Linear Matrix Inequality (LMI)

A linear matrix inequality is an expression of the form

$$F(x) \triangleq F_0 + x_1F_1 + \cdots + x_mF_m > 0 \qquad (1)$$

where

- $x = (x_1, \cdots, x_m) \in \Re^m$,
- $F_0, \cdots, F_m$ are real symmetric matrices, and
- the inequality > 0 in (1) means positive definite, i.e., $u^TF(x)u > 0$ for all $u \in \Re^n$, $u \neq 0$. Equivalently, the smallest eigenvalue of $F(x)$ is positive.

Definition[Linear matrix inequality(LMI)]
A linear matrix inequality is

$$F(x) > 0 \tag{2}$$

where F is an affine function mapping a finite dimensional vector space to the set $\mathbb{S}^n \triangleq \{M : M = M^T \in \Re^{n\times n}\}$, $n > 0$, of real matrices.

remark Recall, from definition, that an affine mapping $F : \mathbb{V} \to \mathbb{S}^n$ necessarily takes the form $F(x) = F_0 + T(x)$ where $F_0 \in \mathbb{S}^n$ and $T : \mathbb{V} \to \mathbb{S}^n$ is a linear transformation. Thus if $\mathbb{V}$ is of dimension m, and $\{e_1, \cdots, e_m\}$ constitutes a basis for $\mathbb{V}$, then we can write

$$T(x) = \sum_{j=1}^{m} x_j F_j$$

where the elements $\{x_1, \cdots, x_m\}$ are such that $x = \sum_{j=1}^{m} x_j e_j$ and $F_j = T(e_j)$ for $j = 1, \cdots, m$. Hence we obtain (1) as a special case.

Remark. The same remark applies to mappings $F : \Re^{m_1 \times m_2} \to \mathbb{S}^n$ where $m_1, m_2 \in \mathbb{Z}^+$. A simple example where $m_1 = m_2$ is the Lyapunov inequality

$$F(X) = A^T X + XA + Q > 0 \ .$$

Here, $A, Q \in \Re^{m \times m}$ are assumed to be given, Q is symmetric, and $X \in \Re^{m \times m}$ is the unknown *matrix*.

In this case, the domain $\mathbb{V}$ of F in definition is equal to $\mathbb{S}^m$. We can view this LMI as a special case of (1) by defining a basis $E_1, \cdots, E_m$ of $\mathbb{S}^m$ and writing $X = \sum_{j=1}^m x_j E_j$:

$$\begin{aligned} F(X) = F\left(\sum_{j=1}^m x_j E_j\right) &= F_0 + \sum_{j=1}^m x_j F(E_j) \\ &= F_0 + \sum_{j=1}^m x_j F_j \end{aligned}$$

which is of the form (1).

Remark. The LMI

$$F(x) = F_0 + xF_1 + \cdots + x_m F_m$$

defines a *convex constraint* on $x = (x_1, \cdots, x_m)$. i.e., the set

$$\mathcal{F} \triangleq \{x : F(x) > 0\}$$

is convex. Indeed, if $x_1, x_2 \in \mathcal{F}$ and $\alpha \in (0, 1)$ then

$$F(\alpha x_1 + (1-\alpha)x_2) = \alpha F(x_1) + (1-\alpha)F(x_2) > 0$$

Convexity has an important consequence: even though the LMI has no analytical solution in general, it can be solved numerically with guarantees of fining a solution when one exists. Although the LMI may seem special, it turns out that many convex sets can be represented in this way.

