\Delta) < 1 \quad \Rightarrow \quad \det(I - M\Delta) \neq 0.$$

So, to make $(I - M\Delta)$ singular, we need

$$\min\{\bar{\sigma}(\Delta) : \ \Delta \in \mathbf{\Delta}, \ \det(I - M\Delta) = 0\} \ \ge \ \frac{1}{\bar{\sigma}(M)}$$

i.e.

$$\mu_\Delta(M) \le \bar{\sigma}(M). \qquad (\star)$$

By definition, we can find unit vectors $u, v \in \mathbb{C}^n$ such that

$$Mv = \bar{\sigma}(M)u \ .$$

Let

$$\Delta := \frac{1}{\bar{\sigma}(M)} vu^*$$

Then $\bar{\sigma}(\Delta) = \frac{1}{\bar{\sigma}(M)}$, and

$$(I - M\Delta)u = u - M\frac{v}{\bar{\sigma}(M)} = u - u = 0$$

so

$$\min\{\bar{\sigma}(\Delta):\ \Delta \in \boldsymbol{\Delta},\ I - M\Delta \text{ singular}\} \leq \frac{1}{\bar{\sigma}(M)}\ ,$$

thus,

$$\mu_\Delta(M) \geq \bar{\sigma}(M)\ . \qquad (\star\star)$$

From $(\star)$ and $(\star\star)$, we have

$$\mu_\Delta(M) = \bar{\sigma}(M)\ . \qquad \square$$

Lemma 11.3 *For* $\boldsymbol{\Delta} = \{\delta I_n :\ \delta \in \mathbb{C}\}$, $\mu_\Delta(M) = \rho(M)$, *where* ρ *denotes the spectral radius, i.e. the largest, in magnitude, eigenvalue.*

Proof. The only $\delta' s$ that can make $I - M\Delta$ singular is the reciprocals of nonzero eigvals of M, i.e. $\delta \neq 0, \frac{1}{\delta} \in \{\lambda_i(M)\}$. The minimum $|\delta|$ is associated with the largest (in magnitude) eigval of M. Thus $\mu_\Delta(M) = \rho(M)$. $\square$

Corollary 11.4 *For* $\boldsymbol{\Delta} = \{\delta I_n :\ \delta \in \mathbb{R}\}$, $\mu_\Delta(M) = \rho_{\mathbb{R}}(M)$, *where* $\rho_{\mathbb{R}}$ *denotes the largest, in magnitude, real eigenvalue (*$\mu_\Delta(M) = 0$ *if* M *has no real eigvalues).*

From these, we conclude that, for

$$\{\delta I_n :\ \delta \in \mathbb{R}\} \subseteq \boldsymbol{\Delta} \subseteq \{\Delta :\ \Delta \in \mathbb{C}^{n\times n}\}\ ,$$

$$\rho_{\mathbb{R}}(M) \leq \mu_\Delta(M) \leq \bar{\sigma}(M)\ .$$

But the gap between these two can be arbitrarily large. Also note that $\rho_{\mathbb{R}}$ is not a continuous function of a matrix, and μ itself can be discontinuous.

Lemma 11.5 *Let* $\boldsymbol{\Delta_1} \subseteq \mathbb{C}^{n_1\times n_1}$, $\boldsymbol{\Delta_2} \subseteq \mathbb{C}^{n_1\times n_1}$, $M \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)}$, *and*

$$\boldsymbol{\Delta} := \{\text{diag}[\Delta_1, \Delta_2] :\ \Delta_i \in \boldsymbol{\Delta_i}\}.$$

Then

$$\mu_{\boldsymbol{\Delta}}(M) \geq \max(\mu_{\boldsymbol{\Delta}_1}(M_{11}), \mu_{\boldsymbol{\Delta}_2}(M_{22}))\ .$$

Proof. Suppose that $\Delta_1 \in \boldsymbol{\Delta_1}$ causes $\det(I - M_{11}\Delta_1) = 0$. Then

$$\Delta := \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} \in \boldsymbol{\Delta}\ ,$$

and

$$\det(I - M\Delta) = \det \begin{bmatrix} I - M_{11}\Delta_1 & 0 \\ -M_{21}\Delta_1 & 0 \end{bmatrix} = 0$$

Therefore,

$$\min\{\bar\sigma(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta)\} \;\le\; \min\{\bar\sigma(\Delta_1) : \Delta_1 \in \mathbf{\Delta}_1, \det(I - M_{11}\Delta_1) = 0\}$$

and

$$\mu_{\mathbf{\Delta}}(M) \ge \mu_{\mathbf{\Delta}_1}(M_{11}) \ .$$

The following can be proved by the same reasoning:

$$\mu_{\mathbf{\Delta}}(M) \ge \mu_{\mathbf{\Delta}_2}(M_{22}) \ . \qquad \Box$$

Theorem 11.6 (Main Loop Theorem) *The following are equivalent:*

$$\mu_{\mathbf{\Delta}}(M) < 1 \quad \Leftrightarrow \quad \begin{cases} \mu_{\mathbf{\Delta}_2}(M_{22}) < 1 \ , \ \textit{and} \\ \max_{\Delta_2 \in \mathbf{\Delta}_2, \ \bar\sigma(\Delta_2) \le 1} \mu_{\mathbf{\Delta}_1}(F_L(M, \Delta_2)) < 1 \end{cases}$$

Proof. ($\Rightarrow$)From the previous lemma,

$$\mu_{\mathbf{\Delta}_2}(M_{22}) \le \mu_{\mathbf{\Delta}}(M) < 1 \ .$$

Now take any $\Delta_2 \in \mathbf{B}_2 := \{\Delta_2 \in \mathbf{\Delta}_2 : \bar\sigma(\Delta_2) \le 1\}$, then $I - M_{22}\Delta_2$ is nonsingular. Now take any $\Delta_1 \in \mathbf{B}_1 := \{\Delta_1 \in \mathbf{\Delta}_1 : \bar\sigma(\Delta_1) \le 1\}$, and let

$$\Delta = \mathrm{diag}[\Delta_1, \Delta_2] \ .$$

Then $\Delta \in \mathbf{\Delta}$, and

$$\bar\sigma(\Delta) = \max(\bar\sigma(\Delta_1), \bar\sigma(\Delta_2)) \le 1 \ .$$

Now using

$$\begin{aligned} \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det D \det(A - BD^{-1}C) \\ &= \det A \det(D - CA^{-1}B) \ , \end{aligned}$$

observe that

$$\begin{aligned} \det(I - M\Delta) &= \det \begin{bmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ -M_{21}\Delta_1 & I - M_{22}\Delta_2 \end{bmatrix} \\ &= \det(I - M_{22}\Delta_2) \det(I - M_{11}\Delta_1 - M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21}\Delta_1) \\ &\ne 0 \ . \end{aligned}$$

Therefore,

$$\det(I - M_{11}\Delta_1 - M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21}\Delta_1) \;=\; \det(I - F_L(M, \Delta_2)\Delta_1) \ne 0 \ ,$$

which leads to

$$\max_{\Delta_2 \in \mathbf{\Delta}_2, \ \bar\sigma(\Delta_2) \le 1} \mu_{\mathbf{\Delta}_1}(F_L(M, \Delta_2)) < 1 \ .$$

And the reverse direction can be proved using the same reasoning. □

From the proof of the main loop theorem, we can also see that the following are equivalent:

$$\mu_{\mathbf{\Delta}}(M) < 1 \quad \Leftrightarrow \quad \begin{cases} \mu_{\mathbf{\Delta}_1}(M_{11}) < 1 \text{ , and} \\ \max_{\Delta_1 \in \mathbf{\Delta}_1,\, \bar{\sigma}(\Delta_1) \leq 1} \mu_{\mathbf{\Delta}_2}(F_L(M, \Delta_1)) < 1 \end{cases} .$$

The main loop theorem forms the basis of μ analysis, whether from a state-space, frequency domain perspective, or a Lyapunov approach. Here $\mathbf{\Delta}_2$ is the structure that the perturbations come from. And $\mathbf{\Delta}_1$ defines a particular property of $F_L(M, \Delta_2)$ being considered. This will become clear with following example.