1. Note that a system of LMIs (i.e. a finite set of LMIs) can be written as a single LMI since

$$\left\{ \begin{array}{c} F_1(x) < 0 \\ \vdots \\ F_K(x) < 0 \end{array} \right\} \text{ is equivalent to}$$

$$F(x) \triangleq \text{diag}[F_1(x), \cdots, F_K(x)] < 0$$

2. Combined constraints (in the unknown x) of the form

$$\left\{ \begin{array}{l} F(x) > 0 \\ Ax = b \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} F(x) > 0 \\ x = Ay + b \;\; \text{for some } y \end{array} \right.$$

where the affine function $F : \Re^m \to \mathbb{S}^n$ and matrices $A \in \Re^{n\times m}$ and $b \in \Re^n$ are given can be lumped into one LMI. More generally, the combined equations

$$\left\{ \begin{array}{l} F(x) > 0 \\ x \in \mathcal{M} \end{array} \right. \tag{3}$$

where $\mathcal{M}$ is an affine subset of $\Re^n$, i.e.

$$\mathcal{M} = x_0 + \mathcal{M}_0 = \{x_0 + m \mid m \in \mathcal{M}_0\}$$

with $x_0 \in \Re^n$ and $\mathcal{M}_0$ a linear subspace of $\Re^n$, can be written in the form of one single LMI. In order to see this, let $e_1, \cdots, e_k \in \Re^n$ be a basis of $\mathcal{M}_0$ and let $F(x) = F_0 + T(x)$ be decomposed as in remark. Then (3) can be rewritten as

$$\begin{aligned}
0 < F(x) &= F_0 + T(x_0 + \sum_{j=1}^{k} x_j e_j) \\
&= \underbrace{F_0 + T(x_0)}_{\text{constant part}} + \underbrace{\sum_{j=1}^{k} x_j T(e_j)}_{\text{linear part}} \\
&= \bar{F}_0 + x_1 \bar{F}_1 + \ldots + x_k \bar{F}_k \\
&\triangleq \bar{F}(\bar{x})
\end{aligned}$$

where $\bar{F}_0 = F_0 + T(x_0)$, $\bar{F}_j = T(e_j)$ and $x = (x_1, \cdots, x_k)$. This implies that $x \in \Re^n$ satisfies (3) if and only if $F(x) > 0$. Note that the dimension of $\bar{x}$ is smaller than the dimension of x.

3. (Schur Complement) Let $F : \mathbb{V} \to \mathbb{S}^n$ be an affine function partitioned to

$$F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{bmatrix}$$

where $F_{11}(x)$ is square. Then

$$F(x) > 0 \quad \text{iff}$$

$$\begin{cases} F_{11}(x) > 0 \\ F_{22}(x) - F_{21}(x) F_{11}^{-1}(x) F_{12}(x) > 0 \end{cases} \tag{4}$$

Note that the second inequality in (4) is a nonlinear matrix inequality in x. It follows that nonlinear matrix inequalities of the form (4) can be converted to LMIs, and nonlinear inequalities (4) define a convex constraint on x.

Types of LMI problems

Suppose that $F, G : \mathbb{V} \to \mathbb{S}^{n_1}$ and $H : \mathbb{V} \to \mathbb{S}^{n_2}$ are affine functions.

Feasibility: The test whether or not there exist solutions x of $F(x) > 0$ is called a feasibility problem. The LMI is called non-feasible if no solutions exist.

Optimization: Let $f : \mathcal{S} \to \Re$ and suppose that $\mathcal{S} = \{x | F(x) > 0\}$. The problem to determine $V_{\text{opt}} = \inf_{x \in \mathcal{S}} f(x)$ is called an optimization problem with an LMI constraint.

Generalized eigenvalue problem: Minimize a scalar $\lambda \in \Re$ subject to

$$\begin{cases} \lambda F(x) - G(x) > 0 \\ F(x) > 0 \\ H(x) > 0 \end{cases}$$

What are LMIs good for?

Many optimization problems in control design, identification, and signal processing can be formulated using LMIs.