Example. Let $\mathbf{\Delta} := \{ \text{diag}[\delta_1 I_n, \Delta_2] : \delta_1 \in \mathbb{C}, \Delta_2 \in \mathbb{C}^{m \times m}\}$. And let $M \in \mathbb{C}^{(n+m)\times(n+m)}$

$$M(s) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} .$$

be a state-space matrix that represents a discrete time system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \, . \end{aligned}$$

Then $\mu_1(A) = \rho(A)$ and $\mu_2(D) = \bar{\sigma}(D)$. Applying the small gain thm yields that the following are equivalent:

1. $\rho(A) < 1$, and
$$\max_{\delta_1 \in \mathbb{C},\, |\delta_1| \leq 1} \bar{\sigma}(D + C\delta_1(I - A\delta_1)^{-1}B) < 1 \, .$$

2. $\bar{\sigma}(D) < 1$, and
$$\max_{\Delta_2 \in \mathbb{C}^{m\times m},\, \sigma(\Delta_2) \leq 1} \rho(A + B\Delta_2(I - D\Delta_2)^{-1}C) < 1 \, .$$

3. $\mu_\Delta(M) < 1$.

The above conditions can be interpreted as the following:

- (By replacing δ_1 by $1/z$ in 1)
  The system is stable, and the $||\cdot||_\infty$ of the transfer ftn from u to y is less than 1:
$$||G||_\infty := \max_{z \in \mathbb{C},\, |z| \geq 1} \bar{\sigma}(D + C(zI - A)^{-1}B) < 1 \, .$$

- 2 implies that $(I - D\Delta_2)^{-1}$ is well-defined for all $\bar{\sigma}(\Delta_2) \leq 1$, and that a robust stability result holds for the uncertain discrete-time system
$$x_{k+1} = (A + B\Delta_2(I - D\Delta_2)^{-1}C)x_k \, .$$

- $\mu_\Delta(M) < 1$.

Here, we see that important properties of linear systems such as robust stability and input-output gains are equivalent to a condition involving the structured singular value.

11.3 Bounds

It was mentioned that

$$\rho_{\mathbb{R}}(M) \le \mu_{\mathbf{\Delta}}(M) \le \bar{\sigma}(M) \ .$$

these bounds alone are not sufficient for our purpose. They can be refined by considering transformations on M that do not affect $\mu_\Delta(M)$, but affect ρ and $\bar{\sigma}$. To do this, let

$$\mathbf{\Delta} := \ \{\mathrm{diag}[\delta_1 I_{r_1}, \cdots, \delta_S I_{r_S}, \Delta_1, \cdots, \Delta_F] : \ \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}\}$$

denote our uncertainty structure and define the following subset of $\mathbb{C}^{n\times n}$:

$$\mathbf{D}_\Delta := \ \{\mathrm{diag}[D_1, \cdots, D_S, d_1 I_{m_1}, \cdots, d_F I_{m_F}] : D_i \in \mathbb{C}^{r_i \times r_i}, D_i = D_i^* > 0, d_j \in \mathbb{R}, d_j > 0\} \ .$$

Take $\Delta \in \mathbf{\Delta}, D \in \mathbf{D}_\Delta$. Then $D\Delta = \Delta D$, i.e. $\Delta = D^{-1}\Delta D$.
Suppose that M and $D \in \mathbf{D}_\Delta$ are given. For any $\Delta \in \mathbf{\Delta}$,

$$\det(I - M\Delta) = \det(I - MD^{-1}\Delta D) = \det(I - DMD^{-1}\Delta) \ ,$$

thus

$$\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1}) \ .$$

And

$$\mu_{\mathbf{\Delta}}(DMD^{-1}) \le \bar{\sigma}(DMD^{-1}) \ .$$

Since this holds for any $D \in \mathbf{D}_\Delta$, we have the following upper bound:

$$\mu_{\mathbf{\Delta}}(M) \le \inf_{D \in \mathbf{D}_\Delta} \bar{\sigma}(DMD^{-1}) \ . \qquad (1)$$

The upper bound can be formulated as a convex optimization problem, so the global minimum can be found. But the upper bound is not always equal to μ.

Theorem 11.7

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathbf{D}_\Delta} \bar{\sigma}(DMD^{-1}) \ \ \textit{if} \ \ 2S + F \le 3.$$

For block structures with $2S + F > 3$, there exist matrices for which μ is less than the infimum(Packard and Doyle, Automatica, 1993).

The following can be proved for the lower bound:

Theorem 11.8

$$\mu_{\boldsymbol{\Delta}}(M) = \max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \rho(M\Delta).$$

Proof. Suppose that $\det(I - M\Delta) = 0$, then $M\Delta$ has an eval 1, and $M\Delta/\bar{\sigma}(\Delta)$ has an eval $\bar{\sigma}(\Delta)$, thus,

$$\rho\left(M\frac{\Delta}{\bar{\sigma}(\Delta)}\right) \geq \frac{1}{\bar{\sigma}(\Delta)} \ .$$

Let $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$ such that $\det(I - M\Delta) = 0$ and $\bar{\sigma}(\Delta) = 1/\mu_{\boldsymbol{\Delta}}(M)$. Then,

$$\rho\left(M\frac{\Delta}{\bar{\sigma}(\Delta)}\right) \geq \frac{1}{\bar{\sigma}(\Delta)} = \mu_{\boldsymbol{\Delta}}(M) \ .$$

Since $\frac{\Delta}{\bar{\sigma}(\Delta)} \in \mathbf{B}_{\boldsymbol{\Delta}}$,

$$\max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \rho(M\Delta) \geq \mu_{\boldsymbol{\Delta}}(M) \tag{2}$$

Now, let $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$ such that $\rho(M\Delta) \geq \mu_{\boldsymbol{\Delta}}(M) := \beta$. Then $M\Delta$ has an eval $\beta e^{j\theta}$, and $M\Delta/\beta e^{j\theta}$ has an eval 1. Thus $\det(I - M\Delta/\beta e^{j\theta}) = 0$. Since , and $\Delta/\beta e^{j\theta} \in \boldsymbol{\Delta}$ and $\bar{\sigma}(\Delta/\beta e^{j\theta}) \leq 1/\beta$,

$$\min\{\bar{\sigma}(\Delta) : \Delta \in \boldsymbol{\Delta}, \det(I - M\Delta) = 0\} \leq 1/\beta \ . \tag{3}$$

(2) and (3) prove this theorem. □

This lower bound can be refined by the following theorem (Doyle,IEEE CDC, 1982):

Theorem 11.9

$$\max_{Q \in \mathbf{Q}_{\Delta}} \rho(QM) = \mu_{\boldsymbol{\Delta}}(M), \tag{4}$$

where $\mathbf{Q}_{\Delta} = \{\Delta \in \boldsymbol{\Delta} : \ \Delta^*\Delta = I_n\}$.