Example. Asymptotic stability of the LTI system

$$\dot{x} = Ax \quad , A \in \Re^{n \times n} \tag{5}$$

Lyapunov said, asymptotically stable iff there exists $X \in \mathbb{S}^n$ such that

$$X > 0, \quad A^T X + XA < 0$$

i.e. equivalent to feasibility of the LMI

$$\begin{bmatrix} X & 0 \\ 0 & -A^T X - XA \end{bmatrix} > 0$$

Example. Determine a diagonal matrix D such that $||DMD^{-1}|| < 1$ where M is some given matrix. Since

$$\begin{aligned} ||DMD^{-1}|| < 1 &\iff D^{-T}M^T D^T DMD^{-1} < I \\ &\iff M^T D^T DM < D^T D \\ &\iff X - M^T XM > 0 \end{aligned}$$

where $X := D^T D > 0$ we see that the existence of such a matrix means the feasibility of LMI.

Example. Let F be an affine function and consider the problem of minimizing $f(x) \triangleq \sigma_{\max}(F(x))$ over x.

$$\begin{aligned} & \lambda_{\max}(F^T(x)F(x)) < \gamma \\ \iff & \gamma I - F^T(x)F(x) > 0 \\ \iff & \begin{bmatrix} \gamma I & F^T(x) \\ F(x) & I \end{bmatrix} > 0 \end{aligned}$$

if we define

$$\bar{x} \triangleq \begin{bmatrix} x \\ \gamma \end{bmatrix}, \quad \bar{F}(\bar{x}) \triangleq \begin{bmatrix} \gamma I & F^T(x) \\ F(x) & I \end{bmatrix}, \quad \bar{f}(\bar{x}) \triangleq \gamma \ ,$$

then $\bar{F}$ is an affine function of $\bar{x}$ and the problem to minimize the maximum eigenvalue of $F(x)$ is equivalent to determining $\inf \bar{f}(\bar{x})$ subject to the LMI $\bar{F}(\bar{x}) > 0$. Hence, this is an optimization problem with a linear objective function $\bar{f}$ and an LMI constraint.

Example(Simultaneous stabilization)
Consider k LTI systems with n-dim state space and m-dim input space:

$$\dot{x} = A_i x + B_i u$$

where $A_i \in \Re^{n\times n}$ and $B_i \in \Re^{n\times m}$, $i \in 1, \cdots, k$. We'd like to find a state feedback law $u = Fx$, $F \in \Re^{m\times n}$ such that the eigenvalues $\lambda(A_i + B_i F)$ lie on the LHP for $i \in 1, \cdots, k$. From the example above, this is solved when we find matrices F and X_i, $i \in 1, \cdots, k$ such that for $i \in 1, \cdots, k$,

$$\begin{cases} X_i > 0 \\ (A_i + B_i F)^T X_i + X_i (A_i + B_i F) < 0 \end{cases} \quad (6)$$

Note that this is *not* a system of LMIs in X_i and F. If we introduce $Y_i = X_i^{-1}$ and $K = FY_i$, then (6) becomes

$$\begin{cases} Y_i > 0 \\ A_i Y_i + Y_i A_i^T + B_i K + K_i^T B_i < 0 \end{cases},$$

which can be further simplified by assuming the existence of a joint Lyapunov function, i.e. $X_i = \cdots = X_k = X$. The joint stabilization problem has a solution if this system of LMIs is feasible.

H_∞ nominal performance

Consider

$$x = Ax + Bu \quad (7)$$
$$y = Cx + Du \quad (8)$$

with state space $X = \Re^n$, input space $U = \Re^m$ and output space $Y = \Re^p$.

proposition If the system (7) is asymptotically stable then $||G||_\infty < \gamma$ whenever there exists a solution $K = K^T > 0$ to the LMI

$$\begin{bmatrix} A^TK + KA + C^TC & KB + C^TD \\ B^TK + D^TC & D^TD - \gamma^2 I \end{bmatrix} < 0. \quad (9)$$

Can compute the H_∞ norm of the transfer function by minimizing $\gamma > 0$ over all variables γ and $K > 0$ that satisfy the LMI.