Unfortunately, $\rho(\cdot)$ can have multiple local maxima, so local search cannot be guaranteed to obtain μ, but can only yield a lower bound. Combining (1) and (4), we obtain

$$\max_{Q \in \mathbf{Q}_{\Delta}} \rho(QM) \ \leq \ \mu_{\boldsymbol{\Delta}}(M) \ \leq \ \inf_{D \in \mathbf{D}_{\Delta}} \bar{\sigma}(DMD^{-1}) \ .$$

Matlab commands `mu` uses a slightly different formulation of the lower bound.
`[bnds,rowd,sens,rowp,rowg] = mu(matin,blk,opt)`
MATIN - input matrix, CONSTANT/VARYING matrix M
BLK - block structure information about Δ
OPT - optional argument for selecting the computation algorithm
BNDS - upper and lower bounds
SENS - sensitivity of $||DMD^{-1}||$ to D scaling
ROWP - perturbation from lower bound
ROWD - D scaling from upper bound
ROWG - G scaling from upper bound

11.4 Structured Robust Stability

How can we use μ for the stability test of a nominally stable system under structured perturbations?

Theorem 11.10 (Robust Stability Theorem) *Let $\beta > 0$. Then the interconnected system shown in Fig. 3 is well-posed and internally stable for all $\Delta(\cdot) \in \mathbf{\Delta}$ with $||\Delta||_\infty < 1/\beta$ iff*

$$\sup_{\omega \in \mathbb{R}} \mu_\Delta(G(j\omega)) \leq \beta \ .$$

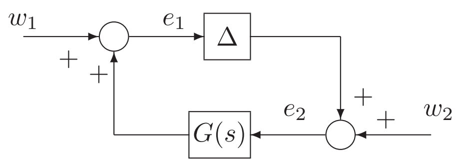

Figure 3: $G - \Delta$ interconnection

Robust stability analysis using μ consists of the following steps:

1. Recast the problem into the feedback loop (i.e. $\Delta - G$ LFT, where G is a known linear sys, and Δ is a structured perturbation).
2. Calculate a frequency response of G.
3. Describe the structure of the perturbation set Δ.
4. Compute $\mu_\Delta(G(j\omega))$ and plot the bounds obtained from the μ calculation.

Let

- β_u := peak (across frequency) of the upper bound of $\mu_\Delta(G(j\omega))$
- β_l := peak (across frequency) of the lower bound of $\mu_\Delta(G(j\omega))$

Then

- for all $\Delta \in \mathbf{\Delta}$ with $\max_\omega \bar{\sigma}(\Delta(j\omega)) < 1/\beta_u$, the perturbed system is stable.
- Moreover, there *is* a particular perturbation satisfying $\Delta \in \mathbf{\Delta}$ with $\max_\omega \bar{\sigma}(\Delta(j\omega)) = 1/\beta_l$, that causes instability.

11.5 Robust Performance

We assume that good performance is equivalent to

$$||T||_\infty := \max_{\omega \in \mathbb{R}} \bar{\sigma}(T(j\omega)) \ \leq 1 ,$$

where T is some weighted CL transfer matrix. In the case of robust performance of uncertain systems, we will take T to be the uncertain transfer ftn from d to e, i.e. $T = F_U(G, \Delta)$ in Fig. 4(b).

From small gain theorem, we know that $||T||_\infty \leq 1$ iff the feedback loop shown in Fig. 4(a) is stable for every stable $\Delta_f(s)$ of the suitable dimension with $||\Delta_f(s)||_\infty \leq 1$. Therefore, we have

$$||F_U(G, \Delta)||_\infty \leq 1 \text{ for all } \Delta \in \boldsymbol{\Delta} \text{ such that } ||\Delta||_\infty < 1$$

iff the LFT shown in Fig. 4(a) is stable for all $\Delta \in \boldsymbol{\Delta}$ and all stable Δ_f such that $||\Delta||_\infty < 1$, $||\Delta_f||_\infty < 1$. But this is exactly a robust stability problem for G on a larger setting (i.e. with the augmented uncertainty) !

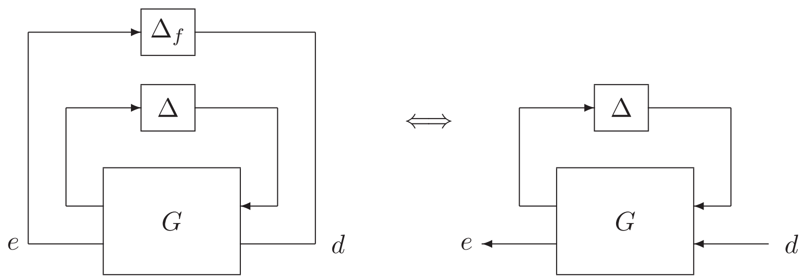

Figure 4: (a) Robust Stability test vs. (b) Robust Performance test

Let G is a stable, real-rational, proper transfer ftn with $q_1 + q_2$ inputs and $p_1 + p_2$ outputs, partitioned as

$$G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} .$$

Let $\Delta \subset \mathbb{C}^{q_1 \times p_1}$ be our block structure under consideration. Define an augmented block structure

$$\boldsymbol{\Delta}_P := \{\text{diag}[\Delta, \Delta_f] : \ \Delta \in \boldsymbol{\Delta}, \ \Delta_f \in \mathbb{C}^{q_2 \times p_2}\} .$$

Theorem 11.11 (Robust Performance as Robust Stability) *Let $\beta > 0$. Then the interconnected system shown in Fig. 3 is well-posed and internally stable, and*

$$||F_U(G, \Delta)||_\infty \ \leq \beta$$

for all $\Delta \in \boldsymbol{\Delta}$ with $||\Delta||_\infty < 1/\beta$, iff

$$\sup_{\omega \in \mathbb{R}} \ \mu_{\boldsymbol{\Delta}_P}(G(j\omega)) \leq \beta .$$

This is a very useful theorem. It says that
we can carry RP analysis by introducing a fictitious uncertainty block across the disturbance/error channels and carrying out a RS analysis.

Robust performance analysis using μ consists of the following steps:

1. Recast the problem into the feedback loop (i.e. $\Delta - G$ LFT).
2. Calculate a frequency response of G.
3. Describe the structure of the perturbation set Δ.
4. Augment Δ with a fictitious uncertainty block Δ_f to generate an extended uncertainty set Δ_P.
5. Compute $\mu_{\Delta_P}(G(j\omega))$ and plot the bounds obtained from the μ calculation.

Let β denote the peak value of the μ-plot, i.e.,

$$\max_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}_P}(G(j\omega)) \; := \beta.$$

Then

- for all $\Delta \in \mathbf{\Delta}$ with $\max_\omega \bar{\sigma}(\Delta(j\omega)) < 1/\beta$,
  the perturbed system is stable and $||F_U(G,\Delta)||_\infty \; \leq \beta$.
- Moreover, there *is* a particular perturbation satisfying $\Delta \in \mathbf{\Delta}$ with $\max_\omega \bar{\sigma}(\Delta(j\omega)) = 1/\beta$, that causes either $||F_U(G,\Delta)||_\infty \; = \beta$ or instability.

Unfortunately, exact computation of μ is not possible, so what can we say using bounds?

Let

- $\beta_u :=$ peak (across frequency) of the upper bound of $\mu_{\Delta_P}(G(j\omega))$
- $\beta_l :=$ peak (across frequency) of the lower bound of $\mu_{\Delta_P}(G(j\omega))$

Then

- for all $\Delta \in \mathbf{\Delta}$ with $\max_\omega \bar{\sigma}(\Delta(j\omega)) < 1/\beta_u$,
  the perturbed system is stable and $||F_U(G,\Delta)||_\infty \; \leq \beta_u$.
- Moreover, there *is* a particular perturbation satisfying $\Delta \in \mathbf{\Delta}$ with $\max_\omega \bar{\sigma}(\Delta(j\omega)) = 1/\beta_l$, that causes either $||F_U(G,\Delta)||_\infty \; = \beta_l$ or instability.

Hence the gap between the upper and lower bounds of μ leads to gaps in the inability to precisely determine the robustness.