H_2 nominal performance

We take impulsive inputs of the form $u(t) = \delta(t)e_i$ with e_i the i_{th} basis vector in the standard basis of the input space $\Re^m$. $(i = 1 \cdots m)$. With zero initial conditions, the corresponding output $y_i \in \mathcal{L}_2$ and is given by

$$y_i(t) = \begin{cases} C\exp(At)Be_i & \text{for } t > 0 \\ De_i\delta(t) & \text{for } t = 0 \\ 0 & \text{for } t < 0. \end{cases} .$$

Only if $D = 0$, the sum of the squared norms of all such impulse responses $\sum_{i=1}^{m} ||y_i||_2^2$ is well defined and given by

$$\begin{aligned} \sum_{i=1}^{m} ||y_i||_2^2 &= trace \int_0^\infty B^T \exp(A^t)C^T C\exp(At)B\, dt \\ &= trace \int_0^\infty C\exp(At)BB^T \exp(A^T t)C^T\, dt \\ &= trace \int_{-\infty}^{\infty} G(j\omega)G^*(j\omega)\, d\omega \end{aligned}$$

where G is the transfer function of the system.

proposition Suppose that the system (7) is asymptotically stable (and $D = 0$), then the following statements are equivalent.

(a) $||G||_2 < \gamma$

(b) there exists $K = K^T > 0$ and Z such that

$$\begin{bmatrix} A^T K + KA & KB \\ B^T K & -I \end{bmatrix} < 0; \quad \begin{bmatrix} K & C^T \\ C & Z \end{bmatrix} > 0; \quad (10)$$

$$trace(Z) < \gamma^2 \quad (11)$$

(c) there exists $K = K^T > 0$ and Z such that

$$\begin{bmatrix} AK + KA^T & KC^T \\ CK & -I \end{bmatrix} < 0; \quad \begin{bmatrix} K & B \\ B^T & Z \end{bmatrix} > 0; \quad (12)$$

$$trace(Z) < \gamma^2 \quad (13)$$

pf. note that $||G||_2 < \gamma$ is equivalent to requiring that the controllability gramian $W_c := \int_0^\infty \exp(At)BB^T exp(A^T t)\, dt$ satisfies

$$trace(CWC^T) < \gamma^2.$$

Since the controllability gramian is the unique positive definite solution to the Lyapunov equation

$$AW + WA^T + BB^T = 0$$

this is equivalent to saying that there exists $X > 0$ such that

$$AX + XA^T + BB^T < 0; \qquad trace(CXC^T) < \gamma^2.$$

With a change of variables $K := X^{-1}$, this is equivalent to the existence of $K > 0$ and Z such that

$$A^T K + KA + KBB^T K < 0; \quad CK^{-1}C^T < Z;$$

and

$$trace(Z) < \gamma^2.$$

Now, using Schur complements for the first two inequalities yields that $||G||_2 < \gamma$ is equivalent to the existence of $K > 0$ and Z such that

$$\begin{bmatrix} A^T K + KA & KB \\ B^T K & I \end{bmatrix} < 0; \quad \begin{bmatrix} K & C^T \\ C & Z \end{bmatrix} > 0;$$

and

$$trace(Z) < \gamma^2 \, .$$

The equivalence with (12) is obtained by the observation that $||G||_2 = ||G^T||_2$.

Therefore, the smallest possible upper bound of the H2-norm of the transfer function can be calculated by minimizing the criterion $trace(Z)$ over the variables $K > 0$ and Z that satisfy the LMIs defined by the first two inequalities in (10) or (12).

Controller Synthesis

Let

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u \\ z_\infty &= C_\infty x + D_{\infty 1} w + D_{\infty 2} u \\ z_2 &= C_2 x + D_{21} w + D_{22} u \\ y &= C_y x + D_{y1} w\end{aligned}$$

and

$$\begin{aligned}\dot{x}_K &= A_K x_K + B_K y \\ u &= C_K x_K + D_K y\end{aligned}$$

be state-space realizations of the plant $P(s)$ and the controller $K(s)$ respectively.

Denoting by $T_\infty(s)$ and $T_2(s)$ the CL TF from w to z_∞ and z_2, respectively, we consider the following multi-objective synthesis problem:

Design an output feedback controller $u = K(s)y$ such that

- H_∞ performance: maintains the H_∞ norm of T_∞ below γ_0.

- H_2 performance: maintains the H_2 norm of T_2 below ν_0.

- Multi-objective H_2/H_∞ controller design: minimizes the trade-off criterion of the form $\alpha||T_\infty||_\infty^2 + \beta||T_2||_2^2$ with some $\alpha, \beta \geq 0$.

- Pole placement: places the CL poles in some prescribed LMI region $\mathcal{D}$.

Let the following denote the corresponding CL state-space eqns,

$$
\begin{aligned}
\dot{x_{cl}} &= A_{cl}x_{cl} + B_{cl}w \\
z_\infty &= C_{cl1}x_{cl} + D_{cl1}w \\
z_2 &= C_{cl2}x_{cl} + D_{cl2}w
\end{aligned}
$$

then our design objectives can be expressed as follows:

- H_∞ performance: the CL RMS gain from w to z_∞ does not exceed γ iff there exists a symmetric matrix X_∞ such that

$$
\begin{bmatrix} A_{cl}X_\infty + X_\infty A_{cl}^T & B_{cl} & X_\infty C_{cl1}^T \\ B_{cl}^T & -I & D_{cl1}^T \\ C_{cl1}X_\infty & D_{cl1} & -\gamma^2 I \end{bmatrix} < 0
$$

$$
X_\infty > 0
$$

- H_2 performance: the LQG cost from w to z_2 does not exceed ν iff $D_{cl2} = 0$ and there exists a symmetric matrices X_2 and Q such that

$$\begin{bmatrix} A_{cl}X_2 + X_2A_{cl}^T & B_{cl} \\ B_{cl}^T & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} Q & C_{cl2}^T X_2 \\ X_2 C_{cl2}^T & X_2 \end{bmatrix} > 0$$

$$trace(Q) < \nu^2$$

- Pole placement: the CL poles lie in the LMI region $\mathcal{D} := \{z \in \mathbb{C} : L + Mz + M^T\bar{z} < 0\}$ with $L = L^T = [\lambda_{ij}]_{1\le i,j\le m}$ and $M = [\mu_{ij}]_{1\le i,j\le m}$ iff there exists a symmetric matrix X_{pol} such that

$$\begin{aligned} [\lambda_{ij}X_{pol} + \mu_{ij}A_{cl}X_{pol} + \mu_{ji}X_{pol}A_{cl}^T]_{1\le i,j\le m} &< 0 \\ X_{pol} &> 0\,. \end{aligned}$$

For tractability, we seek a single Lyapunov matrix $X := X_\infty = X_2 = X_{pol}$ that enforces all three sets of constraints. Factorizing X as

$$X = \begin{bmatrix} R & I \\ M^T & 0 \end{bmatrix} \begin{bmatrix} 0 & S \\ I & N^T \end{bmatrix}^{-1}$$

and introducing the transformed controller variables:

$$\begin{aligned} \mathcal{B}_K &:= NB_K + SB_2D_K \\ \mathcal{C}_K &:= C_KM^T + D_KC_yR \\ \mathcal{A}_K &:= NA_KM^T + NB_KC_yR + SB_2C_KM^T \\ &\quad +S(A + B_2D_KC_y)R \ , \end{aligned}$$

the inequality constraints on X are turned into LMI constraints in the variables $R, S, Q, \mathcal{A}_K, \mathcal{B}_K, \mathcal{C}_K$ and D_K. And we have the following suboptimal LMI formulation of our multi-objective synthesis problem:

Minimize $\alpha\gamma^2 + \beta trace(Q)$ over $R, S, Q, \mathcal{A}_K, \mathcal{B}_K, \mathcal{C}_K, D_K$
and γ^2 satisfying:

$$\left[\begin{array}{cc} AR + RA^T + B_2\mathcal{C}_K + \mathcal{C}_K^T B_2^T & \mathcal{A}_K + A + B_2 D_K C_y \\ \star & A^T S + SA + \mathcal{B}_K C_y + C_y^T \mathcal{B}_K^T \\ \star & \star \\ C_\infty R + D_{\infty 2}\mathcal{C}_K & C_\infty + D_{\infty 2} D_K C_y \end{array}\right.$$

$$\left[\begin{array}{c} Q \\ \star \\ \star \end{array}\right.$$

$$\left[\lambda_{ij}\begin{bmatrix} R & I \\ I & S \end{bmatrix} + \mu_{ij}\left[\begin{array}{cc} AR + B_2\mathcal{C}_K & A \\ \mathcal{A}_K & S \end{array}\right.\right.$$

$$\mu_{ji}\left[\begin{array}{c} (AR + B_2\mathcal{C}_K \\ (A + B_2 D_K C \end{array}\right.$$

Given optimal solutions γ^*, Q^* of this LMI problem, the closed loop performances are bounded by

$$||T||_\infty \le \gamma^*, \quad ||T||_2 \le \sqrt{trace(Q^*)}\ .$$

This has been implemented by the matlab command "`hinfmix`".

Reference

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- Scherer C, Weiland S. Linear matrix inequalities in control. Lecture notes of DISC Course

- LMI Control Toolbox, Gahinet, Nemirovski, Laub, Chilali, Mathworks

Affine combinations of linear systems

Often models uncertainty about specific parameters is reflected as uncertainty in specific entries of the state space matrices A, B, C, D. Let $p = (p_1, ..., p_n)$ denote the parameter vector which expresses the uncertain quantities in the system and suppose that this parameter vector belongs to some subset $\mathcal{P} \subset \Re^n$. Then the uncertain model can be thought of as being parameterized by $p \in \mathcal{P}$ through its state space representation

$$\dot{x} = A(p)x + B(p)u \tag{14}$$

$$y = C(p)x + D(p)u \, . \tag{15}$$

One way to think of equations of this sort is to view them as a set of linear time-invariant systems as parameterized by $p \in \mathcal{P}$. However, if p is time, then (14) defines a linear time varying dynamical system and it can therefore also be viewed as such. If components of p are time varying and coincide with state components then (14) is better viewed as a nonlinear system.

Of particular interest will be those systems in which the system matrices affinely depend on p. This means that

$$\begin{aligned} A(p) &= A_0 + p_1 A_1 + \cdots + p_n A_n && (16) \\ B(p) &= B_0 + p_1 B_1 + \cdots + p_n B_n && (17) \\ C(p) &= C_0 + p_1 C_1 + \cdots + p_n C_n && (18) \\ D(p) &= D_0 + p_1 D_1 + \cdots + p_n D_n \,. && (19) \end{aligned}$$

Or, written in a more compact form

$$S(p) = S_0 + p_1 S_1 + ... + p_n S_n$$

where

$$S(p) = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix}$$

is the system matrix associated with (14). We call these models *affine parameter dependent* *models*. In MATLAB such a system is represented with the routines psys and pvec. For $n = 2$ and a parameter box

$$\mathcal{P} \triangleq \{(p_1, p_2) | \, p_1 \in [p_1^{\mathrm{min}}, p_1^{\mathrm{max}}], p_2 \in [p_2^{\mathrm{min}}, p_2^{\mathrm{max}}]\}$$

the syntax is

```
affsys = psys( p, [s0, s1, s2] );
p = pvec( 'box', [p1min p1max ; p2min
p2max])
```

where `p` is the parameter vector whose i-th component ranges between `pimin` and `pimax`. Bounds on the rate of variations, $\dot{p}_i(t)$ can be specified by adding a third argument "`rate`" when calling "`pvec`".

See also the following routines:

- `pdsimul` for time simulations of affine parameter models

- `aff2pol` to convert an affine model to an equivalent polytopic model

- `pvinfo` to inquire about the parameter vector